Variable forgetting in preference relations over propositional domains

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Abstract

Representing (and reasoning about) preference relations over combinatorial domains is computationally expensive. We argue that for many problems involving such preferences, it is relevant to simplify them by projecting them on a subset of variables. We investigate several possible definitions, focusing without loss of generality on propositional (binary) variables. We then define the notion of independence of a preference relation from a set of propositional variables.

1 Introduction

Decision-making problems are concerned with managing agents' preferences. Crucial tasks include modelling, elicitation, aggregation (especially when a common decision has to be made among several agents) and optimization. The computational difficulty of these tasks depends on the size and the structure of the space of alternatives. When the latter is small, preferences can be represented explicitly, by simply ranking alternatives, and the above tasks can be implemented in an easy way. However, in many real-world applications, domains have a combinatorial structure, i.e., an alternative consists of a value of a given domain for each one of a given set of variables. In that case, managing agents' preferences can be an enormous computational burden. This has led some researchers to work on compact preference representation languages.

For some problems it might be relevant to process preference relations (already elicited, and represented in some compact form), so as to simplify it and make it more compact, even if this results in a loss of information. Especially, it may be helpful to *project* a preference relation on a subset of the variables. This way of summarizing a preference relation is relevant in particular when some variables are more important than others, or when some variables should be assigned prior to others. Consider for instance a group decision making scenario. Rather than aggregating the whole preference relations before finding out an optimal assignment of variables, which generally is computationally intractable, it may be a good idea to focus on "primary" variables first, project the preference relation on those variables, aggregate them, decide on the values to be assigned to those variables, and only then consider secondary variables. Such a decomposition of the problem makes it much more tractable (furthermore, it can be argued that human decision makers would probably decompose the problem as well in such a way, which suggests it is cognitively relevant, even if it does not prove anything regarding the best way to automatize it).

Projection operations have not been considered much as far as preference relations are concerned, but there is a huge amount of work about projecting probability distributions represented in compact languages such as Bayesian networks (projection being then referred to as *marginalization*), and more generally projecting valuation functions [Shenoy, 1989; Kohlas & Shenoy, 2000], as well as projecting sets of constraints (as in Constraint Satisfaction Problems), and formulae of propositional logic such as the forgetting operation considered in [Lin & Reiter, 1994]. In this paper we aim at defining similar projection operations for ordinal preference relations, namely, preorders on a set of alternatives. For the sake of simplicity, we focus on combinatorial domains formed from *binary* variables. Section 2 introduces some natural definitions and study their properties. Section 3 makes precise a connection between some notions of projection and the notion of forgetting from propositional logic. The starting point of Section 4 is the study of conditions under which projecting a preference relation can be done without loss of information, which will lead to several notions of independence of a preference relation from a set of variables. Section 5 points out related work and further issues. Due to space limitations, most proofs are omitted.

2 Projection of a preference relation over a set of variables

2.1 Preference relations

Let V be a finite set of propositional variables. For any (possibly improper) subset X of V, an X-*alternative* is an element of 2^X, that is, an assignment of a binary truth value to each one of the variables in X. X-alternatives are denoted by $\vec{x}$, $\vec{x}\,'$ etc. If X and Y are disjoint subsets of V then the concatenation of $\vec{x} \in 2^X$ and $\vec{y} \in 2^Y$ is the $X \cup Y$-alternative, denoted by $\vec{x}\vec{y}$, assigning values to variables of X (resp. Y) as $\vec{x}$ (resp. $\vec{y}$) does.

