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How Hard is it to Compute Majority-Preserving Judgment Aggregation Rules?

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Abstract. Several recent articles have studied judgment aggregation rules under the point of view of the normative properties they satisfy. However, a further criterion to choose between rules is their computational complexity. Here we review a few rules already proposed and studied in the literature, and identify the complexity of computing the outcome.

1 Introduction

Given a collection of judgments, cast on a set of logically related issues by different agents, a judgment aggregation problem is the problem of finding a coherent collective set of judgments that is representative of the individual judgment collection. Judgment aggregation is particularly interesting as it can be seen as a general framework for abstract aggregation that encompasses preference aggregation [9]. Judgment aggregation has its origins in law and has been studied in economy science, political science, but also in artificial intelligence and multiagent systems.

In most papers in the literature on judgment aggregation, the aggregation examples involve a small number of issues and a small number of agents that cast judgments on those issues. However, if we want to be able to implement the aggregation rules that these papers justify by axiomatic properties, we have to be able to compute the aggregate outcome. If the number of issues and agents is more than a few units, the computation of the outcome may not be computationally easy. Only a few papers have considered judgment aggregation under the point of view of computation.

Endriss et al. [13] analyze the complexity of computing the outcome of a judgment aggregation procedure ("winner determination") and strategic manipulation for three specific procedures: the quota rules [7], the premised-based procedure [11], and a distance-based rule (see further). They also consider another problem, which is independent of the procedure used: given an agenda, how complex is it to determine whether this agenda is "safe", that is, whether the issue-by-issue majoritarian aggregation is guaranteed to output a consistent result?

Baumeister *et al.* [4, 3] go further on the computational aspects of strategic manipulation for premise-based procedures and quota rules, by considering various forms of manipulations and investigating the parameterized complexity of these problems with respect to natural parameters; they also consider control by the chair and bribery in judgment aggregation, by generalizing some notions from voting, but also by defining a new problem, specific to jugment aggregation, namely, control by bundling judges.

Alon et al. [2] introduce and give a computational study of the notion of control by bundling issues. Alon et al. [1] study the related issue of finding a consensual subset of issues.

Here we choose to leave aside the questions of strategic behaviour such as manipulation, control and bribery, and focus on the more basic question of computing the outcome of a procedure, called "winner determination" in [13] by analogy with voting theory: given a collection of judgments and a judgment aggregation rule, how difficult is it to compute the collective judgment set(s)? We go beyond the three rules considered by Endriss et al. and investigate the complexity of winner determination for several voting rules that have been introduced in the literature, the common point of which being that they are all *majority-preserving* (if the issue-by-issue majoritarian aggregation leads to a consistent judgment set, then the output should consist of this judgment set), and *neutral* (they treat all issues equally, unlike *e.g.*, the premise-based procedure). We show that the complexity of winner determination for these rules lies at the first or second levels of the polynomial hierarchy. One of the interests of our work is that it establishes interesting connections with other fields of AI, especially belief revision and nonmonotonic reasoning.

The paper is structured as follows. In Section 2 we give the necessary background, first on judgment aggregation in general, and then on the judgment aggregation rules that we study. In Section 3 we introduce and discuss different computational problems. In Section 4 we consider the rules one by one and address the complexity of computing the outcome. Finally, Section 5 discusses the significance of our results and points to future research directions.

2 Background

We first give the basics of judgment aggregation and then we introduce the judgment aggregation rules whose computational properties we explore. Due to space limitations we do not recall any background about the polynomial hierarchy (see, *e.g.*, Chapter V.1 of [24]).

2.1 Judgment aggregation: general definitions

Let $\mathcal{L}$ be a set of well-formed propositional logical formulas, including $\top$ (tautology) and $\bot$ (contradiction). For any finite subset S of formulas of $\mathcal{L}$, $\bigwedge(S)$ denotes the conjunction of all formulas in S.

An *issue* is a pair of formulas $\varphi, \neg\varphi$ where $\varphi \in \mathcal{L}$ and φ is neither a tautology nor a contradiction. An *agenda* $\mathcal{A}$ is a finite set of issues, and has the form $\mathcal{A} = \{\varphi_1, \neg\varphi_1, \ldots, \varphi_m, \neg\varphi_m\}$. The *preagenda* $[\mathcal{A}]$ associated with $\mathcal{A}$ is $[\mathcal{A}] = \{\varphi_1, \ldots, \varphi_m\}$. A *sub-agenda* is a subset of issues from $\mathcal{A}$, that is, a subset of $\mathcal{A}$ of the form $\{\varphi_j, \neg\varphi_j \mid j \in J\}$. A *sub-preagenda* is a subset of $[\mathcal{A}]$.

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A *judgment* on $\varphi \in [\mathcal{A}]$ is one of φ or $\neg\varphi$. A *judgment set* J is a subset of $\mathcal{A}$ that is *complete* if and only if for each $\varphi \in [\mathcal{A}]$, either $\varphi \in J$ or $\neg\varphi \in J$.

Constraints can be specified to explicitly represent logical dependencies enforced on agenda issues. Since we have a finite $\mathcal{L}$, without loss of generality we can assume that the constraints consist of *one* propositional formula (typically the conjunction of several simpler constraints). The constraint associated to an agenda $\mathcal{A}$ is thus a consistent formula $\Gamma \in \mathcal{L}$. When not otherwise specified, Γ is the tautology $\top$. Involving constraints in judgment aggregation has already been considered in a few places, such as [10, 16]. We reproduce here the definitions of two special constraints, the transitivity (Tr) and the dominance (W) constraint, used to prove relations between voting rules and judgment aggregation rules [20].

