# Graded Paraconsistency – Reasoning with Inconsistent and Uncertain Knowledge

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**Abstract**

We propose an introduction of *gradation* into a paraconsistent logic, namely Da Costa's $C_1^{++}$. This gradation is modelled by possibility and necessity degrees, which are issued from fuzzy set theory and which are an ordinal approach to uncertainty. This new framework allows for both a local and graded view of inconsistency and may be applied to Artificial Intelligence problems such as nonmonotonic reasoning and multi-source reasoning.

Keywords: possibility theory, graded inconsistency, artificial intelligence.

## 1 Introduction

Among all the approaches that have been used in Artificial Intelligence to deal with uncertain knowledge, possibility theory turns out to be rather versatile because it can representent information in a more qualitative way than, for example, probability theory: Indeed, it is equivalent to work with "quantitative" possibility theory (which means using possibility and necessity measures and possibility distributions, which map formulas or worlds to [0,1]) or with its qualitative counterpart (where qualitative necessity and possibility relations are preorders over a logical language and qualitative possibility distributions are just preorders on the set of worlds). Actually, its connection to various qualitative formalisms in logic and Artificial Intelligence has been established, notably with epistemic entrenchment relations in [DP 91a], conditional logics in [Bou 92] [FHL 94], System Z in [BDP 92]. Noteworthy, the use of possibility theory in Artificial Intelligence covers nonmonotonic reasoning, belief revision, inconsistency handling, inheritance and default rules handling, temporal reasoning, constraint satisfaction,...

In Artificial Intelligence, paraconsistent logics have been used for modelling contradiction in knowledge-based systems and thus enable some non-trivial inferences from inconsistent knowledge bases. Besnard and Laenens [BL 94a] discuss the use and the interest of paraconsistent logics in knowledge representation and reasoning, especially by comparison with nonmonotonic reasoning and belief revision. A recent application of Belnap's logic to model-based diagnosis was proposed by Arieli and Avron [AA 96].

Now, possibility theory brings in something more that should be fruitfully exploited as complementary to paraconsistency. Hence, we study how to integrate possibility theory with paraconsistent logics. Our work comes from the following two facts:

- paraconsistent logics propose a *local* view of inconsistency, by localising the contradictions on some literals or formulas of the language.
- possibility theory proposes a *graded* but global view of inconsistency, namely by measuring the contradiction in a knowledge base by a so-called *inconsistency degree* (cf Section 2).

Now, as we will show on motivating exemples, some applications need both a local and graded view of inconsistency. This is why it is valuable to integrate paraconsistent logics with a numerical theory of uncertainty. The reason why we focus in this paper on possibility theory rather than another theory of uncertainty, is its qualitative nature (as it amounts to a "numerical account" of preordering relations over formulas or worlds), which should make it a priori simpler to generalize than more quantitative approaches such as probability theory or belief functions.

After recalling the basics of possibility theory and its graded handling of inconsistency, we will briefly show how possibility and necessity functions can be extended from classical to non-classical logics [BL 94b]. Then, in Section 3, we take a case study, that is, we choose a paraconsistent logic, namely da Costa's $C_1^{++}$, and study the properties of its graded, possibilistic extension. In Section 4, we discuss applications to Artificial Intelligence (namely, multisource reasoning and knowledge fusion, default reasoning, flexible decision). Lastly, we mention further developments of our work.

## 2 Necessity and possibility functions: the need for paraconsistency

### 2.1 Classical necessity and possibility functions

The natural presentation of necessity and possibility functions (see [Zad 78] for instance) shows that possibility theory consists in meta-level definitions over classical logic (with the symbol $\perp$ for falsity and $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ as connectives), which respect completely the structure of classical logic.

**Definition 1** *Let $\mathcal{L}$ be a classical propositional language and $\vdash$ the consequence relation from classical propositional logic. A necessity function is a mapping $N$ from $\mathcal{L}$ to $[0,1]$ satisfying the following axioms*[1]

$(Taut)$ *if* $\vdash \varphi$ *then* $N(\varphi) = 1$

$(Eq_N)$ *if* $\vdash \varphi \leftrightarrow \psi$ *then* $N(\varphi) = N(\psi)$

$(Conj)$ $N(\varphi \wedge \psi) = \min(N(\varphi), N(\psi))$

The dual functions of necessity functions are called possibility functions. They can be defined by three axioms about contradiction, equivalence and disjunction:

**Definition 2** *A possibility function is a mapping $\Pi$ from $\mathcal{L}$ to $[0,1]$ such that*

$(Contr)$ *if* $\vdash \neg\varphi$ *then* $\Pi(\varphi) = 0$

$(Eq_\Pi)$ *if* $\vdash \varphi \leftrightarrow \psi$ *then* $\Pi(\varphi) = \Pi(\psi)$

$(Disj)$ $\Pi(\varphi \vee \psi) = \max(\Pi(\varphi), \Pi(\psi))$

With the assumption that $\mathcal{L}$ is built from a finite number of propositional variables, "classical" necessity/possibility functions can be semantically defined by means of possibility distributions: A possibility distribution $\pi$ is simply a function from the set $\Omega$ of all interpretations for $\mathcal{L}$ to $[0,1]$. The possibility function induced by $\pi$ is defined by

$$\Pi(\varphi) = \sup\{\pi(\omega) | \omega \models \varphi\}$$

[1] Many definitions of necessity functions include the extra axiom $(Inc)$ if $\vdash \neg\varphi$ then $N(\varphi) = 0$ but not all: for example, in [DLP 94] the quantity $N(\varphi) > 0$, reflects a degree of (partial) inconsistency. Even without $(Inc)$ –and similarly for $(Taut)$– a derived property called dominance: $\vdash \varphi \rightarrow \psi \Rightarrow N(\varphi) \leq N(\psi)$ ensures that inconsistencies (resp. tautologies) have anyway the lowest (resp. highest) necessity degree. The reason why we require $(Taut)$ and not $(Inc)$ concerns the characterization of necessity functions in terms of possibility distributions. Finally, $(Inc)$ must be absent for $Incons(KB)$ (see below) to be possibly larger than 0.

and the necessity function induced by $\pi$ is defined by

$$N(\varphi) = \inf\{1 - \pi(\omega) | \omega \models \neg\varphi\} = 1 - \Pi(\neg\varphi)$$

(with the conventions $\sup \emptyset = 0$ and $\inf \emptyset = 1$ that also applies in subsequent sections). It can then be proved that $\Pi$ (resp. $N$) is a possibility (resp. necessity) function, and that if $\mathcal{L}$ is generated by a finite number of propositional variables then any possibility (resp. necessity) function on $\mathcal{L}$ is induced by a possibility distribution.

## 2.2 Possibilistic logic and inconsistency handling

Possibility theory, as well as its qualitative counterparts such as epistemic entrenchment relations [GM 88], ranked knowledge bases [Pea 90] or rational closure [Leh 89] provide a relativized treatment of inconsistency, since the latter becomes a gradual notion. A possibilistic knowledge base [DLP 94] consists of a set of constraints

$$KB = \{(\varphi_i \ \alpha_i), i = 1..n\}$$

where $(\varphi_i \ \alpha_i)$ is a syntactic notation for the semantical constraint $N(\varphi_i) \geq \alpha_i$.