A V*-preference relation* R, sometimes denoted as $\geq_R$, is a preorder, that is, a reflexive and transitive relation, over 2^V. The *strict preference* $>_R$ associated with R is the strict preorder defined by $\vec{v} >_R \vec{v}\,'$ if and only if $R(\vec{v}, \vec{v}\,')$ and not $R(\vec{v}\,', \vec{v})$. The *indifference relation* $\sim_R$ associated with R is the equivalence relation defined by $\vec{v} \sim_R \vec{v}\,'$ if and only if $R(\vec{v}, \vec{v}\,')$ and $R(\vec{v}\,', \vec{v})$. If neither $R(\vec{v}, \vec{v}\,')$ nor $R(\vec{v}\,', \vec{v})$ then $\vec{v}$ and $\vec{v}\,'$ are *incomparable* w.r.t. R, denoted by $Q_R(\vec{v}, \vec{v}\,')$. If R is connected (that is, $Q_R = \emptyset$), then it is a *complete preference relation.*

As for notation, R^* denotes the transitive closure of a relation R over 2^V.

For any V-preference relation R and any partition $\{X, Y, Z\}$ of V, X is preferentially independent from Y given Z w.r.t. R if and only if for all $\vec{x}, \vec{x}\,' \in 2^X$, all $\vec{y}, \vec{y}\,' \in 2^Y$ and all $\vec{z} \in 2^Z$, $R(\vec{x}\vec{y}\vec{z}, \vec{x}\,'\vec{y}\vec{z})$ implies $R(\vec{x}\vec{y}\,'\vec{z}, \vec{x}\,'\vec{y}\,'\vec{z})$. If $Z = \emptyset$ then we say that X is preferentially independent from $V \setminus X$ w.r.t. R.

2.2 Lower and upper projections

Informally, the projection of a V-preference relation R on $X \subseteq V$ is a preference relation over 2^X obtained from R so as to be as close as possible from the original relation R.

Definition 1 (lower and upper projections) *Let R be a V-preference relation and $X \subseteq V$. Let $Y = V \setminus X$;*

- $R_L^{\downarrow X}$*, called the* lower projection *of R on X, is the binary relation over X defined as follows: $R_L^{\downarrow X}(\vec{x}, \vec{x}\,')$ holds if and only if $R(\vec{x}\vec{y}, \vec{x}\,'\vec{y})$ holds for all $\vec{y} \in 2^Y$;*
- $R_U^{\downarrow X}$*, called the* upper projection *of R on X, is the transitive closure of the binary relation R' over X such that $R'(\vec{x}, \vec{x}\,')$ holds if and only if $R(\vec{x}\vec{y}, \vec{x}\,'\vec{y})$ holds for some $\vec{y} \in 2^Y$.*

Some first properties, where R and R' are V-preference relations and $X, Y \subseteq V$, are the following ones:

Proposition 1

1. *$R_L^{\downarrow X}$ and $R_U^{\downarrow X}$ are X-preference relations;*
2. *if R is complete then $R_U^{\downarrow X}$ is complete;*
3. *if $R \subseteq R'$ then $R_L^{\downarrow X} \subseteq (R')_L^{\downarrow X}$ and $R_U^{\downarrow X} \subseteq (R')_U^{\downarrow X}$;*
4. *$(R \cap R')_L^{\downarrow X} = R_L^{\downarrow X} \cap (R')_L^{\downarrow X}$ and $(R \cap R')_U^{\downarrow X} \subseteq R_U^{\downarrow X} \cap (R')_U^{\downarrow X}$;*
5. *$((R \cup R')^*)_U^{\downarrow X} = (R_U^{\downarrow X} \cup (R')_U^{\downarrow X})^*$ and $(R \cup R')_L^{\downarrow X} \supseteq (R_L^{\downarrow X} \cup (R')_L^{\downarrow X})^*$;*
6. *$(R_U^{\downarrow X})_U^{\downarrow Y} = (R_U^{\downarrow Y})_U^{\downarrow X}$ and $(R_L^{\downarrow X})_L^{\downarrow Y} = (R_L^{\downarrow Y})_L^{\downarrow X}$.*

Of course, $R_L^{\downarrow X} \subseteq R_U^{\downarrow X}$ and a question that comes naturally is when $R_L^{\downarrow X}$ and $R_U^{\downarrow X}$ are the same.

Proposition 2 *For any V-preference relation R and any $X \subseteq V$, $R_L^{\downarrow X} = R_U^{\downarrow X}$ if and only if X is preferentially independent from $V \setminus X$ w.r.t. R.*

Note that, when R is complete, $R_U^{\downarrow X}$ is obviously complete as well but $R_L^{\downarrow X}$ may fail to be complete.