The preference agenda $\mathcal{A}_C = \{x_i P x_j \mid 1 \leq i < j \leq q\}$ is a special type of agenda associated with a set of alternatives $C = \{x_1, \ldots, x_q\}$. For the preference agenda we define the Tr and W constraints as:

- $Tr = \bigwedge_{i,j,k \in \{1,\ldots,m\}} \big((x_i P x_j) \wedge (x_j P x_k) \rightarrow (x_i P x_k)\big)$
- $W = \bigvee_{i \leq m} \bigwedge_{j \neq i} (x_i P x_j)$

A judgment set J (and more generally, a set of propositional formulas) is Γ-*consistent* if and only if $J \cup \{\Gamma\} \not\models \bot$. Let $\mathcal{D}(\mathcal{A}, \Gamma)$ be the set of all Γ-consistent judgment sets (for agenda $\mathcal{A}$) and $\mathbb{D}(\mathcal{A}, \Gamma) \subset \mathcal{D}(\mathcal{A}, \Gamma)$ be the set of all judgment sets that are also *complete*. We omit specifying $\mathcal{A}$ and Γ when they are clear from the context.

For $I \subseteq \mathcal{A}$, we define $\texttt{Comp}_{\mathcal{A},\Gamma}(I)$ as the set of all complete and consistent judgment sets containing I, i.e. $\texttt{Comp}_{\mathcal{A},\Gamma}(I) = \{J \in \mathbb{D}(\mathcal{A}, \Gamma) \mid I \subseteq J,\}$. For $S = \{I_1, \ldots, I_k\}$ with $I_1 \subseteq \mathcal{A}, \ldots, I_k \subseteq \mathcal{A}$, we define $\texttt{Comp}_{\mathcal{A},\Gamma}(S) = \cup_{I \in S} \texttt{Comp}_{\mathcal{A},\Gamma}(I)$.

A *profile* $P = \langle J_1, \ldots, J_n \rangle \in \mathbb{D}^n(\mathcal{A}, \Gamma)$ is a collection of complete, Γ-consistent individual judgment sets. Given a sub-agenda Y, the projection of J on Y is $J^{\downarrow Y} = J \cap Y$. Given a profile $P = \langle J_1, \ldots, J_n \rangle$, the projection of P on Y is $P^{\downarrow Y} = \langle J_1^{\downarrow Y}, \ldots, J_n^{\downarrow Y} \rangle$. Lastly, we define $N(P, \varphi)$ as $N(P, \varphi) = |\{i \mid J_i \in P \text{ and } \varphi \in J\}|$.

An *irresolute judgment aggregation rule*, for n voters, is a function $F_\Gamma : \mathbb{D}^n(\mathcal{A}, \Gamma) \rightarrow 2^{\mathbb{D}(\mathcal{A},\Gamma)} \setminus \{\emptyset\}$, *i.e.*, F_Γ maps a profile of complete judgment sets to a nonempty set of consistent and complete judgment sets. When Γ is omitted, *i.e.*, when we note F instead of F_Γ, we assume that F is defined for any possible constraint Γ (F then defines a family of judgment aggregation rules – one for each Γ – but by a slight abuse of language we use F for a judgment aggregation rule).

The majoritarian judgment set associated with profile P contains all elements of the agenda that are supported by a majority of judgment sets in P, *i.e.*,

$$m(P) = \{\varphi \in \mathcal{A} \mid N(P, \varphi) > \frac{n}{2}\}.$$

A profile P is (Γ)-*majority-consistent* if and only if $m(P)$ is Γ-consistent.

A judgment aggregation rule F is *majority-preserving* if and only if for every agenda $\mathcal{A}$, for every $\Gamma \in \mathcal{L}$, for every majority-consistent profile P based on $\mathcal{A}$ and Γ, we have $F_\Gamma(P) = \texttt{Comp}_{\mathcal{A},\Gamma}(m(P))$.

A judgment aggregation rule F is neutral if for any permutation σ of the issues of the preagenda, we have $F(P_\sigma) = F(P)_\sigma$, where P_σ and $F(P)_\sigma$ are obtained from P and $F(P)$ by replacing everywhere every issue φ (resp. $\neg\varphi$) by $\sigma(\varphi)$ (resp. $\neg\sigma(\varphi)$).

Given a set of formulas Σ, $S \subseteq \Sigma$ is a maximal Γ-consistent subset of Σ if S is Γ-consistent and no S' such that $S \subset S' \subseteq \Sigma$ is Γ-consistent; and $S \subseteq \Sigma$ is a maxcard (for "maximal cardinality") Γ-consistent subset of Σ if S is Γ-consistent and no $S' \subseteq \Sigma$ such that $|S| < |S'|$ is Γ-consistent. With $max(S, \Gamma, \subseteq)$ we denote the set of all maximal Γ-consistent subsets of S, while $max(S, \Gamma, |.|)$ denotes the set of all maxcard Γ-consistent subsets of S.

Lastly, we give the example of an agenda and profile which we will make use of in this paper. We make use of the following classical notation (see Figure 1): we have one column for each element of the preagenda $[A]$ and one row for each judgment set of each class of identical judgment sets; $+$ (resp. $-$) in column φ and row J means that $\varphi \in J$ (resp. $\neg\varphi \in J$). Thus, for instance, in the profile of Example 1 below, J_i contains $\{\neg x_j, y_j, (x_j \wedge y_j) \rightarrow \varphi_j\}$ for all $j \neq i$, and $\{x_j, y_j, (x_j \wedge y_j) \rightarrow \varphi_i\}$ for $j = i$; and J_{i+1} contains $\{x_j, \neg y_j, (x_j \wedge y_j) \rightarrow \varphi_j\}$ for all $j \neq i$ and $\{x_j, y_j, (x_j \wedge y_j) \rightarrow \varphi_i\}$ for $j = i$.

