Twenty-Five Years of Preferred Subtheories

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Abstract. In the seminal paper [6], Gerd Brewka argued that ranking a set of default rules without prerequisites, and selecting extensions according to a lexicographic refinement of the inclusion ordering proves to be a natural, simple and efficient way of dealing with the multiple extension (or "subtheories") problem. This natural idea has been reused, discussed, revisited, reinvented, adapted many times in the AI community and beyond. Preferred subtheories do not only have an interest in default reasoning, but also in reasoning about time, reasoning by analogy, reasoning with compactly represented preferences, judgment aggregation, and voting. They have several variants (but arguably not so many). In this short paper I will say as much as I can about preferred subtheories in sixteen pages.

1 Prioritized Default Theories and Preferred Subtheories

Preferred subtheories were introduced in [6] as a way of representing and exploiting priorities between default rules. Their starting point was the THEORIST system [28] for default reasoning. In Poole's system – equivalent to the restriction of Reiter's default logic to normal defaults without prerequisites – a default theory is a set of facts F plus a set of hypotheses Δ (both composed of logical formulas) and an extension is the set of logical consequences of a set-inclusion maximal subset D of Δ such that $D \cup F$ is consistent. In spite of the (apparently drastic) restriction to normal defaults without prerequisites, this system is able to deal adequately with many of the standard default reasoning examples from the literature, but not all, because of the impossibility of expressing priorities between defaults. Let me reuse this example from [6], suggested to Gerd Brewka by Ulrich Junker.

> "Usually one has to go to a meeting.
> This rule does not apply if somebody is sick, unless he only has a cold.
> The rule is also not applicable if somebody is on vacation."

As shown by [6] (Section 3), given that the person is sick, the natural writing of this example in Poole's system generates two extensions: one where she has to attend the meeting and one where she does not. In order to avoid this, one would need to

> "(...) look down in the hierarchy of exceptions and block defaults lower in the hierarchy. (...) the number of defaults may increase heavily in cases where more exceptions and exceptions of exceptions are involved"

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which is arguably unpleasant and inefficient. The core of the problem is the impossibility to express that a default has a priority over an other default – in this case, when an agent only has a cold and is on vacation, the rule that someone on vacation does not have to attend should have priority over the rule that someone who has only a cold has to attend. This inability to represent priorities in Poole's system was Brewka's motivation for generalizing it by introducing explicit priorities among defaults. By convention, degree 1 corresponds to the highest priority defaults.

Definition 0. *A* ranked default theory T *is a tuple* $(T_1, \ldots, T_n)$ *where each* T_i *is a set of classical first-order (possibly open) formulas. Without loss of generality, we assume that all formulas appearing in* T *are different.*[1]

The meeting example is expressed as

$$\begin{array}{ll} T_1 = \{ & cold \rightarrow sick, vacation \rightarrow \neg r_1, cold \rightarrow \neg r_2, \\ & r_2 \wedge sick \rightarrow \neg r_1, r_1 \rightarrow meeting \quad \} \\ T_2 = \{r_2\} & \\ T_3 = \{r_1\} & \end{array}$$

while the classical Tweety story is expressed as

$$\begin{array}{l} T_1 = \{bird(tweety), \forall x.penguin(x) \rightarrow bird(x)\} \\ T_2 = \{penguin(x) \rightarrow \neg flies(x)\} \\ T_3 = \{bird(x) \rightarrow flies(x)\} \end{array}$$

Now, it remains to define the *preferred subtheories* of a ranked default theory. This beautiful yet simple notion has several equivalent characterizations, each of which can be used as a definition. Below we give no less than six definitions; two others will come in Section 2.

We first define a *subtheory* of T as a tuple $S = (S_1, \ldots, S_n)$ with $S_i \subseteq T_i$ for each i, and such that $\cup_i S_i$ is consistent. By abuse of language, we also consider S as a subset of T, that is, we sometimes note $\delta \in S$ for ($\delta \in S_i$ for some i). The first defin
give is Brewka's original defin2(io)-7.0999999(n)-223.899993(*S is a preferred subtheory of T*)9999(])-5.5(.)-233(I)-5.5(n)-23
iff for all $k = 1, \ldots, n$ S

$$_1 \cup \ldots \cup S_k \textit{ is a maximal consistent subset of } T_1 \cup \ldots \cup T_k.$$

To paraphrase the definition in the author's terms:

> "(...) to obtain a preferred subtheory of T we have to start with any maximal consistent subset of T_1, add as many formulas from T_2 as coasistently can be added (in any possible way),01and continue this process for $T_3, \ldots, T_n$."

This explanation does not in fact correspond to Definition 1, but to the following equivalent, more constructive definition with a clear algorithmic flavour, which is also

[1] T his is without loss of generality, because if a fomula appears several times, all its occurrences except one can be rewritten into syntactically different, equivalent formulas. We could have chosen to allow some formulas to appear several times, but then each T_i shoul be defined as a multiset rather than a set, and this would be slightly more complicated.

the definition used for prioritized removal in prioritized base revision [25]. Given two sets of formulas F and G, we say that $G' \subseteq G$ is maximal F-consistent if $G' \cup F$ is consistent and for all G'' such that $G' \subset G'' \subseteq G$, $G'' \cup F$ is inconsistent.

Definition 2 (preferred subtheories, second definition). *S is a* preferred subtheory *of T iff for all $i = 1, \ldots, n$, S_i is a maximal $(S_1 \cup \ldots S_{i-1})$-consistent subset of T_i.*

In the meeting example, if we add the facts $F = \{cold, vacation\}$ to T_1, then $meeting$ is not derived from the preferred subtheory $F \cup T_1$; but if we add only $F' = \{cold\}$, then the preferred subtheory is $F' \cup T_1 \cup \{r_1\}$, and $meeting$ is derived, and if we don't add any fact, then the preferred subtheory is $T_1 \cup \{r_1, r_2\}$, and again, $meeting$ is derived. In the Tweety example, the only preferred subtheory is $T_1 \cup T_2$ and allows to derive $\neg flies(Tweety)$.