2.3 Optimistic and pessimistic projections

The following definitions exhibit some extra specific notions of a projection.

Definition 2 (optimistic/pessimistic projections) *Let R be a V-preference relation and $X \subseteq V$. Let $Y = V \setminus X$;*

- $R_{StrongOpt}^{\downarrow X}$*, the* strong optimistic projection *of R on X, is defined by: $R_{StrongOpt}^{\downarrow X}(\vec{x}, \vec{x}\,')$ if and only if $\exists \vec{y}$ $\forall \vec{y}\,'$, $R(\vec{x}\vec{y}, \vec{x}\,'\vec{y}\,')$;*
- $R_{WeakOpt}^{\downarrow X}$*, the* weak optimistic projection *of R on X, is defined by: $R_{WeakOpt}^{\downarrow X}(\vec{x}, \vec{x}\,')$ if and only if $\forall \vec{y}\,'$ $\exists \vec{y}$ $R(\vec{x}\vec{y}, \vec{x}\,'\vec{y}\,')$;*
- $R_{StrongPess}^{\downarrow X}$*, the* strong pessimistic projection *of R on X, is defined by: $R_{StrongPess}^{\downarrow X}(\vec{x}, \vec{x}\,')$ if and only if $\exists \vec{y}\,'$ $\forall \vec{y}$, $R(\vec{x}\vec{y}, \vec{x}\,'\vec{y}\,')$;*
- $R_{WeakPess}^{\downarrow X}$*, the* weak pessimistic projection *of R on X, is defined by: $R_{WeakPess}^{\downarrow X}(\vec{x}, \vec{x}\,')$ if and only if $\forall \vec{y}$ $\exists \vec{y}\,'$ $R(\vec{x}\vec{y}, \vec{x}\,'\vec{y}\,')$.*

It is easily checked that these four relations are transitive. The optimistic projections focus on finding some possibility to have $\vec{x}$ dominating $\vec{x}\,'$ whatever the context for $\vec{x}\,'$. The pessimistic projections focus on finding some possibility to have $\vec{x}\,'$ dominated by $\vec{x}$ whatever the context for $\vec{x}$.

When R is complete, $R_{StrongOpt}^{\downarrow X}$ and $R_{WeakOpt}^{\downarrow X}$ coincide, as well as $R_{StrongPess}^{\downarrow X}$ and $R_{WeakPess}^{\downarrow X}$, and all four are complete. In this case, $R_{StrongOpt}^{\downarrow X}(\vec{x}, \vec{x}\,')$ (and equivalently $R_{WeakOpt}^{\downarrow X}(\vec{x}, \vec{x}\,')$) iff the best alternatives extending $\vec{x}$ are at least as good as the best alternatives extending $\vec{x}\,'$, whereas $R_{StrongPess}^{\downarrow X}(\vec{x}, \vec{x}\,')$ (and equivalently $R_{WeakPess}^{\downarrow X}(\vec{x}, \vec{x}\,')$) if and only if the worst alternatives extending $\vec{x}$ are at least as good as the worst alternatives extending
$vx\,'$. These criteria are reminiscent of those used in qualitative decision theory (see e.g. [Brafman & Tennenholtz, 1997; Dubois & Prade, 1995] – with the slightly different interpretation that X-alternatives represent possible decisions and elements of $(V \setminus X)$-alternatives represent possible states of the world.

Proposition 3 *We have the following inclusions.*

- $R_L^{\downarrow X} \subseteq R_{StrongOpt}^{\downarrow X} \subseteq R_{WeakOpt}^{\downarrow X} \subseteq R_U^{\downarrow X}$;
- $R_L^{\downarrow X} \subseteq R_{StrongPess}^{\downarrow X} \subseteq R_{WeakPess}^{\downarrow X} \subseteq R_U^{\downarrow X}$.