Example 1 *Let $\Delta \subset \mathcal{L}$ be a set of formulas, $\Delta = \{\varphi_1, \ldots, \varphi_p\}$. To Δ we associate a pre-agenda $[\mathcal{A}_\Delta]$ of p elements constructed in the following manner*

$$[\mathcal{A}_\Delta] = \{x_i, y_i, (x_i \wedge y_i) \rightarrow \varphi_i \mid i \in [1, p], \varphi_i \in \Delta\}.$$

We construct a profile $P_\Delta = \langle J_1, J_2, \ldots, J_{2p-1}, J_{2p} \rangle$ where judgment sets J_{2j} and J_{2j+1} for i odd and $j \in [1, 2p]$ consist of the following judgments:

- *$\{\neg x_j, y_j, (x_j \wedge y_j) \rightarrow \varphi_j\} \subset J_i$ for all $j \neq i$, and $\{x_j, y_j, (x_j \wedge y_j) \rightarrow \varphi_i\} \subset J_i$ for $j = i$;*
- *$\{x_j, \neg y_j, (x_j \wedge y_j) \rightarrow \varphi_j\} \subset J_{i+1}$ for all $j \neq i$ and $\{x_j, y_j, (x_j \wedge y_j) \rightarrow \varphi_i\} \subset J_{i+1}$ for $j = i$.*

The profile P_Δ is also depicted in Figure 1. Observe that the judgment sets J_i and J_{i+1} are consistent if and only if the formula $\varphi_i \in \Delta$ is consistent. More precisely, the subset $\{x_i, y_i, (x_i \wedge y_i) \rightarrow \varphi_i\}$ is consistent if and only if φ_i is consistent. For the same reason, $m(P_\Delta)$ is consistent if and only if Δ is consistent.

	x_1	y_1	$(x_1 \wedge y_1) \rightarrow \varphi_1$	x_2	y_2	$(x_2 \wedge y_2) \rightarrow \varphi_2$	$\cdots$	x_p	y_p	$(x_p \wedge y_p) \rightarrow \varphi_p$
J_1	$+$	$+$	$+$	$+$	$-$	$+$	$\cdots$	$+$	$-$	$+$
J_2	$+$	$+$	$+$	$-$	$+$	$+$	$\cdots$	$-$	$+$	$+$
J_3	$+$	$-$	$+$	$+$	$+$	$+$	$\cdots$	$+$	$-$	$+$
J_4	$-$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$-$	$+$	$+$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$
J_{2p-1}	$+$	$-$	$+$	$+$	$-$	$+$	$\cdots$	$+$	$+$	$+$
J_{2p}	$-$	$+$	$+$	$-$	$+$	$+$	$\cdots$	$+$	$+$	$+$
$m(P_\Delta)$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$	$+$	$+$	$+$

Figure 1. Profile P_Δ obtained for a set of formulas Δ.

2.2 Judgment aggregation rules

In this work we focus on neutral and majority-preserving judgment aggregation rules. This rules out rules that are not neutral, such as premise-based and conclusion-based rules, and rules that are not majority-consistent, such as scoring rules introduced recently in [8]. These rules occur under different names in the literature, and sometimes with slightly different (but equivalent) definitions. We give the definition for each of them, reusing the names from [18, 20]. For more details and intuitions behind these rules, as well as for detailed examples, we invite the reader to consult [18, 19, 20].

In the rest of this Section, let $P = \langle J_1, \ldots, J_n \rangle$, $P \in \mathbb{D}^n(\mathcal{A}, \Gamma)$.

Definition 1 (Maximal & maxcard sub-agenda rules) *The maximal sub-agenda (MSA) and the maxcard sub-agenda (MCSA) rules are defined as follows: for every agenda $\mathcal{A}$, for every $\Gamma \in \mathcal{L}$, for every profile P based on $\mathcal{A}$ and Γ,*

$$MSA_\Gamma(P) = \texttt{Comp}_{\mathcal{A},\Gamma}(max(m(P), \Gamma, \subseteq)), \quad (1)$$

$$MCSA_\Gamma(P) = \texttt{Comp}_{\mathcal{A},\Gamma}(max(m(P), \Gamma, |.|)). \quad (2)$$

The MSA rule is called "Condorcet admissible set", and the $MCSA$ "Slater rule", in [23]. The $MCSA$ rule is also equivalent to the ENDPOINT$_d$ rule from [21] defined for d being the Hamming distance.

Definition 2 (Ranked agenda) *Let $\succsim_P$ be the weak order on $\mathcal{A}$ defined by: for all $\psi, \psi' \in \mathcal{A}$, $\psi \succsim_P \psi'$ if and only if $N(P, \psi) \geq N(P, \psi')$. For $\mathcal{A} = \{\psi_1, \ldots, \psi_{2m}\}$ and a permutation σ of $\{1, \ldots, 2m\}$, let $>_\sigma$ be the linear order on $\mathcal{A}$ defined by $\psi_{\sigma(1)} > \ldots > \psi_{\sigma(2m)}$. We say that $>_\sigma$ is compatible with $\succsim_P$ if $\psi_{\sigma(1)} \succsim_P \ldots \succsim_P \psi_{\sigma(2m)}$. The ranked agenda rule RA_Γ is defined as $J \in RA_\Gamma(P)$ if and only if there exists a permutation σ such that $>_\sigma$ is compatible with $\succsim_P$ and such that $J = J_\sigma$ is obtained by the following procedure:*

- *$S := \emptyset$;*
- *for $j = 1, \ldots, 2m$ do*
- *if $S \cup \{\psi_{\sigma(j)}\}$ is Γ-consistent, then $S := S \cup \{\psi_{\sigma(j)}\}$;*
- *$J_\sigma := S$.*

This rule is similar, although not exactly the same, as the "leximin rule" in [23]. See also [14] for a similar rule.

Definition 3 (Maxweight sub-agenda) *The maxweight sub-agenda rule MWA is defined as*

$$MWA_\Gamma(P) = \underset{J \in \mathbb{D}(\mathcal{A},\Gamma)}{argmax} \sum_{\varphi \in J} N(P, \varphi). \quad (3)$$

The MWA rule is called "Median rule" by Nehring *et al.* [23] and "Simple scoring rule" in [8]. The MWA rule is equivalent [18] to the PROTOTYPE$_d$ for d being the Hamming distance and the "Distance-based procedure" of [13].