Here is another example with more than one preferred subtheory: $T = T_1 \cup T_2 \cup T_3$ with $T_1 = \{a \vee b, a \rightarrow c\}, T_2 = \{\neg a, \neg b\}, T_3 = \{\neg c\}$. T has two preferred subtheories: $T_1 \cup \{\neg a, \neg c\}$ and $T_1 \cup \{\neg b\}$.

Two dual notions of provability from a default theory can be defined: given a default theory T, formula α is *strongly provable* from T if for every preferred subtheory S of T we have $S \models \alpha$, and *weakly provable* from T if for some preferred subtheory S of T we have $S \models \alpha$. These notions come back to Rescher [29] and have been used and discussed many times afterwards, under different names such as credulous and skeptical inferences, in various areas such as nonmonotonic reasoning, belief revision, inconsistency-tolerant reasoning, argumentation, and beyond (see, *e.g.*, [9,4]). In this short paper we focus on subtheories and won't discuss inference again.

The third definition is the basis of Brewka's second generalization of Poole's system [6], introduced for priority orders between defaults that are strict partial orders (see Section 2.1). It says that a preferred subtheory can be obtained by consistently adding formulas in any possible order that respects the priority relation. Given two defaults δ, δ' of T, let $r(\delta)$ be the integer i such that $\delta \in T_i$.[2] A ranking of T is a bijective mapping σ from $\{1, \ldots, |T|\}$ to T; for all $i \leq |T|$ we note $\sigma(i) = \delta_i$. We say that σ *respects* T iff for all $\delta_i, \delta_j \in T$, $r(\delta) < r(\delta')$ implies $i < j$.

Definition 3 (preferred subtheories, third definition). *S is a* preferred subtheory *of T if there is a ranking σ of T respecting T, such that $S = S_\sigma$, where S_σ is defined inductively by:*

$\Sigma_0 = \emptyset$
for $i = 1, \ldots, n$ ***do***
    if $\Sigma_{i-1} \cup \{\delta_i\}$ *is consistent* ***then***
        $\Sigma_i = \Sigma_{i-1} \cup \{\delta_i\}$
    else
        $\Sigma_i = \Sigma_{i-1}$
    end if
end for
return $S_\sigma = \Sigma$

[2] Recall that we assumed that $T_i \cap T_j = \emptyset$ for $i \neq j$.

The next definition is based on the "discrimin" order (the terminology comes from [16]); it appears under different forms in [21] (there the definition works also for partially ranked default theories), [11] (under the name "democratic"), and [19] (in the context of soft constraint satisfaction problems).

Definition 4 (preferred subtheories, fourth definition). *Let S and S' be two subtheories of T. Define $MinIndex(S \setminus S') = \min\{j | S_j \setminus S'_j \neq \emptyset\}$. We say that S is discrimin-preferred to S' with respect to T, denoted by $S \succ_T^{discrimin} S'$, if $MinIndex(S \setminus S') < MinIndex(S' \setminus S)$. Finally, S is a* preferred subtheory *of T if there is no consistent subtheory S' of T such that $S' \succ_T^{discrimin} S$.*

The next definition we give is from [17,3].

Definition 5 (preferred subtheories, fifth definition). *Let S and S' be two subtheories of T. We say that S is preferred to S' with respect to T, denoted by $S \succ_T S'$, if and only if there is some $k \leq n$ such that*

- *for all $i \leq k$, $S_i = S'_i$;*
- *$S_k \supset S'_k$.*

Finally, S is a preferred subtheory *of T if there is no subtheory S' of T such that $S' \succ_T S$.*

The last definition is semantical, as it is based on a preference relation over interpretations. Let PS be the set of propositional symbols on which the formulas of T are defined. Given an interpretation $I \in 2^{PS}$, and a default theory, we define $Sat(T_i, I) = \{\delta \in T_i \mid I \models \delta\}$ and $Sat(T, I) = (Sat(T_1, I), \ldots, Sat(T_n, I))$. Note that $Sat(T, I)$ is a subtheory of T.

Definition 6 (preferred subtheories, sixth definition). *Given two interpretations I, $I' \in 2^{PS}$, we say that I is preferred to I' with respect to T, denoted by $I \succ_T I'$, if and only if there is some $k \leq n$ such that*

- *for all $i \leq k$, $Sat(T_k, I) = Sat(T_k, I')$.*
- *$Sat(T_i, I) \supset Sat(T_i, I')$.*

Finally, I is a preferred model with respect to T iff there is no I' such that $I' \succ_T I$, and S is a preferred subtheory *of T if there exists a preferred model I with respect to T such that $Sat(T, I) = S$.*

Proposition 1. *Definitions 1, 2, 3, 4, 5 and 6 are equivalent.*

This result is more or less a "folklore" result[3], in the sense that most equivalences are already known without there being an well-identified reference for them. Still, some equivalences have been proven in [3,16] (and probably elsewhere, I apologize

[3] Ulrich Junker made me notice that "folklore" may be understood by some people in a pejorative way (*e.g.*, for unproven claims). It should be clear that the meaning I give here to this word is the same as there:
`http://en.wikipedia.org/wiki/Mathematical_folklore`

for missed references). I however give a proof (in Appendix), not only for the sake of completeness, but also because I cannot see a place where all these definitions are assembled and proven equivalent.

While the notion of preferred subtheory is based on set inclusion, there is a natural variant, defined in [3], based on cardinality. Our definition is a variant of the *second* definition of a preferred subtheory. If X and Y are two sets of formulas, a *maxcard* X-consistent subset of Y is a X-consistent subset Z of Y such that for all $Z' \subseteq Y$, Z' is X-consistent implies $|Z'| \leq |Z|$.