2.4 Examples

To begin with, here is an illustration involving the above ideas. Consider a company which is about to move. Presumably, there are a few options to choose from. The Head of the Public Relation Department may prefer the Headquarters to be a new building located downtown rather in some suburb, which he still prefers to an old building downtown, and the least he cares for is an old building in some suburb. The Head of the Accounting Department may wish first a new building downtown, second, an old building downtown, third, an old building in a suburb, all that preferred to a new building in a suburb. The Head of the Legal Department may prefer the Headquarters to be located dowtown, whereas new or old are incomparable in his view (whatever the location).

All this can be formalized using two variables, one for location (x for downtown, so that $\bar{x}$ stands for suburb) and one for generation (y for new, so that $\bar{y}$ stands for old). As regards the Head of the Legal Department, his preferences R can be depicted by the following Hasse diagram of R (arrows point from a more preferred alternative towards a less preferred one):[1]

$$\begin{array}{ccc} xy & & x\bar{y} \\ \downarrow & \times & \downarrow \\ \bar{x}y & & \bar{x}\bar{y} \end{array}$$

We now give many more examples.

Example 1

$$R: \begin{array}{ccc} xy & & x\bar{y} \\ \downarrow & & \downarrow \\ \bar{x}y & & \bar{x}\bar{y} \end{array}$$

All projections on x coincide and are equal to the preference relation $x > \bar{x}$.

All projections on y coincide and are equal to the preference relation in which y and $\bar{y}$ are incomparable.

Example 2

$$R: \begin{array}{ccc} xy & \rightarrow & x\bar{y} \\ & & \\ \bar{x}y & \leftarrow & \bar{x}\bar{y} \end{array}$$

All projections on x coincide and are equal to the preference relation where x and $\bar{x}$ are incomparable.

$R_L^{\downarrow\{y\}}$ as well as $R_{StrongOpt}^{\downarrow\{y\}}$ and $R_{StrongPess}^{\downarrow\{y\}}$ are equal to the preference relation in which y and $\bar{y}$ are incomparable, while $R_U^{\downarrow\{y\}}$ as well as $R_{WeakOpt}^{\downarrow\{y\}}$ and $R_{WeakPess}^{\downarrow\{y\}}$ are equal to the preference relation $y \sim \bar{y}$.

[1] For the sake of notation, when we specify a preference relation explicitly, we omit pairs coming from reflexivity and transitivity. For instance, the relation denoted by $x > \bar{x}$ is, more rigorously, the relation $\{(x, \bar{x}), (x, x), (\bar{x}, \bar{x})\}$, while the relation denoted by $x \sim \bar{x}$ is, more rigorously, the relation $\{(x, \bar{x}), (\bar{x}, x), (x, x), (\bar{x}, \bar{x})\}$.

Example 3

$$R: \begin{array}{ccc} & xy & \\ & \downarrow & \\ & \bar{x}\bar{y} & \\ \swarrow & & \searrow \\ \bar{x}y & & x\bar{y} \end{array}$$

$R_L^{\downarrow\{x\}}$ is the preference relation in which x and $\bar{x}$ are incomparable; $R_U^{\downarrow\{x\}}$ is the preference relation $x \sim \bar{x}$; $R_{StrongOpt}^{\downarrow\{x\}}$ and $R_{WeakOpt}^{\downarrow\{x\}}$ are equal to the preference relation $x > \bar{x}$; $R_{StrongPess}^{\downarrow\{x\}}$ is the preference relation in which x and $\bar{x}$ are incomparable, while $R_{WeakPess}^{\downarrow\{x\}}$ is the preference relation in which $x \sim \bar{x}$.

Things are symmetric for the projections on y.

Example 4

$$R: \begin{array}{ccc} & xy & \\ \swarrow & & \searrow \\ \bar{x}y & & x\bar{y} \\ \searrow & & \swarrow \\ & \bar{x}\bar{y} & \end{array}$$

All projections on x (resp. y) are equal to the preference relation $x > \bar{x}$ (resp. $y > \bar{y}$).