The following rule corresponds to the FULL$_d$ rule in [21] for the choice of the Hamming distance.

Definition 4 (Minimal number of atomic changes) *Given two profiles $P = \{J_1, ..., J_n\}$ and $Q = \{J'_1, ..., J'_n\}$, let $d_H(P, Q) = \sum_{i=1}^{n} |J_i \setminus J'_i|$. Then*

$$MNAC_\Gamma(P) = \{\texttt{Comp}_{\mathcal{A},\Gamma}(m(P')) \mid m(P') \in \mathcal{D}(\mathcal{A}, \Gamma) \text{ and } d_H(P, P') \leq d_H(P, Q) \text{ for all } Q \in \mathbb{D}^n(\mathcal{A}, \Gamma)\} \quad (4)$$

Intuitively, MNAC looks for a minimal number of elementary changes in the profile (where an elementary change consists in switching a judgment on an issue for some voter) so as to render it Γ-consistent.

We should make clear that there is a slight difference in the definitions of rules MSA and $MCSA$ we give here and as they are defined in [18, 20]. Here we define the rules to always produce complete judgment sets, while in [18, 20], these rules can produce incomplete judgment sets. However, the definitions we choose here considerably simplify the study of their properties (including computational properties); see [19].

Example 2 *We illustrate the presented rules with an example. Consider $[\mathcal{A}] = \{p \wedge r, p \wedge s, q, p \wedge q, t\}$, $\Gamma = \top$, and the profile given in Figure 2. Observe that for this profile, the $m(P)$ is not consistent. The collective judgments from each of the five rules we consider are also given on the Figure.*

Voters	$\{$ $p \wedge r$,	$p \wedge s$,	q,	$p \wedge q$,	t $\}$
$J_1 \times 6$	+	+	+	+	+
$J_2 \times 4$	+	+	-	-	+
$J_3 \times 7$	-	-	+	-	-
$m(P)$	+	+	+	-	+
$MSA_\top(P)$	+	+	+	+	+
	+	+	-	-	+
	-	-	+	-	+
$MCSA_\top(P)$	+	+	+	+	+
	+	+	-	-	+
$RA_\top(P)$	-	-	+	-	+
$MWA_\top(P)$	+	+	+	+	+
$MNAC_\top(P)$	+	+	+	+	+

Figure 2. Profile example illustrating different judgment aggregation rules.

3 The problems

In voting theory, the computational issues of winner determination have been vastly explored for many different rules. There, a winner determination problem is composed of a set of alternatives from which a winner or a subset of winners has to be selected, a profile (generally consisting of a preference ranking for each voter), and the key question is whether a given alternative is among the winners. The winner determination problem in judgment aggregation is not as straightforward to define as in voting. Perhaps the main difficulty is that in judgment aggregation a "winner" could both be a single judgment and a set of judgments[3].

A possible approach to winner determination (taken in [17]) consists in asking whether a particular judgment set is among the judgment sets of the output: given a profile P, a judgment set J and a judgment aggregation rule F, is $J \in F(P)$? Endriss *et al* [13] consider a more general notion of winner determination problem by considering subsets of the agenda, and define winner determination as the following decision problem: given an agenda subset $S \subset \mathcal{A}$ and a profile P, is there a $J \in F(P)$ such that J contains S? We take a similar approach as in [13], with two differences: instead of asking whether a given subset $S \subset \mathcal{A}$ is contained in some $J \in F(P)$, we ask if a given *element* φ of the agenda is contained in *all* output judgment sets. The reason for considering elements φ of the agenda will be clearer after reading the paper: for all rules we consider, we are able to obtain hardness results even for this simple case, and considering arbitrary subsets of the agenda does not make the problem more complex. The reason for requiring that *all* judgment sets of the output contain φ is because we find it more natural, and is without loss of generality, as the 'existential' problems are dual of these 'universal' problems, as explained at the end of this Section.

We end this Section by defining the problems more formally.

[3] This distinction is reminiscent of the distinction between *social choice functions* and *social welfare functions* in preference aggregation: the former select a winner or a set of winners whether the latter output a collective preference relation.

Consider a fixed, irresolute judgment aggregation rule F_Γ. The decision problem WINNER DETERMINATION(F), that we abbreviate in WD(F), is defined as follows:

Input Agenda $\mathcal{A}$, constraint Γ, profile $P \in \mathbb{D}^n(\mathcal{A}, \Gamma)$, $\alpha \in \mathcal{A}$.
Output Is it the case that $\alpha \in J$ for every $J \in F_\Gamma(P)$?

We pay special attention to the case where there is no constraint, *i.e.*, $\Gamma = \top$. The associated problem CONSTRAINT-FREE WINNER DETERMINATION(F), abbreviated into CF-WD(F), is defined as:

Input Agenda $\mathcal{A}$, profile $P \in \mathbb{D}^n(\mathcal{A}, \top)$, $\alpha \in \mathcal{A}$.
Output Is it the case that $\varphi \in J$ for every $J \in F_\top(P)$?

Observe that WD(F) is at least as hard as CF-WD(F).

Lastly, because $F_\Gamma(P)$ is a set of complete judgment sets, we have the following equivalence:

(A) $\alpha \in J$ for *some* $J \in F_\Gamma(P)$ if and only if it is not the case that $\neg\alpha \in J$ for *all* $J \in F_\Gamma(P)$.

Therefore, if WD(F_Γ) is in a complexity class C for a given rule F then the corresponding 'existential' problem (is it the case that $\alpha \in J$ for *some* $J \in F_\Gamma(P)$?) is in coC.