Definition 7 (cardinality-preferred subtheories, first definition). *S is a* C-preferred subtheory *of T iff for all $k = 1, \ldots, n$, S_i is a maxcard $(S_1 \cup \ldots S_{i-1})$-consistent subset of T_i.*

Again we have equivalent definitions, but less than for preferred subtheories. The second definition has been proposed by [17,3] under the name "lexicographic preferred subbases" and in [24] under the name "lexicographic closure".

Definition 8 (cardinality-preferred subtheories, second definition). *S is a C-preferred subtheory of T if there is no subtheory S' of T such that $S' \succ_T^C S$, where $S' \succ_T^C S$ if for some $k \leq n$ we have*

- *for all $i \leq k$, $|S_i| = |S'_i|$;*
- $|S_k| > |S'_k|$.

Definition 9 (cardinality-preferred subtheories, third definition). *Define $I' \succ_T^C I$ if and only if there is some $k \leq n$ such that*

- *for all $i \leq k$, $|Sat(T_i, I)| = |Sat(T_i, I')|$.*
- $|Sat(T_k, I)| > |Sat(T_k, I')|$.

Then S is a C-preferred subtheory of T if $S = Sat(T, I)$ for some C-preferred model I with respect to T, where I is C-preferred w.r.t. T if there is no I' such that $I' \succ_T^C I$.

Proposition 2. *Definitions 7, 8 and 9 are equivalent.*

This is again a "folklore" result. We omit the proof, which is similar to the proof of Proposition 1.

While Definitions 3 and 4 do not seem to be adaptable to cardinality-preferred subtheories, Definition 1 can, but interestingly, leads to a more conservative notion, based on first-order stochastic dominance:

Definition 10 (SD-preferred subtheories, first definition). *S is an* SD-preferred subtheory *of T iff for all $k = 1, \ldots, n$, $S_1 \cup \ldots \cup S_k$ is a maxcard consistent subset of $T_1 \cup \ldots \cup T_k$.*

Again is is possible to give two equivalent definitions (which we omit).

Let $PST(T)$ be the set of preferred subtheories of T, $CPST(T)$ be the set of C-preferred subtheories of T, and $SDPST(T)$ be the set of SD-preferred subtheories of T. Then we have these straightforward facts:

1. $PST(T) \supseteq CPST(T) \neq \emptyset$.
2. if $SDPST(T) \neq \emptyset$ then $CPST(T) = SDPST(T)$.

Sometimes the set of SD-preferred subtheories is empty. Let $T = (\{a \wedge b\}, \{\neg a, \neg b\})$. T has a single C-preferred subtheory, namely $S = (\{a \wedge b\}, \emptyset)$. However S is not a SD-preferred subtheory of T, because $\{a \wedge b\}$ is not a maxcard subset of $\{a \wedge b, \neg a, \neg b\}$.

2 What For? Where Do Priorities Come From?

One key question is, where do these priorities come from, what do they correspond to? As we will see below, there is not a single but a lot of different interpretations of priorities, in various domains of knowledge representation, reasoning, and decision making, which in turn correspond to various understandings of preferred subtheories. I will review here several such interpretations – no less then five, and I'm sure I forget some. Two of these interpretations will allow us to derive new equivalent characterizations of preferred subtheories, in case the reader would think we don't have enough with the six already stated.

2.1 Default Reasoning

The interpretation that Brewka had in mind in [6] was default reasoning. Priorities there correspond to a precedence order bearing on the application of default rules, and allowing to choose between multiple extensions. The examples he uses (two of which are quoted in Section 1) are of that kind: the rule that penguins do not fly has precedence over the rule that birds fly, in the sense that when both are "candidate for application", the first one should be applied first (which, here, implies that the second one will *not* be applied). While [6] deals with normal defaults without prerequisites, also called supernormal defaults, he goes further in [7] and extends the framework to normal defaults.

Brewka argues that there are two kinds of priorities: *explicit* and *implicit* priorities, that I'd prefer to call *exogeneous* and *endogeneous*. Quoting from [7]:

> A number of different techniques for handling priorities of defaults have been developed. Two main types of approaches can be distinguished:
> 1. approaches which handle explicit priority information that has to be specified by the user and is not part of the logical language (...)
> 2. approaches which handle implicit priority information based on the specificity of defaults (...).
>
> (...) For real world applications it seems unrealistic to assume that all relevant priorities can be specified by the user explicitly. On the other hand, specificity as the single preference criterion is (...) insufficient in many cases.

As a consequence, he argues that both types of priorities should be handled together in an homogeneous way.

Deriving priorities from specificity relations between default rules originates in the work on conditionals by [1] and was given more attention in a number of papers starting from Pearl's System Z [27]. This systematic construction of priorities from the expression of defaults is beautiful and elegant, but insufficient when the defaults are not ordered into a single specificity hierarchy: for instance, if Δ contains $\delta_1 =$ birds fly, δ_2 = birds that can be seen in Antarctica don't fly, δ_3 = birds that can be seen in Antarctica because they escaped from a ship fly, δ_4 = birds that can be seen in Antarctica because they escaped from a ship but had their wings broken during the trip don't fly, δ_5 = lions eat meat, δ_6 = vegetarian lions don't eat meat, then System-Z will produce the following ranking: $\delta_4 \sim \delta_6 > \delta_3 \sim \delta_5 > \delta_2 > \delta_1$. While it does make sense to order $\delta_4, \delta_3, \delta_2$ and δ_1 this way, and similarly, to rank δ_6 over δ_5, does it make sense to give δ_5 and δ_3 the same rank, and *a fortiori*, that δ_5 should have priority over δ_2? Of course not: either the order between $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ and $\{\delta_5, \delta_6\}$ should be given exogeneously (by some expert in zoology, for instance), or there should be no order between them. For this, a generalization of preferred subtheories to partially ordered defaults is proposed in [6]. It is a generalization of Definition 3: instead of starting from a complete weak order over defaults, we start from a *partial order* $>$ between defaults and we say that a bijective mapping σ from $\{1, \ldots, |T|\}$ to T *respects* $(T, >)$ iff for all $\delta, \delta' \in T$, $\delta > \delta'$ implies $\sigma^{-1}(\delta) < \sigma^{-1}(\delta')$. The rest of the definition is unchanged.