A possibilistic knowledge base is partially inconsistent if it leads to enforce $N(\bot) > 0$; stated otherwise, the inconsistency degree of $KB$ is defined by

$$\begin{aligned} Incons(KB) &= \max_{S \subseteq KB, S \vdash \bot} \min_{(\varphi_i \alpha_i) \in S} \alpha_i \\ &= \min\{N(\bot), N \ satisfies \ KB\} \end{aligned}$$

Any formula below this level, i.e. any $\varphi_i$ where $\alpha_i \leq Incons(KB)$, is then inhibited (it is "drown" by the inconsistency [BCDLP 93]). This shows that the notion of inconsistency in possibilistic logic and its qualitative counterpart is gradual but *global*. The inconsistency level measures to what extent the knowledge base is inconsistent, but does not locate the inconsistency. The aforementioned "drowning effect" is a consequence of this global treatment of inconsistency. One way to cope with it is to consider the knowledge base syntactically [Bre 89] [Neb 91] [BCDLP 93], by selecting among maximal sub-bases of $KB$ using a criterion involving the $\alpha_i$'s. However, these syntactical approaches do not have (yet) any semantics in terms of uncertainty measures nor paraconsistent logics.

## 2.3 The need for a graded and local view of inconsistency

Using paraconsistent logics for handling inconsistent knowledge bases enables a local treatment of inconsistency, by locating the inconsistency on some formulas. Yet, these paraconsistent approaches do not allow for any graduality in the inconsistency, which implies some loss of information if the initial knowledge was pertained with uncertainty.

Possibilistic logic allows for a gradual but global treatment of inconsistency, where conflicts are solved only by comparing the uncertainty level of the pieces of information with the inconsistency level of the knowledge base, while a pure paraconsistent approach localizes inconsistency, but conflicts cannot be ranked according to uncertainty, importance, priority, normality as done in rank-based systems. Thus paraconsistency does not "solve" the conflicts. We propose to apply the results of Section 2 to a given paraconsistent logic, $C_1^{++}$ [daC 74], to handle both uncertain and inconsistent knowledge, and with a local treatment of inconsistency. We give two motivating examples, one about fusion of uncertain information (multi-source reasoning) and one about reasoning with default rules.

**Example 1** (multi-source reasoning) This example is borrowed from [Cho 94]. Two witnesses report their observations about a murderer. The first witness (W1) is certain that the murderer was a woman with blond hair, and believes (with some uncertainty) that she was wearing a Chanel suit, glasses, and was driving a BMW. The second witness (W2) is certain that the murderer was a woman with brown hair, almost certain that she was not wearing glasses, and believes (with some uncertainty) that she was driving a Fiat.

W1 `female` (sure), `blond-hair` (sure), `drives-BMW` (unsure), `wear-glasses` (unsure), `wear-Chanel-suit` (unsure)

W2 `female` (sure), `brown-hair` (sure), `drives-Fiat` (unsure), ¬`wear-glasses` (almost sure)

What would we like to conclude about the following statements?

- Both witnesses agree that the murderer was female and are completely sure; so we want to conclude the murderer was female.
- No contradiction either about `wear-Chanel-suit` since witness 2 does not know anything.
- Strong contradiction about the colour of the murderer's hair; we wish to conclude neither blond nor brown but we want to keep in mind that these literals are "strongly subject to inconsistency" (knowing the constraint ¬(`blond-hair` ∧ `brown-hair`)).
- Contradiction about wear-glasses: the contradiction is weaker than the one above since the first witness is unsure; moreover, since the second witness's information is prioritary to the first witness's we would like to solve the conflict (by concluding ¬`wear-glasses`).
- Weak contradiction again, about the car; however, since the witnesses are both uncertain, we do not want to conclude anything.

**Example 2** (reasoning with default rules)
A large field of research in Artificial Intelligence consists in modelling and reasoning with default rules of the form "typically, A's are B's". A well-known

approach to default rules, namely Goldszmidt and Pearl's System Z [Pea 90], ranks automatically the rules according to their specificity level (more specific rules are more prioritary – see the example below). This approach can be equivalently recast in the framework of possibilistic logic [BDP 92], modelling rule levels by necessity functions.

However System Z and similar approaches suffer from the problem of "blocking of property inheritance" (an instance of the abovementioned "drowning effect"), namely, some pieces of information which do not have anything to do with the contradictions of the knowledge bases are inhibited by higher level inconsistencies. This is clearly due to the fact that it is not enough to handle inconsistency in a graded way, but it is also necessary to view it as a local notion, which is lacking from procedures like System Z. Clearly, paraconsistency brings this local view, while necessity functions keep the graded view.

Let us consider the following set of default rules:
$\Delta = \{\texttt{penguin} \to \texttt{bird}, \texttt{penguin} \to \neg\texttt{fly}, \texttt{bird} \to \texttt{fly}, \texttt{bird} \to \texttt{wings}\}$
Applying Pearl's ranking procedure to $\Delta$ gives the specificity ordering

$$\Delta_1 = \{\texttt{penguin} \to \texttt{bird}, \texttt{penguin} \to \neg\texttt{fly}\};$$

$$\Delta_0 = \{\texttt{bird} \to \texttt{fly}, \texttt{bird} \to \texttt{wings}\}.$$

That is, the rules $\texttt{penguin} \to \texttt{bird}$ and $\texttt{penguin} \to \neg\texttt{fly}\}$ are more specific, therefore more prioritary, than the two other ones. Now, adding the fact $\texttt{penguin}$ to $\Delta$ enables us to infer $\neg\texttt{fly}$ but $\texttt{wings}$ is not deduced (it is "drown" by the inconsistency appearing at rank 1). This particular case of the drowning effect is known as the property of "inheritance blocking" (see [BCDLP 93]).

Considering $\Delta$ as a set of formulas for the logic $C_1^{++}$, we obtain

$$\Delta \cup \{\texttt{penguin}\} \vdash_{C_1^{++}} \{\texttt{fly}, \neg\texttt{fly}, \texttt{wings}\}.$$

Thus, we avoid the drowning effect but we do not take into account priorities (induced by specificity) such as $\texttt{penguin} \to \neg\texttt{fly}$ over $\texttt{bird} \to \texttt{fly}$ and we conclude that $\texttt{fly}$ is not well-behaved.

What we would like is to take advantage of the localisation of inconsistency, as done by paraconsistent entailment, and the priority between formulas, which would lead us to infer $\{\texttt{fly}, \texttt{wings}\}$ but not $\neg\texttt{fly}$.

Note that prioritized syntax-based approaches based on the selection of maximal consistent subsets of the knowledge base guided by the priorities solve the drowning effect but do not tell anything about where the contradictions are localized; so, for instance, the conclusion $\neg\texttt{wear-glasses}$ is not relativised by the fact that it is (weakly) subject to inconsistency.