4 The results

Many of our hardness results use reductions from problems in knowledge representation and reasoning, and in particular, in nonmonotonic reasoning and belief revision.

A supernormal default theory[4] is a pair $D = \langle \Delta, \beta \rangle$ with $\Delta = \{\varphi_1, \ldots, \varphi_p\}$, where $\langle \varphi_1, \ldots, \varphi_p \rangle \in \mathcal{L}^n$ and $\beta \in \mathcal{L}$. A formula $\alpha \in \mathcal{L}$ is a skeptical consequence of D, denoted by $D \mid\!\sim_\forall \alpha$, if and only if for all $S \in max(\Delta, \beta, \subseteq)$ we have $S \wedge \beta \models \alpha$, and a maxcard skeptical consequence of D, denoted by $D \mid\!\sim_\forall^C \alpha$, if and only if for all $S \in max(\Delta, \beta, |.|)$ we have $S \wedge \beta \models \alpha$. Skeptical inference is Π_2^p-complete [15], even if $\beta = \top$. It is straightforward to show that skeptical inference remains Π_2^p-complete under the restriction that $\alpha = \varphi_i$ for some i (because $\langle \Delta, \beta \rangle \mid\!\sim_\forall \alpha$ if and only if $\langle \Delta \cup \{\alpha\}, \beta \rangle \mid\!\sim_\forall \alpha$). Maxcard skeptical inference is Θ_2^p-complete [22], even if $\beta = \top$.[5] Again, maxcard skeptical inference remains Θ_2^p-complete under the restriction that $\alpha = \varphi_i$ for some i. Under these restrictions $\beta = \top$ and $\alpha = \varphi_i$, because a maximal consistent subset of Δ is consistent with α if and only if it contains α, and because $\varphi_1, \ldots, \varphi_p$ play symmetric roles, the problem SKEPTICAL INFERENCE (in supernormal default theories) becomes

Input $\Delta = \langle \varphi_1, \ldots, \varphi_p \rangle$ with consistent $\varphi_i \in \mathcal{L}$.
Output Is it the case that for every maximal (reps. maxcard) consistent subset S of Δ, we have $\varphi_i \in S$?

Proposition 1 *Both* WD(MSA) *and* CF-WD(MSA) *are* Π_2^p-*complete.*

Proof. We show membership of WD(MSA) to Π_2^p by giving an nondeterministic algorithm that shows that the complement problem $\overline{\text{WD}(MSA)}$ is in Σ_2^p.

The role of steps 2 and 3 of Algorithm 2 is to check that S is a maximal Γ-consistent subset of $m(P)$.

[4] "Supernormal" defaults are also called "normal defaults without prerequisites" [25].

[5] The problem in [22] is actually called CARDINALITY-MAXIMIZING BASE REVISION, but both problems are straightforwardly reducible to each other.

Algorithm 1: Membership of $\overline{\text{WD}(MSA)}$ to Σ_2^p

Input: Agenda $\mathcal{A}$, judgment profile P, $\alpha \in \mathcal{A}$.
Output: **true** if and only if $\exists J$ s.t. $J \in MSA_\Gamma(P)$ and $\alpha \notin J$

1 guess a subset S of $m(P)$ and a complete judgment set $J \supseteq S$;
2 check that S is Γ-consistent;
3 check that for each $\varphi \in m(P) \setminus S$, $S \cup \{\varphi\}$ is Γ-inconsistent;
4 check that $\alpha \notin J$.

Π_2^p-hardness of CF-WD(MSA) is proven by a polynomial reduction of SKEPTICAL INFERENCE with the restrictions $\beta = \top$ and $\alpha \in \Delta$. With any instance $\langle \Delta, \alpha \rangle$ of SKEPTICAL INFERENCE we associate an instance of CF-WD(MSA). For a set of formulas Δ we construct a pre-agenda and profile as in Example 1.

Clearly, $m(P_\Delta) = \{x_1, y_1, x_1 \wedge y_1 \rightarrow \varphi_1, \ldots, x_p, y_p, x_p \wedge y_p \rightarrow \varphi_p\}$. Now, we claim that $S \subseteq m(P_\Delta)$ is a maximal consistent subset of $m(P_\Delta)$ if and only if S is of the form

$$S = \bigcup_{i \in I} \{x_i, y_i, x_i \wedge y_i \rightarrow \varphi_i\} \cup \bigcup_{j \notin I} Z_j$$

where $I \subseteq \{1, \ldots, p\}$ is such that $\Delta_I = \{\varphi_i, i \in I\}$ is a maximal consistent subset of Δ and for each $j \notin I$, Z_j contains exactly two elements among $\{x_j, y_j, x_j \wedge y_j \rightarrow \varphi_j\}$. First, for any such subset S, $\bigwedge(S)$ is equivalent to $\bigwedge_{j \notin I} \bigwedge Z_j \wedge \bigwedge_{j \in I} x_i \wedge y_i \wedge \varphi_i$; it is consistent, because Δ_I is consistent. Second, adding one more element of $m(P_\Delta)$ to such an S makes it inconsistent, since it would imply $\Delta_i \wedge \varphi_j$ for some $j \notin I$, and the latter is inconsistent because Δ_I is a maximal consistent subset of Δ. Therefore, any such S is a maximal consistent subset of $m(P)$. Now, assume S is a maximal consistent subset of $m(P_\Delta)$. If S does not contain at least two among x_i, y_i and $x_i \wedge y_i \rightarrow \varphi_i$ for every $i = 1, \ldots, n$, then it is not maximal consistent, because we can add one of these without creating an inconsistency. Therefore, for each $i = 1, \ldots, n$, S contains either the three formulas x_i, y_i and $x_i \wedge y_i \rightarrow \varphi_i$, or exactly two of them. Let I be the set of indices i such that S contains all three formulas x_i, y_i and $x_i \wedge y_i \rightarrow \varphi_i$. S implies Δ_I, therefore Δ_I is consistent. Suppose that Δ_I is not maximal consistent: then there exists $j \notin I$ such that $\Delta_{I \cup \{j\}}$ is consistent; but then we can add x_j or y_j or $x_j \wedge y_j \rightarrow \varphi_j$ to S (whichever of the three is not in S) without creating an inconsistency, which contradicts the assumption that S is a maximal consistent subset of $m(P)$.