2.2 Goal-Based Preference Representation

So far we considered a ranked base as being composed of *beliefs*; these beliefs may take the form of facts with some degree of reliability, facts that persist through time with some degree of certainty (see further), rules with possible exceptions, actions with normal and exceptional effects, and so on, but in all cases they deal with an agent's doxastic and epistemic state (her beliefs, her knowledge). Now, ranked bases can also be used with a totally different meaning, so as to express the *preferential state* of an agent, that is, her preferences, goals, desires. The difference between beliefs and preferences is paramount to decision theory – in standard decision theory, beliefs are expressed by probability distributions over states of the world whereas preferences are expressed by utility values over possible consequences of the acts.

Because of this, in this subsection we change the terminology – and notation. A *ranked goal base*, or *prioritized goal base*, is defined exactly as a stratified belief base: it is a tuple $(G_1, \ldots, G_n)$ where each G_i is a set of classical formulas, representing the agent's goals of priority degree i – with the convention that lower indexes correspond to more important goals.

Prioritized goals bases prove to be a very efficient way of representing succinctly preferences over *combinatorial* domains of solutions to a decision problem. Let me quote [8]:

> "By a solution we mean an assignment of a certain value d to each variable v in given set of variables V such that d is taken from the finite domain of v. (...) [In] the Boolean case where the values for each variable are true or false (...), solutions (...) correspond to interpretations in the sense of classical propositional logic. (...)
> We are (...) looking for ways of specifying preferences among such models in a concise yet flexible way. (...)

The number of models is exponential in the number of variables. For this reason it is, in general, impossible for a user to describe her preferences by enumerating all pairs of the preference relation among models. This is where logic comes into play."

Using prioritized bases for succinct preference representation has been discussed in a few papers that all appeared around the same time: [8] defines a general rank-based preference representation language (see further); [22] focuses on the complexity of the computational tasks; and [12] on the expressivity and the succinctness of these languages. Note that here we are no longer interested in preferred subtheories themselves, but in the preference relation between solutions: again quoting [8],

"Traditionally, logic is used for proving theorems. Here, we are not so much interested in logical consequence, we are interested in whether a model satisfies a formula or not."

Thus, the definition that makes most sense here is the sixth one, which we rewrite here into: $I \succ_G I'$ if and only if there is some $k \leq n$ such that $Sat(G_k, I) \supset Sat(G_k, I')$ and for all $i \leq k$, $Sat(G_i, I) = Sat(G_i, I')$. Moreover, $I \sim_G I'$ if and only if $Sat(G_i, I) = Sat(G_i, I')$ for all $i \leq n$, and $I \succeq_G I'$ if $I \succ_G I'$ or $I \sim_G I'$.

The two cardinality-based notions are now rewritten as follows:

- $I' \succ_T^C I$ if and only if there is some $k \leq n$ such that $|Sat(G_k, I)| > |Sat(G_k, I')|$ and for all $i \leq k$, $|Sat(G_i, I)| = |Sat(G_i, I')|$. Moreover, $I \sim_G^C I'$ if and only if for all $i \leq n$, $|Sat(G_i, I)| = |Sat(G_i, I')|$; and $I \succeq_G^C I'$ if $I \succ_G^C I'$ or $I \sim_G^C I'$.
- $I' \succeq_G^{SD} I$ if and only if for all $k \leq n$, $|Sat(G_1 \cup \ldots \cup G_k, I)| \geq |Sat(G_1 \cup \ldots \cup G_k, I')|$.

The following implications are parts of the "folklore": $I' \succeq_G^{SD} I$ implies $I' \succeq_G^C I$, and $I' \succeq_G I$ implies $I' \succeq_G^C I$. Note also that $\succeq_G^C$ is a complete weak order, whereas $\succeq_G$ and $\succeq_G^{SD}$ are partial orders.

These three ways of deriving a preference relation from a prioritized goal base can be characterized utility-theoretically. Given a goal base $G = (G_1, \ldots, G_n)$ with $G_i = \{g_i^j, j = 1, \ldots, m_i\}$, we say that $(u_i^j | i = 1, \ldots, n; j = 1, \ldots, m_i)$, where each u_i^j is a strictly positive real number, is a utility vector for G.

Given a utility vector $\boldsymbol{u}$ for G, and an interpretation I, define

$$u_G(I) = \sum \{u_i^j \mid i \leq n;\ j \leq m_i;\ I \models g_i^j\},$$

that is, each goal induces a fiwed reward if it is satisfied by I, and 0 if not.

We now consider three restrictions on utility vectors. A utility vector $\boldsymbol{u}$ for G is

- *uniform* if for all $i \leq n$ and all $j, j' \leq m_i$, we have $u_i^j = u_i^{j'}$.
- *faithful* if for all $i < k \leq n$, $j \leq m_i$, $l \leq m_k$, we have $u_i^j > u_k^l$.
- *big-stepped* if for all $i \leq n$ and all $j \leq m_i$, we have $u_i^j > \sum_{k=i+1}^{n} \sum_{l=1}^{m_k} u_k^l$.

Note that any big-stepped vector is faithful.[4] The next result gives one more characterization of preferred subtheories and its two variants.

[4] The terminology "big-stepped" comes from [15].

Proposition 3. *Let G be a goal base and I, I' two interpretations.*

1. *$I \succ_G I'$ if and only if $u_G(I) > u_G(I')$ holds for all big-stepped vectors $\mathbf{u}$ for G.*
2. *$I \succ_G^C I'$ if and only if $u_G(I) > u_G(I')$ holds for all uniform and big-stepped vectors $\mathbf{u}$ for G.*
3. *$I \succ_G^{SD} I'$ if and only if $u_G(I) > u_G(I')$ holds for all uniform and faithful vectors $\mathbf{u}$ for G.*

Point 1 leads to a seventh definition of a preferred subtheory:

Definition 11 (preferred subtheories, seventh definition). *S is a* preferred subtheory *of T if and only if $S = Sat(T, I)$ for some interpretation I such that for all big-stepped vectors $\mathbf{u}$ for T, there is no I' such that $u_T(I') > u_T(I)$.*

Once these different semantics for defining a preference relation from a prioritized goal base are defined, they can be combined: [8] defines a language allowing to express Boolean combinations of prioritized goals bases, possibly with different semantics.