# 3 Graded paraconsistency: a case study

## 3.1 Possibility and necessity functions over non-classical logics

The definition of possibility and necessity functions (Section 2) consists, as already mentioned, in meta-level definitions over classical logic (as they use a classical propositional language and the classical consequence relation). This suggests that similar functions can be defined on other logics than classical logic; so, replacing $(\mathcal{L}, \vdash)$ by $(\mathcal{L}, \vdash_L)$ where $L$ is a given non-classical logic, we can look for a definition of possibility/necessity functions on the logic $L$ (this approach is developed in a more general setting in [BL 94b]). We deal with a classical propositional language, built from a finite list of propositional variables and the connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$. The only varying parameter is then the consequence relation $\vdash_L$.

## 3.2 The paraconsistent logic $C_1^{++}$

$C_1^{++}$ [daC 74] is a paraconsistent logic, that is, a logic in which a contradiction $\varphi \wedge \neg\varphi$ fails to entail other arbitrary contradictions $\psi \wedge \neg\psi$. As an extension of the well-known paraconsistent logic $C_1$ [daC 74] (actually, $C_1^{++}$ is $C_1$ supplemented with the axiom schema $\varphi \rightarrow \neg\neg\varphi$), $C_1^{++}$ retains all inference patterns of classical logic that are not based on negation. By contrast, some inference patterns of classical logic that do appeal to negation are not preserved. The idea is that positive information is fundamental: Positive formulas and inferences contribute to state what the facts are whereas negative formulas and inferences are merely constraints (in the sense of integrity constraints for databases). Accordingly, $C_1$ (hence $C_1^{++}$) allows us to elicit all and only the formulas responsible for a given contradiction ([CL 92] [BL 94a]).

Regarding notation, we use $\varphi^\circ$ as an abbreviation for $\neg(\varphi \wedge \neg\varphi)$. We also use # to denote any of $\wedge, \vee, \rightarrow$.

A valuation-based semantics for $C_1^{++}$ [CL 92] (extending one [Alv 84] for $C_1$) is given in Section 3.2.3 as we now reproduce the original axiomatic presentation of $C_1^{++}$ that consists of the axioms: 1. $\varphi \rightarrow (\psi \rightarrow \varphi)$; 2. $(\varphi \rightarrow \psi) \rightarrow [(\varphi \rightarrow (\psi \rightarrow \sigma)) \rightarrow (\varphi \rightarrow \sigma)]$; 3. $\varphi \wedge \psi \rightarrow \varphi$; 4. $\varphi \wedge \psi \rightarrow \psi$; 5. $\varphi \rightarrow (\psi \rightarrow \varphi \wedge \psi)$; 6. $\varphi \rightarrow \varphi \vee \psi$; 7. $\varphi \rightarrow \psi \vee \varphi$; 8. $(\varphi \rightarrow \sigma) \rightarrow [(\psi \rightarrow \sigma) \rightarrow (\varphi \vee \psi \rightarrow \sigma)]$; 9. $\varphi \vee \neg\varphi$; 10. $\neg\neg\varphi \leftrightarrow \varphi$; 11. $\psi^\circ \rightarrow [(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi)]$; 12. $(\varphi^\circ \wedge \psi^\circ) \rightarrow ((\varphi \vee \psi)^\circ \wedge (\varphi \wedge \psi)^\circ \wedge (\varphi \rightarrow \psi)^\circ)$ together with the single inference rule

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}.$$

## 3.3 $C_1^{++}$-necessity and possibility functions

They are defined like necessity and possibility functions, replacing $\vdash$ by $\vdash_{C_1^{++}}$.

**Definition 3** *A $C_1^{++}$-necessity function is a mapping $N : \mathcal{L} \to [0,1]$ for which $(Taut)$, $(Eq_N)$ and $(Conj)$ obtain wrt $C_1^{++}$.*

**Proposition 1** *Let $N$ be a $C_1^{++}$-necessity function. Then $N$ satisfies:*

*(N1) If $\vdash_{C_1^{++}} \varphi \to \psi$ then $N(\psi) \geq N(\varphi)$*

*(N2) $N(\psi) \geq \min(N(\varphi), N(\varphi \to \psi))$*

*(N3) If $\varphi_1, \ldots, \varphi_n \vdash_{C_1^{++}} \psi$ then $N(\psi) \geq \min_{i=1..n} N(\varphi_i)$*

*Proof of Proposition 1* For (N1), notice that $\vdash_{C_1^{++}} \varphi \leftrightarrow (\varphi \wedge \psi)$ whenever $\vdash_{C_1^{++}} \varphi \to \psi$, then apply $(Eq_N)$ and $(Conj)$. For (N2), notice that $\vdash_{C_1^{++}} (\varphi \wedge (\varphi \to \psi)) \to \psi$ and apply $(Conj)$ together with (N1). For (N3), notice first that if $\varphi_1, \ldots, \varphi_n \vdash_{C_1^{++}} \psi$ then $\vdash_{C_1^{++}} \varphi_1 \wedge \ldots \wedge \varphi_n \to \psi$. Next, apply (N1). Last, repeatedly apply $(Conj)$. ∎

Proposition 1 shows that $C_1^{++}$-necessity functions are compatible with deduction in $C_1^{++}$.

**Proposition 2** *(N4) $N(\varphi \vee \psi) \geq \max(N(\varphi), N(\psi))$*

*Proof of Proposition 2* As $\vdash_{C_1^{++}} \varphi \to (\varphi \vee \psi)$ holds, the result immediately follows from (N1) (Proposition 1). ∎

In the context of paraconsistent necessity functions, formulas of the form $\varphi^\circ$ are obviously of interest. Here are a few properties about them.

**Proposition 3** *If $N$ is a $C_1^{++}$-necessity function then*

*(N5) $N(\neg\varphi^\circ) = \min(N(\varphi), N(\neg\varphi))$*

*(N6) $N(\neg\varphi) \geq \min(N(\psi^\circ), N(\varphi \to \psi), N(\varphi \to \neg\psi))$*

*(N7) $N((\varphi \# \psi)^\circ) \geq \min(N(\varphi^\circ), N(\psi^\circ))$*

*Proof of Proposition 3* For (N5), notice that $\vdash_{C_1^{++}} \varphi \leftrightarrow \neg\neg\varphi$ and apply $(Eq_N)$ together with $(Conj)$. For (N6), notice that $\vdash_{C_1^{++}} (\psi^\circ \wedge (\varphi \to \psi) \wedge (\varphi \to \neg\psi)) \to \neg\varphi$ and apply $(Conj)$ together with (N1). For (N7), first use (N2) to obtain $N((\varphi\#\psi)^\circ) \geq \min(N(\varphi^\circ \wedge \psi^\circ), N(\varphi^\circ \wedge \psi^\circ \to (\varphi\#\psi)^\circ))$. As $\vdash_{C_1^{++}} \varphi^\circ \wedge \psi^\circ \to (\varphi\#\psi)^\circ$, $(Taut)$ then gives $N(\varphi^\circ \wedge \psi^\circ \to (\varphi\#\psi)^\circ) = 1$. Hence, $N((\varphi\#\psi)^\circ) \geq N(\varphi^\circ \wedge \psi^\circ)$. In view of $(Conj)$, the result follows. ∎

**Proposition 4** *(N8) $N$ is a classical necessity function iff $N(\varphi^\circ) = 1$ for all $\varphi$.*

*Proof of Proposition 4* ($\Rightarrow$) $\varphi^\circ$ is a valid schema of classical logic and $N$ being a classical necessity function, it follows that $N(\varphi^\circ) = 1$.