Lastly, if $\Delta \mid\!\sim_\forall \varphi_i$, then any maximal consistent subset S of $m(P_\Delta)$ contains φ_i, and using the claim above, every judgment set in $MSA_\top(P_\Delta)$ contains x_i. Conversely, if some maximal consistent subset S of $m(P_\Delta)$ does not contain φ_i, then again using the claim above, some judgment set in $MSA_\top(P_\Delta)$ does not contains x_i. Therefore, $\Delta \mid\!\sim_\forall \varphi_i$ if and only if every $J \in MSA_\top(P_\Delta)$ contains φ_i.

□

Proposition 2 WD($MCSA$) *and* CF-WD($MCSA$) *are both* Θ_2^p-*complete.*

Proof sketch. We show membership of $\overline{\text{WD}(MCSA)}$ to Θ_2^p by giving an algorithm that shows that the complement problem $\overline{\text{WD}(MCSA)}$ is in Θ_2^p.

Θ_2^p-hardness of CF-WD($MCSA$) is proven by a polynomial reduction of MAXCARD SKEPTICAL INFERENCE, with the restrictions $\alpha \in \Delta$ and $\beta = \top$. The reduction is the same as for MSA

Algorithm 2: Membership of $\overline{\text{WD}(MCSA)}$ to Θ_2^p

1 find the cardinality K of a maximal Γ-consistent subset of $m(P)$ by dichotomy on $\{1,\dots,m\}$
2 guess a subset S of $m(P)$ of cardinality K and a complete judgment set $J \supseteq S$;
3 check that S is Γ-consistent;
4 guess an interpretation M;
5 check that M satisfies $S \wedge \Gamma$;
6 check that $\alpha \notin J$.

above. (For the first line of the algorithm see for instance the proof of Theorem 5.14 in [22].) Recall that for each maximal subset S of $m(P)$, S has the form $S = \bigcup_{i \in I}\{x_i, y_i, x_i \wedge y_i \rightarrow \varphi_i\} \cup \bigcup_{j \notin I} Z_j$, where $\Delta_I = \{\varphi_i, i \in I\}$ is a maximal consistent subset of Δ and for each $j \notin I$, z_j is either x_j or y_j. Now, $|S| = 3|I| + 2(n - |I|)$; therefore, S is a maxcard consistent subset of $m(P)$ if $|I|$ is maximal, that is, if $\Delta_I = \{\varphi_i, i \in I\}$ is a maxcard consistent subset of Δ. The rest of the proof is similar to the proof for MSA above. □

Proposition 3 *WD(RA) and CF-WD(RA) are both Π_2^p-complete.*

Proof. Membership of WD(RA) to Π_2^p is shown with the following algorithm:

Algorithm 3: Membership of $\overline{\text{WD}(RA)}$ to Σ_2^p

1 guess a judgment set J;
2 guess a permutation σ on $\mathcal{A}$;
3 check that $J_\sigma = J$;
4 check that $\alpha \notin J$.

Π_2^p-hardness of CF-WD(RA) is proven by a polynomial reduction of SKEPTICAL INFERENCE. The proof – like the proof of Proposition 1 – uses the profile of Example 1. The proof is very similar to the proof of Proposition 1. For every i, $N(P, x_i) = N(P, y_i) = p + 1$ and $N(P, x_i \wedge y_i \rightarrow \varphi_i) = 2p$. Therefore, the judgment sets in $RA(P)$ have the following form: they contain $x_i \wedge y_i \rightarrow \varphi_i$ for all $i = 1, \dots, p$, and then contain $\{x_i, y_i\}$ for all $i \in I$ and exactly one of x_i and y_i for $i \notin I$, where I is a subset of $\{1, \dots, p\}$ such that Δ_I is a maximal consistent subset of Δ. The rest of the proof goes exactly like in the proof of Proposition 1. □

We now consider the MWA rule. MWA is equivalent to the so-called "distance-based procedure" in [13]; it is shown in [13] (Theorem 9) that the problem that we call WINNER DETERMINATION SUBSET(F) is Θ_2^p-complete:

Input Agenda $\mathcal{A} \subset \mathcal{L}$, judgment profile $P \in \mathbb{D}^n(\mathcal{A}, \top)$, $S \subset \mathcal{L}$.
Output Is there a $J \in F_\top(P)$ such that $S \subseteq J$?

Due to Remark (A) at the end of Section 3 and the fact that coΘ_2^p = Θ_2^p, the universal version of the problem (do all output judgment sets contain S?) is Θ_2^p-complete as well. The only thing that remains to prove is that the problem remains Θ_2^p-hard when S is restricted to a singleton, which we state now.