Since prioritized goal bases can be used for representing compactly preferences over combinatorial domains, they can be used efficiently in several domains where preference play a role and where domains are typically of this kind, such as planning [20], game theory [5] or voting [22].

2.3 Reliability

We now come back to the primary interpretation of ranked bases as *belief bases*. Perhaps the most obvious interpretation of a ranked belief base is that each formula is a piece of information that has been provided by some unreliable source. This is also the interpretation at work in prioritized merging [14], where preferred subtheories and C-preferred subtheories are used for defining prioritized merging operators. Let $B = (B_1, \ldots, B_n)$ where $B_i = \{b_i^j \mid j = 1, \ldots, m_i\}$. For every formula b_i^j in B_i we define a source σ_i^j with reliability degree $p_i^j \in (\frac{1}{2}, 1)$ for all i, j (sources have a bias towards reliability, and no source is perfectly reliable). The reliability of a source is the likelihood that it tells the truth about p_i^j, that is $p_i^j = Prob(\sigma_i^j : b_i^j \mid b_i^j) = Prob(\sigma_i^j : \neg b_i^j \mid \neg b_i^j)$, where $\sigma_i^j : \varphi$ is the event "σ_i^j says φ". Let $\sigma : B$ be the conjunction of all events $\sigma_i^j : b_i^j$: informally, B is observed if all sources give the formulas that are contained in B. Now, let $S = (S_1, \ldots, S_n)$ be a consistent subbase of B. The likelihood of observing B given that the "true" subbase of B (the one composed of the fomulas of T that are true in the actual world) is S is

$$Prob(\sigma : B \mid S) = \prod_{(i,j):b_i^j \in S} p_i^j \prod_{(i,j):b_i^j \notin S} (1 - p_i^j)$$

Now we have

$$\begin{aligned}\log Prob(s : B|S) &= \textstyle\sum_{(i,j):b_i^j \in S} \log p_i^j + \sum_{(i,j):b_i^j \notin S} \log(1 - p_i^j) \\ &= \textstyle\sum_{(i,j)|i \leq n, j \leq m_i} \log(1 - p_i^j) + \sum_{(i,j):b_i^j \in S} \log\left(\frac{p_i^j}{1-p_i^j}\right) \\ &= \boldsymbol{\alpha} + \textstyle\sum_{(i,j):b_i^j \in S} \log\left(\frac{p_i^j}{1-p_i^j}\right)\end{aligned}$$

where $\boldsymbol{\alpha}$ is a constant, independent of S. Define $\succ_p$ as: $S \succ_p S'$ if and only if $Prob(\sigma :$

$B \mid S) \geq Prob(\sigma : B \mid S')$. Now, let $u_i^j = \log\left(\frac{p_i^j}{1-p_i^j}\right)$. We have that $S \succ_p S'$ if and only if $\sum_{(i,j):b_i^j \in S} u_i^j > \sum_{(i,j):b_i^j \in S'} u_i^j$; furthermore, if $S = Sat(T, I)$ and $S' = Sat(T, I')$, then $S \succ_p S'$ if and only if $u(I_S) > u(I_{S'})$. This correspondence allows to translate the conditions of Proposition 3 in probabilistic terms. Say that p is

– *uniform* if for all $i \leq n$ and all $j, j' \leq m_i$, we have $p_i^j = p_i^{j'}$.
– *faithful* if for all $i < k \leq n$, $j \leq m_i$, $l \leq m_k$, we have $p_i^j > p_k^l$.
– *big-stepped* if for all $i \leq n$ and all $j \leq m_i$, we have $\frac{p_i^j}{1-p_i^j} > \prod_{k=i+1}^{n} \prod_{l=1}^{m_k} \frac{p_k^l}{1-p_k^l}$.

Corollary 1. *Let B be a goal base and S, S' two subbases of B.*

1. *$S \succ_B S'$ if and only if $S \succ_p S'$ holds for all big-stepped vectors $\mathbf{p}$ for B.*
2. *$S \succ_B^C S'$ if and only if $S \succ_p S'$ holds for all uniform, big-stepped vectors $\mathbf{p}$ for B.*
3. *$S \succ_B^{SD} S'$ if and only if $S \succ_p S'$ holds for all uniform, faithful vectors $\mathbf{p}$ for B.*

Point 1 leads to an eighth definition of a preferred subtheory:

Definition 12 (preferred subtheories, eighth definition). *S is a* preferred subtheory *of T if there is no consistent subtheory S' of T such that $S \succ_p S'$ holds for all big-stepped vector $\mathbf{p}$ for B.*

2.4 Time, Space, Analogy

A context where prioritized defaults oocur in a natural way is that of *time-stamped data bases*: there, priorities correpond to recency, and a fact observed at time $t-1$ is more likely to have persisted until t than a fact observed at time $t-2$.