Example 5

$$R: \begin{array}{c} xy \\ \downarrow \\ x\bar{y} \\ \downarrow \\ \bar{x}\bar{y} \\ \downarrow \\ \bar{x}y \end{array}$$

All projections on x are equal to the preference relation $x > \bar{x}$.

$R_L^{\downarrow\{y\}}$ is the preference relation in which y and $\bar{y}$ are incomparable; $R_U^{\downarrow\{y\}}$ is the preference relation $y \sim \bar{y}$; the optimistic projections on y (which coincide because R is complete) are equal to the preference relation $y > \bar{y}$; the pessimistic projections on y (which coincide, again because R is complete) are equal to the preference relation $\bar{y} > y$.

Observe that R is a formal representation of the preferences expressed by the Head of the Accounting Department. That all projections on x (the location variable) amount to $x > \bar{x}$ indeed illustrates that the Head of the Accounting Department favors the Headquarters being located downtown: His preference old vs. new is only next to location, and depends on what the location is (see the fact that the various projections on y do not coincide). The lower projection on y shows that the Head of the Accounting Department does not inconditionnally prefer old to new (or vice-versa). The upper projection on y shows that the preferences of the Head of the Accounting Department include a situation such that he prefers new to old (downtown) and a situation such that he prefers old to new (suburb).

Example 6

$$R: \begin{array}{ccc} xy & \sim & x\bar{y} \\ & \downarrow & \\ \bar{x}y & \sim & \bar{x}\bar{y} \end{array}$$

All projections on x are equal to the preference relation $x > \bar{x}$. All projections on y are equal to the preference relation $y \sim \bar{y}$.

3 Connection to propositional logic

L_V is the propositional language built up from V, together with the usual connectives and the Boolean constants $\top$ and $\bot$. Formulas of L_V are denoted by Greek letters φ, ψ, θ, etc. $Var(\varphi)$ denotes the set of propositional variables occurring in φ.

We make use of the next two notions from [Lin, 2001] where $\varphi \in L_V$ and $X \subseteq V$:

- the *strongest necessary condition* of φ on X is the strongest formula ψ of L_V such that $Var(\psi) \subseteq X$ and $\varphi \models \psi$;
- the *weakest sufficient condition* of φ on X is the weakest formula ψ of L_V such that $Var(\psi) \subseteq X$ and $\psi \models \varphi$.

The strongest necessary condition (resp., weakest sufficient condition) of φ on X is denoted by $\exists(V \setminus X).\varphi$ (resp., $\forall(V \setminus X).\varphi$). Actually, $\exists(V \setminus X).\varphi$ is usually known as the *forgetting* of $V \setminus X$ in φ.

A V-preference relation is *bipartite* if and only if there exists $G \subseteq 2^V$ such that for all $\vec{v}$, $\vec{v}\,' \in 2^V$, then $R(\vec{v}, \vec{v}\,')$ holds if and only if $\vec{v} \in G$ or $\vec{v}\,' \in 2^V \setminus G$; the *characteristic formula θ_R* of a bipartite V-preference relation R is the propositional formula – unique up to logical equivalence – whose set of models is exactly G (in symbols, $Mod(\theta_R) = G$).

So, a bipartite preference relation R can be represented by a propositional formula. Then, it is worthwhile investigating how can some notions of a projection over bipartite preference relations be similarly captured by propositional formulas. The connection is most significant when considering optimistic and pessimistic projections (note that if R is bipartite, it is complete and then strong and weak notions coincide.)

Proposition 4 *Let R be a bipartite preference relation whose characteristic formula is θ_R. Let $X \subseteq V$ and $Y = V \setminus X$. Then*

- $R^{\downarrow X}_{WeakOpt} = R^{\downarrow X}_{StrongOpt}$ *is the bipartite relation whose characteristic formula is $\exists(V \setminus X).\theta_R$.*
- $R^{\downarrow X}_{WeakPess} = R^{\downarrow X}_{StrongPess}$ *is the bipartite relation whose characteristic formula is $\forall(V \setminus X).\theta_R$.*