Proposition 4 *WD(MWA) and CF-WD(MWA) are both Θ_2^p-complete.*

Proof sketch. Membership is a corollary of Proposition 9 in [13]. For hardness, we use a reduction from MAXCARD SKEPTICAL INFERENCE with the same profile as for Propositions 1 and 2. $MWA(P_\Delta)$ consists of all judgment sets containing all formulas x_i, y_i, x_i and $y_i \rightarrow \varphi_i$ for $i \in I$ where S_I is some maxcard consistent subset of Δ, plus, for each $i \notin I$, x_i and $y_i \rightarrow \varphi_i$ and exactly one of x_i and y_i. Every $J \in MWA(P_\Delta)$ contains x_i iff $\Delta \mid\!\sim_\forall^C \varphi_i$. □

Proposition 5 *WD$(MNAC)$ and CF-WD$(MNAC)$ are Θ_2^p-complete.*

Proof. Membership is along the same lines as for Proposition 2. For hardness, we give a reduction from MAXCARD SKEPTICAL INFERENCE. Given $\Delta = \langle \varphi_1, \dots, \varphi_n \rangle$, let us build the following instance of CF-WD$(MNAC)$: $\mathcal{A} = \{\varphi_1 \vee x_1, \varphi_1 \vee \neg x_1, \dots, \varphi_1 \vee x_1, \varphi_n \vee \neg x_n\}$, where $x_1, y_1, \dots, x_p, y_p$ are $2p$ fresh propositional symbols (not appearing in $\varphi_1, \dots, \varphi_p$); and P consists of $2p$ individual judgment sets as given in Figure 3. Every J_i is a consistent individual judgment set. Let K be the cardinality of a maxcard-consistent subset of Δ. We claim that $\min\{d(P,Q) \mid Q \text{ majority-consistent}\} = n - K$. $m(P)$ contains $x_i \vee \varphi_i$ and $\neg x_i \vee \varphi_i$ for all i, and thus implies φ_i for each i. If Q is majority consistent then $\{\varphi_i \mid m(Q) \models \varphi_i\}$ must be consistent, therefore[6] $|m(P) \triangle m(Q)| \geq n - K$; because changing the majority judgment on an issue implies changing at least one individual judgment on that issue, $\min\{d(P,Q) \mid Q \text{ majority-consistent}\} \geq n - K$. Now, let S be a maxcard consistent subset of Δ (*i.e.*, such that $|S| = K$). For each $i \notin S$, switching judgment set J_{2i-1} about $\varphi_i \vee x_i$ from $\varphi_i \vee x_i$ to $\neg(\varphi_i \vee x_i)$ suffices to reach majority-consistency; hence $\min\{d(P,Q) \mid Q \text{ majority-consistent}\} \geq n - K$. More precisely, all sets of minimal changes from P to a majority-consistent Q are of this form: for some maxcard consistent subset S of Δ, for each $\varphi \notin S$, change $\varphi_i \vee x_i$ to $\neg(\varphi_i \vee x_i)$ or $\varphi_i \vee \neg x_i$ to $\neg(\varphi_i \vee \neg x_i)$ in one of the individual judgments where it is possible.

	$\varphi_1 \vee x_1$	$\varphi_1 \vee \neg x_1$	$\varphi_2 \vee x_2$	$\varphi_2 \vee \neg x_2$	$\dots$	$\varphi_n \vee x_n$	$\varphi_n \vee \neg x_n$
J_1	+	+	+	−	$\dots$	+	−
J_2	+	+	−	+	$\dots$	−	+
J_3	+	−	+	+	$\dots$	+	−
J_4	−	+	+	+	$\dots$	−	+
$\dots$							
J_{2n-1}	+	−	+	−	$\dots$	+	+
J_{2n}	−	+	−	+	$\dots$	+	+
$m(P)$	+	+	+	+	$\dots$	+	+

Figure 3. The P profile used in the reduction proof for $MNAC$.

Assume that $\varphi_1 \notin S$ for some maxcard consistent subset of Δ. Then, there will be a Q such that $d(P,Q) = n - K$ and such that $\varphi_1 \vee x_1 \notin m(Q)$, therefore there will be $Q \in MNAC(P)$ such that $\varphi_1 \vee x_1 \notin m(Q)$. Conversely, assume $\varphi_1 \in S$ for every maxcard consistent subset S of Δ. Then no set of minimal changes from P to Q involves a switch of $\varphi_1 \vee x_1$, therefore for all $Q \in MNAC(P)$ we have $\varphi_1 \vee x_1 \in m(Q)$. We conclude that $\Delta \mid\!\sim_\forall^C \varphi_1$ if and only if every $J \in MNAC(P)$ contains $\varphi_1 \vee x_1$. □

There was one more majority-preserving rule defined in [18, 20], namely the *Young rule for judgment aggregation*: given a profile P,

[6] Recall that $\triangle$ denotes the symmetric difference between two sets, not to be confused with the set Δ.

$Y_\Gamma(P)$ is defined as the majoritarian judgment sets of all maxcard Γ-consistent subprofiles of P. Given that this rule generalizes the Young voting rule (see [20]), and given that winner determination for the Young rule is Θ_2^p-complete ([26] for the strong version of the rule and [5] for the original one), we might think that obtaining Θ_2^p-complete for WD(Y) is almost straightforward. However, and surprisingly, it is not, because of the focus on a single element of the agenda in our definition of WD (and so far we do not have a proof).

5 Conclusion

We have established a number of complexity results for winner determination in judgment aggregation (see Table 1), focusing on a family of rules that have received some attention in the literature but, apart of the MWA rule, had not been studied from the point of view of computation.

In all cases, we have started to prove that CF-WD is C-hard for some complexity class (Σ_2^p or Θ_2^p). This allows to conclude that any superproblem of CF-WD who belongs to C is C-complete. This applies of course to WD, as we have said already, but also to the more general problem where we ask if all output judgment sets contain a given subset S of the agenda (which is the 'universal' version of the winner determination notion in [13]). This, however, does not apply to the restriction of the latter problem to complete judgment sets, for which we generally have a complexity fall.

Also, we know that specifying judgment aggregation rules to the preference agenda and imposing one of the two constraints Tr or W leads to recovering voting rules which are, in many cases, well-known rules [20]. Such a specialization sometimes comes with no complexity gap (for instance, MWA vs. Kemeny) but sometimes with one: for instance, winner determination for RA is Π_2^p-complete, whereas the rules obtained by the specialization to the preference agenda are: for $\Gamma = W$, maximin (for which winner determination is polynomial) and for $\Gamma = Tr$, ranked pairs, for which winner determination is NP-complete [6].