Example 1

$$\begin{aligned} now &: a \vee b \\ now - 1 &: a \rightarrow c \\ now - 2 &: \neg a, \neg b \\ now - 3 &: \neg c \end{aligned}$$

If we focus on what holds now, then this scenario gives us the ranked default theory $(T_1 = \{a \vee b\}, T_2 = \{a \rightarrow c\}, T_3 = \{\neg a, \neg b\}, T_4 = \{\neg c\})$ – with two preferred subtheories $\{a \vee b, a \rightarrow c, \neg a, \neg c\}$ and $\{a \vee b, a \rightarrow c, \neg b\}$.. However, default persistence does not only work forward but also backward: if $a \vee b$ holds now, by default it holds also at $now - 1$, etc. If we focus on what holds at $now - 3$, we get the ranked default theory $(T_1 = \{\neg c\}, T_2 = \{\neg a, \neg b\}, T_3 = \{a \rightarrow c\}, T_4 = \{a \vee b\})$ with one preferred subtheory $T_1 \cup T_2 \cup T_3$. If we focus on what holds at $now - 1$, this becomes more complicated: should we have the ranked default theory $(T_1 = \{a \rightarrow c\}, T_2 = \{a \vee b, \neg a, \neg b\}, T_3 = \{\neg c\})$, that is, should the information at now and the information at $now - 2$ count equally, or should we rather have a partially ordered default theory $a \rightarrow c > a \vee b, a \rightarrow c > \neg a, \neg b > \neg c\}$, and apply the second generalization of [6]? (In both cases we get three preferred subtheories $\{a \rightarrow c, a \vee b, \neg a, \neg c\}$, $\{a \rightarrow c, a \vee b, \neg b\}$ and $\{a \rightarrow c, \neg a, \neg b, \neg c\}$.

Other natural examples involve reasoning about spatial observations, about case-labelled facts (reasoning by analogy, case-based reasoning), about ontologies. A more general framework where priorities come from distances between 'labels' (such as time points, points in space, cases, classes) and where observations are labelled, is described in [2].

2.5 Judgment Aggregation and Voting

Given a set of formulas $A = \{\alpha_1, \neg\alpha_1, \ldots, \alpha_m, \neg\alpha_m\}$ closed under negation (called the *agenda*), a *judgment set* is a consistent subset of A containing, for all i, either α_i or $\neg\alpha_i$, and a *profile* is a collection of n individual judgment sets. An *(irresolute) judgment aggregation rule* maps a profile into a set of collective judgment sets. As common in social choice, there is a tension between respecting majority and requiring consistency of the collective judgment sets.

An interesting family of judgment aggregation rules is composed of rules that are based on the *support* of a profile, that is, the vector containing, for each element of the agenda, the number of individual judgment sets that contain it. For instance, if $A = \{p, \neg p, q, \neg q, p \wedge q, \neg(p \wedge q)\}$, and $P = \langle J_1, J_2, J_3, J_4, J_5, J_6, J_7\rangle$ where $J_1 = J_2 = J_3 = \{p, q, p \wedge q\}$, $J_4 = J_5 = \{\neg p, q, \neg(p \wedge q)\}$ and $J_6 = J_7 = \{p, \neg q, \neg(p \wedge q)\}$, the support vector associated with P is $s_P = \langle 5, 2, 5, 2, 3, 4\rangle$. Now, define the prioritized base $T(P)$ where priorities correspond to strength of support: in our example, $T_1(P) = \{p, q\}$ (support 5), $T_2(P) = \{\neg(p \wedge q)\}$ (support 4), $T_3(P) = \{p \wedge q\}$ (support 3), and $T_4(P) = \{\neg p, \neg q\}$ (support 2). Given a profile P, Nehring et al. [26] define a *supermajority efficient judgment set* as (reformulated in my terms) a SD-undominated subtheory of $T(P)$, and define the so-called *leximin* judgment aggregation rule as the set of C-preferred subtheories of $T(P)$, while Lang et al. [23] define the so-called *ranked agenda* rule as the set of preferred subtheories of $T(P)$. See also [18] for a discussion on these rules.

These connections between judgment aggregation rules and preferred theories and their variants carry on to *voting rules*, which is not surprising given that preference aggregation can be see as a specific case of judgment aggregation. The *ranked pairs* voting rule [30] thus corresponds to the ranked agenda rule, when the agenda consists of propositions of the form xPy ("x is preferred to y"), where x and y range over a set of *candidates*, together with the transitivity constraint bearing on judgment sets. In other terms, this means that *the ranked pairs voting rule can be seen as a specific application of preferred subtheories*. This is is probably the first time that this connection between this well-known voting rule (and the corresponding judgment aggregation rule) is mentioned; interestingly, the ranked pairs rule and preferred subtheories have been invented roughly at the same time, in two research areas that were (at the time) totally disconnected from each other. Let me end up with an example.

Example 2. Let the set of candidates be $C = \{a, b, c, d\}$ and the 38-voter profile P consisting of 5 votes $abdc$ (with the usual convention that $abdc$ is a shorthand for $a \succ b \succ d \succ c$), 7 votes $cdab$, 8 votes $bcad$, 7 votes $dabc$, 4 votes $dcab$, 3 votes $cbda$, 2 votes $bacd$, 1 vote $dbca$ and 1 vote $acdb$. The pairwise majority matrix is

	a	b	c	d
a	$-$	24	15	16
b	14	$-$	23	18
c	23	15	$-$	21
d	22	20	17	$-$

and the corresponding prioritized base is $T(P) = (T_1(P), T_2(P), \ldots, T_{11}(P)))$, where $T_1(P) = Trans$ is the transitivity constraint, $T_2(P) = \{aPb\}$, $T_3(P) = \{bPc, cPa\}$, $T_4(P) = \{dPa\}$, $T_5(P) = \{cPd\}$, $T_6(P) = \{dPb\})$ etc. The preferred subtheories of $T(P)$ are $\{Trans, aPb, bPc, dPa, dPb, aPc, dPc\}$, corresponding to the collective ranking $dabc$ and to the winner d, and $\{Trans, aPb, cPa, dPa, cPd, dPb, cPb\}$, corresponding to the collective ranking $cdab$ and to the winner c.

Note that taking C-preferred subtheories instead of preferred subtheories leads to a refinement of the ranked pairs rules (in our example, the sole winner for this rule is c).