Moreover, if θ_R is logically equivalent to a formula of L_X then

- $R^{\downarrow X}_{U} = R^{\downarrow X}_{WeakOpt} = R^{\downarrow X}_{StrongOpt}$ *is the bipartite relation whose characteristic formula is $\exists(V \setminus X).\theta_R$.*
- $R^{\downarrow X}_{L} = R^{\downarrow X}_{WeakPess} = R^{\downarrow X}_{StrongPess}$ *is the bipartite relation whose characteristic formula is $\forall(V \setminus X).\theta_R$.*

As already mentioned, the deepest result here is with optimistic and pessimistic projections. The basic reason is that the way optimistic and pessimistic projections are defined requires *all* $V \setminus X$-alternatives extending the same context to behave alike (w.r.t. R) hence R can be partitioned along the language (actually, just the variables in $V \setminus X$). Since lower and upper projections have no such definition, the above constraint on θ_R as being logically independent of $V \setminus X$ provides the necessary link between R and its potential partitions along the language.

4 Independence of a preference relation from a set of variables

This section requires a couple of notions, as follows.

If $X \subseteq V$ and $\vec{v} \in 2^V$ then the *X-conjugate of $\vec{v}$*, denoted by $switch(\vec{v}, X)$, is the alternative obtained from $\vec{v}$ by switching the truth value of each $x \in X$ (and leaving the other variables unchanged). When X is a singleton consisting of a single variable x, we drop the curly brackets, writing $switch(\vec{v}, x)$ as the x-conjugate of $\vec{v}$.

Let $switch(R, X)$ be the relation obtained from R by exchanging each alternative $\vec{v}$ with its X-conjugate, that is, $switch(R, X)(\vec{v}, \vec{v}\,')$ if and only if $R(switch(\vec{v}, X), switch(\vec{v}\,', X))$.

4.1 Definitions and properties

The introduction motivates the need to simplify preference relations so that applying one is possible just by handling part of it. Clearly, a projection provides such an abridged version of a preference relation. The question is what conditions, if any, allows us to substitute a projection for the original preference without losing relevant information? A general answer is that projection over $V \setminus X$ is presumably harmless when X can in some sense be dispensed with, i.e. R is independent of X.

Definition 3 *Let R be a V-preference relation and let $X \subseteq V$ and $Y = V \setminus X$.*

I-independence *R is I-independent of X if and only if for all $\vec{x}, \vec{x}\,' \in 2^X$ and all $\vec{y} \in 2^Y$, $\vec{x}\vec{y} \sim_R \vec{x}\,'\vec{y}$.*

Q-independence *R is Q-independent of X if and only if for all $\vec{x}, \vec{x}\,' \in 2^X$ and all $\vec{y} \in 2^Y$, $\vec{x}\vec{y}$ and $\vec{x}\,'\vec{y}$ are incomparable w.r.t. R.*

G-independence *R is G-independent of X if and only if $switch(R, X) = R$.*

We might think of a stronger definition of G-independence, where invariance of R by any permutation on 2^X is required instead of invariance of R by permutations of single variables. Let us first introduce the following definition:

- let σ be a permutation of 2^X; then $\sigma(R)$ is the V-preference relation obtained from R by letting $\sigma(R)(\vec{v}, \vec{v}\,')$ hold if and only if $R(\sigma(\vec{v}), \sigma(\vec{v}\,'))$ holds.

Fortunately, this notion, which may appear stronger at first glance, is equivalent to the one we gave above:

Proposition 5 *R is G-independent from X if and only if $\sigma(R) = R$ holds for every permutation σ of 2^X.*

Interestingly, all three notions above satisfy the property of *decomposability.*

Proposition 6 *For any of the three notions of independence (I, Q and G), R is independent from X if and only if R is independent for every x in X.*

There is at least one interesting notion of independence that fails decomposability, though. It comes from preferential independence:

Definition 4 *Let R be a V-preference relation and let $X \subseteq V$ and $Y = V \setminus X$.*

P-independence *R is P-independent of X if and only if Y is preferentially independent of X w.r.t. R.*

Intuitively, P-independence w.r.t. x means that if you want to compare two alternatives then you do not have to worry about x as long as both alternatives share the same value for x: The outcome would be the same for another value of x. Back to the company illustration, if the preference relation is independent from the variable "logo of the company", then you can compare "old&downtown" against "new&suburban" just by fixing "logo of the company" to whatever value and then directly compare "old&downtown&logo" against "new&suburban&logo" because the outcome would be exactly the same as when comparing "old&downtown&otherlogo" against "new&suburban&otherlogo".