($\Leftarrow$) By Theorem 4 of [DaC 74], for every classical theorem $\psi$ there are finitely many formulas $\varphi_1^\circ, \ldots, \varphi_n^\circ$ such that $\vdash_{C_1^{++}} \varphi_1^\circ \wedge \ldots \wedge \varphi_n^\circ \to \psi$. By $(Taut)$, $N(\varphi_1^\circ \wedge \ldots \wedge \varphi_n^\circ \to \psi) = 1$. Now, it follows from the assumption that $N(\varphi_i^\circ) = 1$ and $(Conj)$ in turn gives $N({\varphi_1}^\circ \wedge \ldots \wedge {\varphi_n}^\circ) = 1$. Using (N1) then yields $N(\psi) = 1$. This takes care of verifying $(Taut)$ for the classical case. Turning to the case of two formulas $\psi$ and $\xi$ which are equivalent in classical logic, $\vdash \psi \leftrightarrow \xi$, it has just been proved that in such a case $N(\psi \leftrightarrow \xi) = 1$. Applying $(Conj)$ to the definition of $\psi \leftrightarrow \xi$ as $(\psi \to \xi) \wedge (\xi \to \psi)$ gives $N(\psi \to \xi) = N(\xi \to \psi) = 1$. Using (N2), $N(\psi) \geq N(\xi)$ as well as $N(\xi) \geq N(\psi)$ so that $N(\psi) = N(\xi)$. That is, $(Eq_N)$ is also verified in the classical case. The last property $(Conj)$ is independent of the logic and the proof is over. ■

$N(\neg\varphi^\circ) = \min(N(\varphi), N(\neg\varphi))$ is the necessity of $\varphi$ "behaving badly"; it can be seen as a measure of the inconsistency inherent in $\varphi$. $C_1^{++}$-necessity functions enable us to rank the formulas not only with respect to their certainty, but also with respect to their inherent inconsistency: $N(\neg\varphi^\circ)$ gives a notion of inconsistency which is both local and gradual. We recover as particular cases:

- Classical necessity functions, so that $N(\neg\varphi^\circ) = N(\neg\psi^\circ)$ for all $\varphi$ and $\psi$. The notion of inconsistency is still gradual but global.
- Standard $C_1^{++}$-valuations, which verify $N(\neg\varphi^\circ) = 0$ or $N(\neg\varphi^\circ) = 1$, for all $\varphi$. The notion of inconsistency is still local but not gradual.

**Definition 4** *A $C_1^{++}$-possibility function is a mapping $\Pi : \mathcal{L} \to [0,1]$ for which $(Contr)$, $(Eq_\Pi)$ and $(Disj)$ obtain wrt $C_1^{++}$.*

**Proposition 5** *$C_1^{++}$-possibility functions enjoy the following property:*

*(P1) If $\vdash_{C_1^{++}} \varphi \to \psi$ then $\Pi(\varphi) \leq \Pi(\psi)$*

*Proof of Proposition 5* If $\vdash_{C_1^{++}} \varphi \to \psi$ then $\vdash_{C_1^{++}} \psi \leftrightarrow (\varphi \vee \psi)$. Apply $(Eq_\Pi)$ and $(Disj)$. ■

**Proposition 6** *If $\Pi$ is a $C_1^{++}$-possibility function then it satisfies:*

*(P2)* $\Pi(\varphi \wedge \psi) \leq \min(\Pi(\varphi), \Pi(\psi))$

*Proof of Proposition 6* As $\vdash_{C_1^{++}} (\varphi \wedge \psi) \to \varphi$, apply (P1) of Proposition 5. ■

The classical duality between necessity functions and possibility functions fails for the functions we have defined here because negation in $C_1^{++}$ is not De Morgan.

In Artificial Intelligence, possibility functions have less significance than necessity functions. So, the fact that $C_1^{++}$-possibility functions enjoy less properties than $C_1^{++}$-necessity functions does not harm too much. In the rest of the paper, we will focus on $C_1^{++}$-necessity functions.

### 3.4 $C_1^{++}$-necessity and possibility functions: semantics

At first, a (paraconsistent) $C_1^{++}$-valuation [CA 77] [CL 92] is a mapping from $\mathcal{L}$ to $\{0,1\}$ such that:

- $v(\varphi) = 0 \Rightarrow v(\neg\varphi) = 1$
- $v(\neg\neg\varphi) = 1 \Leftrightarrow v(\varphi) = 1$
- $v(\psi^\circ) = v(\varphi \rightarrow \psi) = v(\varphi \rightarrow \neg\psi) = 1 \Rightarrow v(\varphi) = 0$
- $v(\varphi \rightarrow \psi) = 1 \Leftrightarrow v(\varphi) = 0$ or $v(\psi) = 1$
- $v(\varphi \wedge \psi) = 1 \Leftrightarrow v(\varphi) = 1$ and $v(\psi) = 1$
- $v(\varphi \vee \psi) = 1 \Leftrightarrow v(\varphi) = 1$ or $v(\psi) = 1$
- $v(\varphi^\circ) = v(\psi^\circ) = 1 \Rightarrow v((\varphi \# \psi)^\circ) = 1$

This gives a sound and complete semantics for $C_1^{++}$ [CA 77] [CL 92], i.e.

- $\vdash_{C_1^{++}} \varphi$ iff $v \models_{C_1^{++}} \varphi$ for all $v$

with the following properties for all $v$

- $v \models_{C_1^{++}} \varphi \leftrightarrow \psi$ iff $(v \models_{C_1^{++}} \varphi) \Leftrightarrow (v \models_{C_1^{++}} \psi)$
- $v \models_{C_1^{++}} \varphi \wedge \psi$ iff $v \models_{C_1^{++}} \varphi$ and $v \models_{C_1^{++}} \psi$

**Definition 5** *A $C_1^{++}$-possibility distribution is a mapping $\pi$ from the set of all $C_1^{++}$-valuations to $[0,1]$.*