	WD	CF-WD
MSA	Π_2^P-c.	Π_2^P-c.
$MCSA$	Θ_2^P-c.	Θ_2^P-c.
MWA	Θ_2^P-h.	Θ_2^P-h.
RA	Π_2^P-c.	Π_2^P-c.
$MNAC$	Θ_2^P-c.	Θ_2^P-c.

Table 1. Complexity of the winner determination problem for judgment aggregation rules.

The high complexity of these judgment aggregation rules should be relativized by the fact that many agendas will in fact contain few potential inconsistencies, and it is not hard to see that winner determination for MSA, $MCSA$, MWA and RA is polynomial when the number of minimal inconsistent subsets of the agenda is bounded by a constant. Another way of escaping intractability consists in defining polynomial approximations of our rules, such as rules based on most representative voters [12].

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REFERENCES

[1] N. Alon, R. Bredereck, J. Chen, S. Kratsch, R. Niedermeier, and G.J. Woeginger, 'How to put through your agenda in collective binary decisions', in *ADT*, pp. 30–44, (2013).
[2] N. Alon, D. Falik, R. Meir, and M. Tennenholtz, 'Bundling attacks in judgment aggregation', in *AAAI*, (2013).
[3] D. Baumeister, G. Erdélyi, O. J. Erdélyi, and J. Rothe, 'Computational aspects of manipulation and control in judgment aggregation.', in *Proceedings of the 3rd International Conference on Algorithmic Decision Theory*, ADT 2013, (2013).
[4] D. Baumeister, G. Erdélyi, and J. Rothe, 'How hard is it to bribe the judges? a study of the complexity of bribery in judgment aggregation', in *Algorithmic Decision Theory*, volume 6992 of *LNCS*, 1–15, Springer, (2011).
[5] F. Brandt, M. Brill, E. Hemaspaandra, and L. Hemaspaandra, 'Bypassing combinatorial protections: Polynomial-time algorithms for single-peaked electorates', in *AAAI*, (2010).
[6] M. Brill and F. Fischer, 'The price of neutrality for the ranked pairs method', in *Proceedings of AAAI-12*, pp. 1299–1305, (2012).
[7] F. Dietrich, 'Judgment aggregation by quota rules majority voting generalized', *Journal of Theoretical Politics*, **19**(4), 391–424, (October 2007).
[8] F. Dietrich, 'Scoring rules for judgment aggregation', MPRA paper, University Library of Munich, Germany, (2012).
[9] F. Dietrich and C. List, 'Arrow's theorem in judgment aggregation', *Social Choice and Welfare*, **29**(1), 19–33, (July 2007).
[10] F. Dietrich and C. List, 'Judgment aggregation under constraints', in *Economics, Rational Choice and Normative Philosophy*, eds., T. Boylan and R. Gekker, Routledge, (2008).
[11] F. Dietrich and P. Mongin, 'The premiss-based approach to judgment aggregation', *Journal of Economic Theory*, **145**(2), 562–582, (2010).
[12] U. Endriss and U. Grandi, 'Binary aggregation by selection of the most representative voter', in *Proceedings of AAAI-14*, (2014).
[13] U. Endriss, U. Grandi, and D. Porello, 'Complexity of judgment aggregation', *Journal Artificial Intelligence Research (JAIR)*, **45**, 481–514, (2012).
[14] P. Everaere, S. Konieczny, and P. Marquis, 'Counting votes for aggregating judgments', in *Proceedings of the 13th International Conference on Autonomous Agents and Multiagent Systems*, AAMAS '14, pp. 1177–1184, (2014).
[15] G. Gottlob, 'Complexity results for nonmonotonic logics', *Journal of Logic and Computation*, **2**(3), 397–425, (1992).
[16] U. Grandi and U. Endriss, 'Lifting integrity constraints in binary aggregation', *Artificial Intelligence*, **199–200**, 45–66, (2013).
[17] W. Jamroga and M. Slavkovik, 'Some complexity results for distance-based judgment aggregation', in *26th Australasian Joint Conference on Artificial Intelligence*, eds., S. Cranefield and A. Nayak, Lecture Notes in Computer Science, p. forthcoming. Springer, (2013).
[18] J. Lang, G. Pigozzi, M. Slavkovik, and L. van der Torre, 'Judgment aggregation rules based on minimization', in *TARK*, pp. 238–246, (2011).
[19] J. Lang, G. Pigozzi, M. Slavkovik, and L. van der Torre, 'Judgment aggregation rules based on minimization', Technical report, Université Paris-Dauphine, (2012).
[20] J. Lang and M. Slavkovik, 'Judgment aggregation rules and voting rules', in *Proceedings of the 3rd International Conference on Algorithmic Decision Theory (ADT 2013)*, volume 8176 of *Lecture Notes in Artificial Intelligence*, pp. 230–244. Springer-Verlag, (2013).
[21] M.K. Miller and D. Osherson, 'Methods for distance-based judgment aggregation.', *Social Choice and Welfare*, **32**(4), 575 – 601, (2009).
[22] B. Nebel, 'How hard is it to revise a belief base?', in *Handbook of Defeasible Reasoning and Uncertainty Management Systems*, eds., D. Dubois and H. Prade, 77–145, Kluwer, (1998).
[23] K. Nehring, M. Pivato, and C. Puppe, 'Condorcet admissibility: Indeterminacy and path-dependence under majority voting on interconnected decisions'. http://mpra.ub.uni-muenchen.de/32434/, July 2011.
[24] Ch. Papadimitriou, *Computational complexity*, Addison-Wesley (Reading, Mass.), 1994.
[25] R. Reiter, 'A logic for default reasoning', *Artificial Intelligence*, **13**(1-2), 81–132, (1980).
[26] J. Rothe, H. Spakowski, and J. Vogel, 'Exact complexity of the winner problem for young elections', *Theory of Computing SystemsComput. Syst.*, **36**(4), 375–386, (2003).