Two further definitions may be thought of, namely:

union independence R is U-independent of X if and only if R is I-independent of X or Q-independent of X.

weak independence R is W-independent of X if and only if for all $\vec{x}, \vec{x}\,' \in 2^X$ and all $\vec{y} \in 2^Y$, $\vec{x}\vec{y}$ and $\vec{x}\,'\vec{y}$ are either indifferent or incomparable w.r.t. R.

Proposition 7 *Let R be a V-preference relation and let $x \in V$. We have the following implications:*

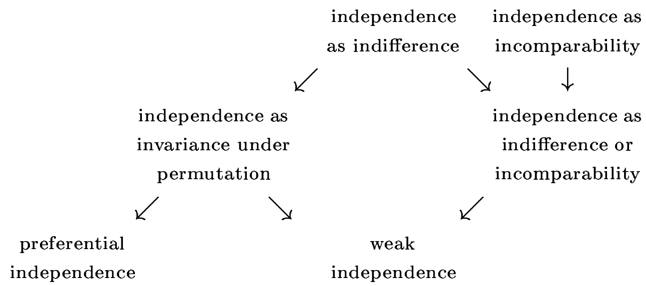
- $Ind_I(R,x) \Rightarrow Ind_G(R,x) \Rightarrow Ind_W(R,x)$;
- $Ind_I(R,x) \Rightarrow Ind_U(R,x) \Rightarrow Ind_W(R,x)$;
- $Ind_Q(R,x) \Rightarrow Ind_U(R,x)$;
- $Ind_G(R,x) \Rightarrow Ind_P(R,\{x\})$.

Note that Ind_G and Ind_U (and Ind_Q) are incomparable, which can be seen on the following two counterexamples:

- $R = \{(x\bar{y}, xy), (\bar{x}y, \bar{x}\bar{y})\}$. Then $switch(R,x) = \{(xy, x\bar{y}), (\bar{x}\bar{y}, \bar{x}y)\}$, therefore we do not have $Ind_G(R,x)$, whereas we have $Ind_Q(R,x)$ and a fortiori $Ind_U(R,x)$. Therefore $Ind_Q(R,x)$ does not imply $Ind_G(R,x)$ and likewise, $Ind_U(R,x)$ does not imply $Ind_G(R,x)$.
- $R = \{(xy, x\bar{y}), (x\bar{y}, xy)\}$. Then $switch(R,x) = R$, however neither $Ind_Q(R,x)$ nor $Ind_I(R,x)$ holds, therefore $Ind_U(R,x)$ does not hold. Therefore $Ind_G(R,x)$ does not imply $Ind_U(R,x)$.

Note also that:

Table 1: Relationships between notions of independence. Arrows point from the more demanding notion to the less demanding.



- With $R = \{(x\bar{y}, xy)\}$, we have $switch(R,x) = \{(xy, \bar{x}y)\}$. Hence, we do not have $Ind_G(R,x)$, whereas we have $Ind_P(R,\{x\})$. That is, $Ind_P(R,\{x\})$ does not imply $Ind_G(R,x)$.
- $Ind_G(R,x)$ does imply $Ind_P(R,\{x\})$ (but this is only because x is a single variable, otherwise $Ind_G(R,X)$ may fail to imply $Ind_P(R,X)$ when X is not a singleton set).

4.2 Examples

Example 1 (continued)

$$R: \begin{array}{cc} xy & x\bar{y} \\ \downarrow & \downarrow \\ \bar{x}y & \bar{x}\bar{y} \end{array}$$

R is Q-independent of y and G-independent of y but not I-independent of y (it is obviously not independent from x for any notion of independence considered).