**Proposition 7** *The function $n_\pi$ defined as*

$$n_\pi(\varphi) = 1 - \sup\{\pi(v) | v \not\models_{C_1^{++}} \varphi\}$$

*is a $C_1^{++}$-necessity function*

*Proof of Proposition 7* If $\vdash_{C_1^{++}} \varphi$ then the soundness of $C_1^{++}$ means that $v \models_{C_1^{++}} \varphi$ holds for all $v$, hence $n_\pi(\varphi) = 1 - \sup \emptyset = 1$. Thus, $(Taut)$ is verified. If $\vdash_{C_1^{++}} \varphi \leftrightarrow \psi$ then $v \models_{C_1^{++}} \varphi$ iff $v \models_{C_1^{++}} \psi$ for all $v$, hence $n_\pi(\varphi) = n_\pi(\psi)$. So, $(Eq_N)$ holds as well. Finally, $n_\pi(\varphi \wedge \psi) = 1 - \sup\{\pi(v) | v \not\models_{C_1^{++}} \varphi \wedge \psi\} = 1 - \sup\{\pi(v) | v \not\models_{C_1^{++}} \varphi \textit{ or } v \not\models_{C_1^{++}} \psi\} = 1 - \max(\sup\{\pi(v) | v \not\models_{C_1^{++}} \varphi\}, \sup\{\pi(v) | v \not\models_{C_1^{++}} \psi\}) = \min(n_\pi(\varphi), n_\pi(\psi))$ so that $(Conj)$ also holds. ■

### 3.5 $C_1^{++}$-necessity orderings

It has been shown [Dub 86] that necessity and possibility functions can be equivalently expressed in purely qualitative terms, with preordering relations. We briefly give a generalization of this result, for the case of $C_1^{++}$-necessity functions (the case for $C_1^{++}$-possibility functions is similar).

**Definition 6** *A $C_1^{++}$-necessity ordering $\succeq$ is a binary relation on $\mathcal{L}$ satisfying the following properties:*

- *if $\varphi \succeq \psi$ and $\psi \succeq \xi$ then $\varphi \succeq \xi$ (transitivity)*
- *if $\vdash_{C_1^{++}} \varphi \to \psi$ then $\psi \succeq \varphi$ (dominance)*
- *$\varphi \wedge \psi \approx \varphi$ or $\varphi \wedge \psi \approx \psi$ (conjunctiveness)*

*where of course $\varphi \approx \psi$ stands for $\varphi \succeq \psi$ and $\psi \succeq \varphi$.*

**Definition 7** *([Dub 86]) A mapping $f$ from $\mathcal{L}$ to $[0,1]$ is said to agree strictly with a relation $\succeq$ on $\mathcal{L}$ iff $\forall \varphi, \psi \in \mathcal{L}$, we have $\varphi \succeq \psi \Leftrightarrow f(\varphi) \geq f(\psi)$.*

**Proposition 8** *The only mappings from $\mathcal{L}$ to $[0,1]$ agreeing strictly with $C_1^{++}$-necessity orderings and also satisfying $(Taut)$ are $C_1^{++}$-necessity functions.*

*Proof of Proposition 8* Given a $C_1^{++}$-necessity ordering $\succeq$, let $f : \mathcal{L} \to [0,1]$ agree strictly with $\succeq$ and satisfy $(Taut)$. If $\vdash_{C_1^{++}} \varphi \leftrightarrow \psi$ then $\varphi \succeq \psi$ and $\psi \succeq \varphi$ follow by (N1) of Proposition 1 (using $\vdash_{C_1^{++}} \varphi \wedge \psi \to \varphi$ together with modus ponens). The fact that $f$ agrees strictly with $\succeq$ then implies $f(\varphi) \geq f(\psi)$ and $f(\psi) \geq f(\varphi)$, that is, $f(\varphi) = f(\psi)$. So, $(Eq_N)$ is proved. For two formulas $\varphi$ and $\psi$, either $\varphi \wedge \psi \succeq \varphi$ or $\varphi \wedge \psi \succeq \psi$ by conjunctiveness. The cases are symmetric, hence, for the time being, it is enough to consider $\varphi \wedge \psi \succeq \varphi$. It follows that $f(\varphi \wedge \psi) \geq f(\varphi)$ because $f$ agrees strictly with $\succeq$. On the other hand, (N1) applies to $\vdash_{C_1^{++}} \varphi \wedge \psi \to \varphi$ to yield $\varphi \succeq \varphi \wedge \psi$. Since $f$ agrees strictly with $\succeq$, $f(\varphi) \geq f(\varphi \wedge \psi)$. Therefore, $f(\varphi) = f(\varphi \wedge \psi)$. Also, (N1) applies to $\vdash_{C_1^{++}} \varphi \wedge \psi \to \psi$ to yield $\psi \succeq \varphi \wedge \psi$, hence $f(\psi) \geq f(\varphi \wedge \psi)$ so that $f(\psi) \geq f(\varphi)$ (in view of $f(\varphi) = f(\varphi \wedge \psi)$ as just shown). Of course, $f(\psi) \geq f(\varphi) = f(\varphi \wedge \psi)$ means that $f(\varphi \wedge \psi) = \min(f(\varphi), f(\psi))$. The symmetric case is similar. In all, $(Conj)$ holds. ■

## 4 Application to reasoning and decision making with uncertain and inconsistent information

In this Section, we give several applications of graded paraconsistency (namely $C_1^{++}$-necessity functions) to artificial intelligence.

## 4.1 Minimum specificity and nonmonotonic inference

The principle of minimum specificity [DP 86] or equivalently, minimum compact ranking [Pea 90] and rational closure [Leh 89] (all these being equivalent, up to the language on which they are defined) induces, from a possibilistic knowledge base, a particular necessity function i.e. the smallest among all necessity functions satisfying the knowledge base. Thanks to the property (N3), we are able to generalize the principle of minimum specificity to $C_1^{++}$-necessity functions:

**Definition 8** *A $C_1^{++}$-possibilistic knowledge base is a finite set*

$$KB = \{(\varphi_i\ \alpha_i), 1 \leq i \leq n\}$$

*where $\varphi_i \in \mathcal{L}$ and $\alpha_i \in [0,1]$. A $C_1^{++}$-necessity function $N$ is said to satisfy $KB$ iff $N(\varphi_i) \geq \alpha_i$ for $i = 1..n$.*

**Definition 9** *Given a $C_1^{++}$-possibilistic knowledge base $KB$, the minimum specificity closure $N_{KB}$ of $KB$ is defined by:*

$$\forall\psi \in \mathcal{L}, N_{KB}(\psi) = \sup\{\beta | KB_\beta \vdash_{C_1^{++}} \psi\}$$

*where $KB_\beta = \{\varphi_i | (\varphi_i\ \alpha_i) \in KB$ and $\alpha_i \geq \beta\}$.*

**Proposition 9** *The minimum specificity closure of a $C_1^{++}$-possibilistic knowledge base is a $C_1^{++}$-necessity function.*