3 Conclusion

We have seen that preferred subtheories and their extensions and variants have had a tremendous impact in the Artificial Intelligence literature and beyond, and are tightly connected to notions that have been developed independently in social choice. If I had more pages, I could talk for instance about the computation of preferred subtheories and inferences therefrom (e.g., [10,13]. A further question is, is logic really useful when defining preferred subtheories? We have seen at least one example (voting) where logic isn't necessary at all. After all, all we use from logic is the notion of *consistency*. When defining the ranked pairs voting rule, a weighted graph plays the role of the ranked base, and acyclicity plays the role of consistency. How can we define an abstract (logic-free) notion of preferred subtheory and what about other applications and/or connections?

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Appendix

Proof of Proposition 1

Proof. Throughout the proof, we say that S is a i-PST of T (where $1 \leq i \leq 6$) if S is a preferred subtheory of T according to Definition i. Let S be a subtheory of T.

- $1 \Rightarrow 2$: Assume that S is not a 2-PST of T; then for some i there is $S'_i \supset S_i$ such that S'_i is $S_1 \cup \ldots \cup S_{i-1}$-consistent. Since $S_1 \cup \ldots \cup S_{i-1} \cup S'_i$ is consistent and $S_1 \cup \ldots \cup S_{i-1} \cup S_i \subset S_1 \cup \ldots \cup S_{i-1} \cup S'_i$, $S_1 \cup \ldots \cup S_i$ is not a maximal consistent subset of $T_1 \cup \ldots \cup T_i$, henceforth, not a 1-PST of T.
- $2 \Rightarrow 1$: Assume that S is not a 1-PST of T; then for some i, $S_1 \cup \ldots \cup S_i$ is not a maximal consistent subset of $T_1 \cup \ldots \cup T_i$. Let $S'_1 \cup \ldots S'_i \supset S_1 \cup \ldots \cup S_i$ be a maximal consistent subset of $T_1 \cup \ldots \cup T_i$ and let $j = \min\{i, S_i \neq S'_i\}$. Then S'_j is a $S_1 \cup \ldots \cup S_{j-1}$ consistent subset of T_j, which implies that S_j is not, and that S is not a 2-PST of T.
- $4 \Rightarrow 2$: Assume that S is not a a 2-PST of T; then for some i there is $S'_i \supset S_i$ such that S'_i is $S_1 \cup \ldots \cup S_{i-1}$-consistent. Let $S' = S_1 \cup \ldots \cup S_{i-1} \cup S'_i$. S' is a subtheory of T and we have $MinIndex(S' \setminus S) = i$ and $MinIndex(S \setminus S') > i$, therefore S is not a a 4-PST of T.
- $5 \Rightarrow 4$: Assume that S is not a 4-PST of T; then for some S' we have $MinIndex(S' \setminus S) \leq MinIndex(S \setminus S')$. Since $MinIndex(S' \setminus S) = MinIndex(S \setminus S')$ is not possible, we must have $MinIndex(S' \setminus S) = k < MinIndex(S \setminus S')$. Now, for all $j < k$ we have $S_j = S'_j$ and $S'_k \supset S_k$, therefore S is not a 5-PST of T.
- $2 \Rightarrow 3$: assume S is a 2-PST of T. Let us construct σ this way: σ considers first formulas of S_1 (in any order), followed by formulas in $T_1 \setminus S_1$ (in any order), then S_2 then $T_2 \setminus S_2$, etc. until $T_n \setminus S_n$. We show by induction on i that after considering all formulas of T_i, we have $\Sigma_{t(i)} = S_1 \cup \ldots \cup S_i$, where $t(i) = |T_1 \cup \ldots \cup T_i|$. This is true for $i = 1$, because S_1 is maximal consistent. Assume it is true for i, i.e., $\Sigma_{t(i)} = S_1 \cup \ldots \cup S_i$. Because S is a 2-PST of T, S_{i+1} is $(S_1 \cup \ldots \cup S_i)$-consistent, therefore, all formulas of S_{i+1} are added, and because it is maximal $(S_1 \cup \ldots \cup S_i)$-consistent, none of the formulas of $T_{i+1} \setminus S_{i+1}$ are added. Therefore, at the end of

step $t(i)$ $\Sigma_{t(i+1)} = S_1 \cup \ldots \cup S_{i+1}$. Applying the induction hypothesis to $i = n$ leads to $S_\sigma = \Sigma_{t(n)} = S$: S is a 3-PST of T.

- $3 \Rightarrow 6$: Let S be a 3-PST of T and let σ such that $S_\sigma = S$. Note that S is necessarily maximal consistent: if there was $\delta \in T \setminus S$ such that $S \cup \{\delta\}$ is consistent, then δ would have been added to Σ when considered; therefore, there exists I such that $Sat(T, I) = S$. Assume S is not a 6-PST of T: then there is $I' \succ_T I$, that is, for some k, we have that for all $i < k$, $Sat(T_i, I) = Sat(T_i, I')$, and $Sat(T_k, I) \subset Sat(T_k, I')$. But then, when the defaults of T_k are considered for addition to Σ, all formulas of $Sat(T_k, I') \setminus Sat(T_k, I)$ should have been added, which contradicts $S_\sigma = S$.
- $6 \Rightarrow 5$: Assume that S is a 6-PST of T: there is an I such that $Sat(T, I) = S$. Assume that S is not a 5-PST of T: then there is an S' such that $S' \succ_T S$. Because $S' \subset S'' \subseteq T$ implies $S' \succ_T S$, there is a maximal consistent subset S'' of T such that $S'' \succ_T S$. Let $S'' = Sat(T, I'')$: then $I'' \succ_T I$, which contradicts the assumption that S is a 6-PST of T.