Example 2 (continued)

$$R: \begin{array}{ccc} xy & \rightarrow & x\bar{y} \\ & & \\ \bar{x}y & \leftarrow & \bar{x}\bar{y} \end{array}$$

R is Q-independent of y but neither G-independent nor I-independent of y.

Example 3 (continued)

$$R: \begin{array}{ccc} & xy & \\ & \downarrow & \\ & \bar{x}\bar{y} & \\ \swarrow & & \searrow \\ \bar{x}y & & x\bar{y} \end{array}$$

R is independent neither of x nor of y, whatever the notion of independence under consideration. Idem for the preference relations of Examples 4 and 5.

Example 6 (continued)

$$R: \begin{array}{ccc} xy & \sim & x\bar{y} \\ & \downarrow & \\ \bar{x}y & \sim & \bar{x}\bar{y} \end{array}$$

R is I-independent and G-independent of y, but not Q-independent of y. (It obviously fails to be independent of x, whatever the notion of independence under consideration.)

4.3 Independence and projection

Proposition 8 *Let $X \subseteq V$. For any X-preference relation R_X there is a unique V-preference relation R such that (a) $Ind_I(R, V \setminus X)$ and (b) $R_L^{\downarrow X}$ coincides with R_X.*

However, there is no such result as regards Q-independence and G-independence. Here are two counterexamples.

- First, consider R_1 to be the reflexive-transitive closure of $\{(xy, \bar{x}y)\}$ and R_2 to be the reflexive-transitive closure of $\{(x\bar{y}, \bar{x}\bar{y})\}$. Both R_1 and R_2 are Q-independent of $Y = V \setminus X$ where $X = \{x\}$. Also, the empty set is the lower projection of R_1 on X. Similarly, the empty set is the lower projection of R_2 on X. Furthermore, R_1 and R_2 have the same upper projection on X, that is the reflexive-transitive closure of $\{(x, \bar{x})\}$.
- Second, consider R_1 to be the reflexive-transitive closure of $\{(xy, \bar{x}y), (x\bar{y}, \bar{x}\bar{y}), (\bar{x}y, x\bar{y}), (\bar{x}\bar{y}, \bar{x}y)\}$ and R_2 to be the reflexive-transitive closure of $\{(xy, \bar{x}y), (x\bar{y}, \bar{x}\bar{y}), (\bar{x}y, \bar{x}\bar{y})\}$. Both R_1 and R_2 are G-independent of Y. R_1 and R_2 have the same lower projection on X, that is $\{(x, \bar{x})\}^*$, which is also their upper projection on X.

Proposition 9 *If R is $Pref$-independent of $V \setminus X$ then each of $R_U^{\downarrow X}$ and $R_L^{\downarrow X}$ coincides with the restriction of R to 2^X.*

5 Conclusion and perspectives

This paper is meant to pave the way towards simplifying and decomposing preference relations over combinatorial structures, by investigating and comparing various notions of projection and independence. It is still a preliminary work and raises many questions.

One of the most salient issues that we did not investigate is about computing the various notions of projection (as well as checking the various notions of independence) when the initial preference relation is represented in a *compact representation language* such as CP-nets [Boutilier *et al.*, 2004] or a language based on propositional logic (see e.g. [Lang, 2004] for a survey). The problem is then the following: given a compact structure (e.g., a CP-net) representing a preference relation R in a compact way, compute another input of the same language (e.g. another CP-net) that represents the projection of R on a given subset of variables X w.r.t. one of the various definitions given in this paper. Clearly, we are looking for algorithms that would perform this computation *directly* (without generating R explicitly, nor even its projection on X). This looks harder than we initially thought and is certainly a promising issue for further research. As to independence, it would be worth investigating the computational complexity of checking, for a given notion of independence and a given representation language, whether a given compactly represented preference relation is independent from a given set of variables (in the same vein as the work in [Lang, Liberatore, & Marquis, 2003] for independence and forgetting in propositional logic).

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