*Proof of Proposition 9* Verification of $(Taut)$, which is easy, is omitted. Consider now $\varphi$ and $\psi$ such that $\vdash_{C_1^{++}} \varphi \leftrightarrow \psi$. Assume $N_{KB}(\varphi) = \alpha$ (the case for $\psi$ is symmetric). Then, $KB_\alpha \vdash_{C_1^{++}} \varphi$. Also, $KB_\alpha \vdash_{C_1^{++}} \varphi \leftrightarrow \psi$ (apply monotonicity to $\vdash_{C_1^{++}} \varphi \leftrightarrow \psi$). By modus ponens, $KB_\alpha \vdash_{C_1^{++}} \psi$. So, $N_{KB}(\psi) \geq \alpha = N_{KB}(\varphi)$. The symmetric case shows $N_{KB}(\varphi) \geq N_{KB}(\psi)$. Hence, $N_{KB}(\varphi) = N_{KB}(\psi)$. I.e., $(Eq_N)$ holds. Using $\vdash_{C_1^{++}} (\varphi \wedge \psi) \leftrightarrow \varphi$ and modus ponens, it can be shown that $KB_\beta \vdash_{C_1^{++}} \varphi \wedge \psi$ implies $KB_\beta \vdash_{C_1^{++}} \varphi$. By monotonicity and adjunction, it can be shown that $KB_\alpha \vdash_{C_1^{++}} \varphi$ implies $KB_\beta \vdash_{C_1^{++}} \varphi$ whenever $\alpha \geq \beta$. Combining both properties, $\sup\{\beta | KB_\beta \vdash_{C_1^{++}} \varphi \wedge \psi\} = \min(\sup\{\beta | KB_\beta \vdash_{C_1^{++}} \varphi\}, \sup\{\beta | KB_\beta \vdash_{C_1^{++}} \psi\})$. Now, $\min(\sup\{\beta | KB_\beta \vdash_{C_1^{++}} \varphi\}, \sup\{\beta | KB_\beta \vdash_{C_1^{++}} \psi\}) = \min(N_{KB}(\varphi), N_{KB}(\psi))$ whereas $N_{KB}(\varphi \wedge \psi) = \sup\{\beta | KB_\beta \vdash_{C_1^{++}} \varphi \wedge \psi\}$. So, $(Conj)$ holds as well. ■

The next result mentions the notion of a $C_1^{++}$-necessity function $N$ being greater than a $C_1^{++}$-necessity function $N'$, written $N \geq N'$, which is of course defined as $N(\varphi) \geq N'(\varphi)$ for all $\varphi$.

**Proposition 10** *(principle of minimum specificity for $C_1^{++}$-necessity functions) Whenever $N$ is a $C_1^{++}$-necessity function, $N$ satisfies $KB$ iff $N \geq N_{KB}$.*

*Proof of Proposition 10* ($\Rightarrow$) Let $N$ be a $L_X$-necessity function satisfying $KB$. According to (N3) of Proposition 1, if $KB_\beta \vdash_{C_1^{++}} \varphi$ then $N(\varphi) \geq \min\{N(\varphi_i)|\varphi_i \in KB_\beta\} = \min\{\alpha_i|\varphi_i \in KB_\beta\} = \beta$. Hence, $N(\varphi) \geq \sup\{\beta|KB_\beta \vdash_{C_1^{++}} \varphi\}$. That is, $N(\varphi) \geq N_{KB}(\varphi)$ ($\Leftarrow$) Let $(\varphi_i\ \alpha_i) \in KB$. Trivially, $\varphi_i \in KB_{\alpha_i}$ and $KB_{\alpha_i} \vdash_{C_1^{++}} \varphi_i$. Then, $\alpha_i \leq \sup\{\beta|KB_\beta \vdash_{C_1^{++}} \varphi_i\} = N_{KB}(\varphi_i) \leq N(\varphi_i)$. ■

Applying the principle of minimum specificity enables us to draw conclusions that takes into account the uncertainty and the inconsistency of the knowledge base. We propose the following definition of a consequence relation.

**Definition 10** $KB \mathrel{|\!\sim} \psi$ *iff* $N_{KB}(\psi) > N_{KB}(\neg\psi^\circ)$.

**Proposition 11** $KB \mathrel{|\!\sim} \psi$ *iff* $N_{KB}(\psi) > N_{KB}(\neg\psi)$

*Proof of Proposition 11* Apply the equality (N5) of Proposition 3, $N_{KB}(\neg\varphi^\circ) = \min(N_{KB}(\varphi), N_{KB}(\neg\varphi))$, to the definition. ■

Intuitively, we deduce $\psi$ from $KB$ iff the certainty of $\psi$ is higher than the inconsistency inherent to $\psi$, or equivalently, iff the certainty of $\psi$ is higher than the certainty of $\neg\psi$. The classical version of $\mathrel{|\!\sim}$ would be defined by: $\varphi \mathrel{|\!\sim}_{KB} \psi$ iff $N_{KB}(\varphi \to \psi) > N_{KB}(\varphi \to \neg\psi^\circ)$, or equivalently iff $N_{KB}(\varphi \to \psi) > N_{KB}(\varphi \to \neg\psi)$. Note that $\mathrel{|\!\sim}$ is nonmonotonic; a more complete study of the properties of $\mathrel{|\!\sim}$ à la Kraus, Lehmann and Magidor [KLM 90], is possible with respect to the (monotonic) logic $C_1^{++}$ instead of classical logic.

Note that when $N$ collapses to a classical necessity function, we have

$$N_{KB}(\neg\varphi^\circ) = N(\neg\psi^\circ) \text{ for all } \varphi, \psi$$

and $\mathrel{|\!\sim}$ is then the classical possibilistic consequence relation [DP 91b].

## 4.2 Knowledge fusion and multisource reasoning

When reasoning with knowledge bases in artificial intelligence, a cause of inconsistency is often the fusion (or combination) of different pieces of information pertaining to the same topic. There are many approaches to knowledge fusion. Some of them resolve inconsistencies using so-called "syntactical" methods [Neb 91], based on the selection of maximal consistent subsets of formulas and making use of levels of reliability attached to the sources to guide this selection. Others use more numerical frameworks such as probability theory, possibility theory or belief functions. In all cases, it is essential that there exist a gradation of certainty/reliability levels – since this feature is present in all approaches to knowledge fusion – and this might be the reason why paraconsistent logics have not been widely applied to knowledge fusion. Yet, the interest of paraconsistency in knowledge fusion is obvious: by locating precisely the inconsistencies, the user is able to know what (s)he can deduce safely and when (s)he has to be cautious. We illustrate this informal discussion by a brief example which is a continuation of Example 1.

**Example**: let us return to Example 1 (Section 2). Taking some $\alpha, \beta$ such that $0 < \alpha < \beta < 1$:

- W1 (first witness):
  $N(\texttt{female}) = 1$; $N(\texttt{brown}) = 1$; $N(\texttt{BMW}) = \alpha$; $N(\texttt{Chanel}) = \alpha$ ; $N(\texttt{glasses}) = \alpha$.

- W2 (second witness):
  $N(\texttt{female}) = 1$; $N(\neg\texttt{brown}) = 1$; $N(\neg\texttt{BMW}) = \alpha$; $N(\neg\texttt{glasses}) = \beta$.

The fusion $KB$ of these two knowledge bases gives the following minimum specificity closure:

- $N_{KB}(\texttt{female}) = 1$; $N_{KB}(\neg\texttt{female}) = 0$; $N_{KB}(\neg\texttt{female}^\circ) = 0$;
- $N_{KB}(\texttt{brown}) = 1$; $N_{KB}(\neg\texttt{brown}) = 1$; $N_{KB}(\neg\texttt{brown}^\circ) = 1$;
- $N_{KB}(\texttt{BMW}) = \alpha$; $N_{KB}(\neg\texttt{BMW}) = \alpha$; $N_{KB}(\texttt{BMW}^\circ) = \alpha$;
- $N_{KB}(\texttt{Chanel}) = \alpha$; $N_{KB}(\neg\texttt{Chanel}) = 0$; $N_{KB}(\neg\texttt{Chanel}^\circ) = 0$;
- $N_{KB}(\texttt{glasses}) = \alpha$; $N_{KB}(\neg\texttt{glasses}) = \beta$; $N_{KB}(\neg\texttt{glasses}^\circ) = \alpha$.