Proof of Proposition 4

Proof. 1. Assume $I \succ_G I'$ and let k such that $Sat(G_j, I) = Sat(G_j, I')$ for all $j < k$, and $Sat(G_k, I) \supset Sat(G_k, I')$. Let $\boldsymbol{u}$ be a big-stepped vector for G. Let $A_i = \sum\{u_i^j \mid j \leq n_i;\ g_i^j \in Sat(G_i, I) \setminus Sat(G_i, I')\} - \sum\{u_i^j \mid j \leq n_i;\ g_i^j \in Sat(G_i, I') \setminus Sat(G_i, I)\}$. We have $u_G(I) - u_G(I') = \sum_{i \leq n} A_i$. Because $Sat(G_j, I) = Sat(G_j, I')$ for all $j < k$, we have (1) $A_i = 0$ for all $i < k$. Because $Sat(G_k, I) \supset Sat(G_k, I')$, there exists some $g_k^l \in Sat(G_k, I) \supset Sat(G_k, I')$. Because $\boldsymbol{u}$ is big-stepped, we have $u_k^l > \sum_{p=k+1}^{n} \sum_{q=1}^{m_p} u_p^q$, which implies (2) $u_k^l > \sum_{p=k+1}^{n} |A_p|$. Now, (1) and (2) imply $u_G(I) - u_G(I') = A_k + \sum_{i>k} A_i > 0$, that is, $u(I) > u(I')$.
Conversely, assume $I \not\succ_G I'$. If $I \sim_G I'$, then clearly $u(I) = u(I')$. If not, then there is a k such that $Sat(G_j, I) = Sat(G_j, I')$ for all $j < k$, and $Sat(G_k, I') \setminus Sat(G_k, I) \neq \emptyset$. Let $g_k^l \in Sat(G_k, I') \setminus Sat(G_k, I)$. Define $\boldsymbol{u}$ as follows: $u_k^l = |B_k|$; for all $l' \neq l$, $u_k^{l'} = 1$; and the other values u_i^j are defined in any way such that $\boldsymbol{u}$ is big-stepped (since we have put constraints on values concerning level k, this is obviously possible to do so). Let A_i be defined as above. Since $\boldsymbol{u}$ is big-stepped, we have, for all $l' \neq l$, $u_k^{l'} = 1 > \sum_{p=k+1}^{n} \sum_{q=1}^{m_p} u_k^l$, which implies $-1 < \sum_{p=k+1}^{n} A_k < 1$. Finally, $A_k \leq -|B_k| + \sum_{j \leq m_k, j \neq l} u_i^j \leq -1$, and $u(I) - u(I') = \sum_{i \leq n} A_i = A_k + \sum_{i>k} A_i < 0$, that is, $u(I) < u(I')$.

2. Assume $I \succ_G^C I'$ and let k such that $|Sat(G_j, I)| = |Sat(G_j, I')|$ for all $j < k$, and $|Sat(G_k, I)| > |Sat(G_k, I')|$. Let $\boldsymbol{u}$ be a uniform, big-stepped vector for G, and let $u_i^j = u_i$ for all $j \leq m_i$. Define A_i, for all $i \leq n$, as above. Then (1) for all $i < k$, $A_i = (2|Sat(G_i, I)| - m_i).u_i - (2|Sat(G_i, I')| - m_i).u_i = 0$, (2) $A_k = 2|Sat(G_k, I)| - m_k).u_k - (2|Sat(G_k, I')| - m_k).u_k = 2(Sat(G_k, I)| - |Sat(G_k, I')| > 2u_k$ and because $\boldsymbol{u}$ is big-stepped, (3) $u_k > \sum_{p=k+1}^{n} |A_p|$. (1), (2) and (3) imply $u_G(I) - u_G(I') > 0$, that is, $u(I) > u(I')$.
Conversely, assume $I \not\succ_G^C I'$. If $I \sim_G^C I'$, then clearly $u(I) = u(I')$. If not, then, because $\succ_G^C$ is total, we have $I' \succ_G^C I$, which using the first part of the proof, implies that for all uniform, big-stepped $\boldsymbol{u}$ for G, we have $u(I) < u(I')$.

3. Assume $I \succ_G^{SD} I'$. Then for all $i \leq n$, $\sum_{j \leq i} |Sat(G_j, I)| \geq \sum_{j \leq i} |Sat(G_j, I)$, and for some k, (1) $\sum_{j \leq k} |Sat(G_j, I)| > \sum_{j \leq k} |Sat(G_j, I)$. Let $\boldsymbol{u}$ be uniform and faithful. Let $\boldsymbol{\alpha}_i = |Sat(G_i, I)|$ and $\beta_i = |Sat(G_i, I')|$. Let V (resp. W) be the multiset containing $\boldsymbol{\alpha}_i$ (resp. β_i) occurrences of u_i for all i, and reorder V and W non-increasingly, that is, $V = \{v_{(1)}, \ldots, v_{(p)}\}$ and $W = \{w_{(1)}, \ldots, w_{(q)}\}$ with $v_{(1)} \geq \ldots \geq v_{(p)}$ and $w_{(1)} \geq \ldots \geq w_{(q)}$. $I \succ_G^{SD} I'$ and the faithfulness of u imply $p \geq q$ and for all i, $v_{(i)} \geq w_{(i)}$. Finally, together with (1) they imply that there is a j such that $v_{(i)} > w_{(i)}$. Now, $u(I) - u(I') = \sum_{i=1}^{q} (v_{(i)} - w_{(i)}) + \sum_{i=q+1}^{p} v_{(i)}$ is a sum of positive terms, with at least one strictly positive term, therefore, $u(I) > u(I')$.
   Conversely, assume $I \not\succ_G^{SD} I'$. If $I \sim_G^{SD} I'$, then $I \sim_G^{C} I'$ and $u(I) = u(I')$. If not, then there is some k such that $\sum_{j \leq k} |Sat(G_j, I)| < \sum_{j \leq k} |Sat(G_j, I)$. Define the uniform, faithful vector $\boldsymbol{u}$ as $u_i = i + (k - i)\epsilon$ for all $i \leq k$ and $u_i = (i - k).\epsilon$ for all $i > k$, where $\epsilon < \frac{1}{k|G|}$. Then $u(I') - u(I) > 1 - \sum_{i<k} |G_i|(k-1)\epsilon > 0$, therefore, $u(I) < u(I')$.