Therefore, we have $KB \mid\!\sim \texttt{female}$, $KB \mid\!\sim \texttt{Chanel}$, $KB \mid\!\sim \neg\texttt{glasses}$; however, $KB \mid\!\not\sim \texttt{BMW}$, $KB \mid\!\not\sim \neg\texttt{BMW}$, $KB \mid\!\not\sim \texttt{brown}$, $KB \mid\!\not\sim \neg\texttt{brown}$.

Suppose now that, according to the police officer, *all female suspects wear glasses* (and he has no doubt about it). Thus the new information $N(\texttt{female} \rightarrow \neg\texttt{glasses}) = 1$ is added to $KB$. From the contradictory items $N(\texttt{female} \rightarrow \neg\texttt{glasses}) = 1$, $N(\texttt{female}) = 1$ and $N(\neg\texttt{glasses}) = \beta$ we get $N_{KB'}(\texttt{glasses}) = 1$, $N_{KB'}(\neg\texttt{glasses}) = \beta$, and now we have $KB' \mid\!\sim \texttt{glasses}$.

While one could argue that the first part of the example could be handled by a simpler ad-hoc system (because necessity values were bearing only on literals), the last part shows that we need to combine necessity functions with a full logical system.

Note that in some contexts, the user may have the possibility to perform knowledge-gathering actions (or tests) to verify some of the dubious pieces of information – the result of a test is the truth value of some propositional variable. It is clear that in such a context, paraconsistent necessity functions are helpful for choosing the next test to perform, i.e., the next variable to test; namely, it appears reasonable to choose the variable with the highest level of inconsistency among the variables of interest.

### 4.3 Handling default rules

As already mentioned in Section 2, System Z and similar approaches to the handling of default rules suffer from the problem of "blocking of property inheritance". Paraconsistency brings a local view, while necessity functions keep the graded view, enabling paraconsistent necessity functions to solve some of the problems of System Z (but not all) as is illustrated by the following example.

**Example 2 (continued):** Consider the fact `penguin` and the rules

$$\begin{aligned}\Delta = \{&\texttt{penguin} \to \texttt{bird}, \texttt{penguin} \to \neg\texttt{fly}, \texttt{bird} \to \texttt{fly}, \texttt{bird} \to \texttt{wings},\\ &\texttt{fly} \to \neg\texttt{live-in-Antarctica}\}.\end{aligned}$$

Applying the Z ranking procedure to $\Delta$ (written with the possibilistic ranking convention) gives the ranking: (for any $\alpha, \beta$ such that $0 < \beta < \alpha < 1$)

$$\Delta_\alpha = \{\texttt{penguin} \to \texttt{bird}, \texttt{penguin} \to \neg\texttt{fly}, \texttt{fly} \to \neg\texttt{live-in-Antarctica}\};$$

$$\Delta_\beta = \{\texttt{bird} \to \texttt{fly}, \texttt{bird} \to \texttt{wings}\}.$$

Then, taking the $C_1^{++}$-minimum specificity closure of $KB = \{\texttt{penguin}\} \cup \Delta$ leads to

- $N_{KB}(\texttt{penguin}) = 1$;
- $N_{KB}(\texttt{bird}) = \alpha$;
  $N_{KB}(\neg\texttt{fly}) = \alpha$;
  $N_{KB}(\texttt{fly} \to \neg\texttt{live-in-Antarctica}) = \alpha$;
- $N_{KB}(\texttt{fly}) = \beta$;
  $N_{KB}(\neg\texttt{fly}^\circ) = \beta$;
  $N_{KB}(\texttt{wings}) = \beta$;
  $N_{KB}(\neg\texttt{live-in-Antarctica}) = \beta$;
- $N_{KB}(\neg\texttt{bird}) = 0$;
  $N_{KB}(\neg\texttt{bird}^\circ) = 0$;
  $N_{KB}(\neg\texttt{penguin}) = 0$;
  $N_{KB}(\neg\texttt{penguin}^\circ) = 0$;
  $N_{KB}(\neg\texttt{wings}) = 0$;
  $N_{KB}(\neg\texttt{wings}^\circ) = 0$;
  $N_{KB}(\texttt{live-in-Antarctica}) = 0$;
  $N_{KB}(\neg\texttt{live-in-Antarctica}^\circ) = 0$.

Therefore, we have

$KB \mid\!\sim \neg\texttt{fly}$ (which is intended);

$KB \mid\!\sim \texttt{wings}$ (which is intended);
$\mid\!\sim$ avoids the drowning effect, in contrast to the classical minimum specificity closure, System Z, and similar systems.

but also

$KB \mid\!\sim \neg$`live-in-Antarctica`

which is not intended.

Let us comment on this unintended conclusion: in spite of the fact that the contradiction between `fly` and $\neg$`fly` has been cut off in favour of $\neg$`fly`, `fly` is still in the knowledge base with $N_{KB}(\texttt{fly}) = \beta$, and consequently, the rule `fly` $\rightarrow \neg$`live-in-Antarctica` applies, which is not intended.

A solution to this problem consists in blocking the application of such rules $\varphi \rightarrow \psi$ such that $KB \mid\!\sim \neg\varphi$. Here is a revised definition of a nonmonotonic inference relation drawn from a set of default rules:

**Definition 11** *Let $KB = F \cup \Delta$, where $F$ is a set of facts and $\Delta = \{\varphi_i \rightarrow \psi_i, i = 1..n\}$ a set of default rules, where each rule is assigned a necessity degree corresponding to its Z-ranking. We define*
$G^0(\Delta) = F \cup \Delta$, *and* $\forall k \geq 0$,

$$\begin{aligned} G^{k+1}(\Delta) &= F \cup \{\varphi_i \rightarrow \psi_i \in G^k(\Delta) \mid N_{G^k(\Delta)}(\varphi_i) > N_{G^k(\Delta)}(\neg\varphi_i^\circ)\} \\ &= F \cup \{\varphi_i \rightarrow \psi_i \in G^k(\Delta) \mid G^k(\Delta) \mid\!\sim \varphi_i\} \end{aligned}$$

*Lastly, let $G^\infty(\Delta) = \cap_{k\geq 0} G^k(\Delta)$. Then $KB \mid\!\sim^\bullet \psi$ iff $G^\infty(\Delta) \mid\!\sim \psi$.*

This definition obviously leads to a fixpoint in a number of iterations which is not larger than the size of the default base. Indeed:

**Proposition 12** *Let $n = |\Delta|$. Then $G^n(\Delta) = G^\infty(\Delta)$.*

*Proof of Proposition 12*: It comes from its definition that $G^{k+1}(\Delta) \subseteq G^k(\Delta)$. Moreover, it is obvious that $G^{k+1}(\Delta) = G^k(\Delta)$ implies $G^k(\Delta) = G^\infty(\Delta)$. Thus, $\forall k$ we have $G^{k+1}(\Delta) \subset G^k(\Delta) \subset ... \subset G^0(\Delta)$ or $G^k(\Delta) = G^\infty(\Delta)$, which entails that $\forall k$, $|G^{k+1}(\Delta) \cap \Delta| \leq n-k-1$ or $G^k(\Delta) = G^\infty(\Delta)$. Applying this invariant to $k = n$ gives the result. ■

**Example 2** (continued): We apply the usual ranking procedure:

$F =$ {`penguin`};

$\Delta =$ {`bird` $\rightarrow$ `fly`, `bird` $\rightarrow$ `wings`,
`penguin` $\rightarrow$ `bird`, `penguin` $\rightarrow \neg$`fly`,
`fly` $\rightarrow \neg$`live-in-Antarctica`}.

$G^1(\Delta) =$ {`penguin`, `bird` $\rightarrow$ `fly`, `bird` $\rightarrow$ `wings`,
`penguin` $\rightarrow$ `bird`, `penguin` $\rightarrow \neg$`fly`}.

Clearly, $G^\infty(\Delta) = G^1(\Delta)$. Therefore, $\Delta \mid\!\sim^\bullet \neg$`fly`. Also, $\Delta \mid\!\sim^\bullet$ `bird` and $\Delta \mid\!\sim^\bullet$ `wings`. Contrastedly, $\Delta \not\mid\!\sim^\bullet \neg$`live-in-Antarctica`.

### 4.4 Group decision making

Paraconsistent necessity functions may be used in a decision-theoretic context, to help localizing and then solving the inconsistencies between the conflictual goals and preferences of several agents. Let us give an example:

**Example 3** Three papers $p_1, p_2, p_3$ have been submitted to a conference. Three reviewers $r_1$, $r_2$, $r_3$ give their opinions about them – the values 1, 0.7 and 0.4 correspond respectively to the opinions "strong accept", "accept" and "leaning to accept".

$r_1$ $(\neg p_1\ 0.4), (\neg p_3\ 1)$
$r_2$ $(\neg p_2\ 0.7), (p_3\ 1)$
$r_3$ $(\neg p_1\ 0.4), (p_2\ 0.4)$

The minimum specificity principle yields
$N(\neg p_1) = 0.4; N(p_1) = 0; N(\neg p_1^\circ) = 0;$
$N(\neg p_2) = 0.7; N(p_2) = 0.4; N(\neg p_2^\circ) = 0.4;$
$N(\neg p_3) = 1; N(p_3) = 1; N(\neg p_3^\circ) = 1;$

Thus, the degree of inconsistency $N(\neg x^\circ)$ associated with an atomic proposition $x$ or more generally to a formula represents the local degree of conflict inherent to $x$. Then, up to now this is not different from what we would conclude in a multi-source reasoning application; now, in a context of decision the aim is to end up with a true decision, *i.e.*, an assignment of a truth value to every propositional variable. In such a context, graded paraconsistency enables us to
(i) measure and localize conflicts between agents (finding out which are the variables the agents agree on, and those they disagree on);
(ii) take partial decisions (assign the variables which are not contradictory, and leave the other ones temporarily unassigned);
(iii) help solving conflicts by taking appropriate actions: the strongest degrees of conflict may lead to the first "discrimination" procedures, which may be for instance a vote among all agents, or a negociation between the conflicting agents, or an arbitration (such the call for an extra reviewer for $p_3$ in our example)[2]. The process (infer + act) can be iterated until all conflicts have been cut off.

## 5 Conclusion and further issues

The introduction of gradation into a paraconsistent logic enabled us to handle inconsistency not only in a local way but also in a graded way (reflecting the fact

[2]Such a position that inconsistencies should call for actions has been first taken up by Gabbay and Hunter [GH 91]

that some inconsistencies are stronger than others). This work has been motivated by some applications to artificial intelligence such as knowledge fusion, nonmonotonic reasoning or group decision making.

The choice of the logic $C_1^{++}$ was guided by our needs; in particular, it is essential that property $(N3)$ should hold, because it expresses the compatibility of necessity functions with deduction. Letting the paraconsistent logic vary and looking for the conditions under which a certain property holds is not in the spirit of this paper, which focuses on artificial intelligence applications. Extending possibility and necessity functions to arbitrary paraconsistent logics, and more generally to non-classical logics, and giving very general properties of these functions, is the topic of another paper [BL 94b].

The choice of possibility theory as a theory of uncertainty was strongly justified by its purely ordinal properties. Indeed, possibility theory can be *equivalently* defined by means of functions mapping formulas the unit real scale, or by means of preordering relations on formulas (see [Dub 86] and Section 3.5). This is due to the fact that possibility theory only uses the ordinal structure of the unit scale (min, max and the order reversal $x \mapsto 1 - x$). Clearly, this does not hold with probability, which is a purely numerical theory of uncertainty. The need for ordinal theories of uncertainty has been advocated by many authors in the Artificial Intelligence community; not only they appear closer to logical structures than purely numerical theories, but, more importantly, they are well-suited to Artificial Intelligence paradigms; especially, possibility theory lays a central role in different subareas of AI such as belief revision or nonmonotonic reasoning. Graded extensions of paraconsistent logics using other numerical (and more "quantitative") theories of uncertainty appear to be a very interesting topic for further research.

The use of paraconsistent necessity functions for knowledge fusion has been evoked in this paper. Interestingly, comparison with other logical approaches to information merging shows us that our approach could be improved so as to cope with specific problems. Namely:
- "syntax-based" approaches (e.g., [BDLPSS 98]) retain maximal consistent sets of pieces of informations, using reliability degrees; thus, they localize inconsistencies on sources of information and not on propositional variables;
- possibilistic approaches [BDP 97] [BCST 97] to information merging combine logical formulas weighted by classical necessity functions, and thus do not view inconsistency as local; the variety of combination operators used enable some reinforcement effects that are not handled in our approach;
- approaches based on distances [KP 98] retain information from different sources *as equally as possible* (thus they view sources as voters in social choice theory), but again do not localize inconsistencies;
- the approach by Cholvy [Cho 95] for merging database using topic-dependent reliability orderings seems closer to our approach; however, the pieces of information to be merged have to be literals and not arbitrary formulas.

All this suggests to refine the use of paraconsistent necessities in order to handle some problems dealt with in the previous approaches, such as topic-dependent necessity functions, non-idempotent combination modes, and notions of equity between sources.


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## References


[Alv 84] E. H. Alves, "Paraconsistent Logic and Model Theory", *Studia Logica* 43, 17-32, 1984.

[AB 75] A. R. Anderson, N. D. Belnap Jr., *Entailment: The Logic of Relevance and Necessity*, Vol. 1, Princeton University Press, 1975.

[AA 96] O. Arieli., A. Avron, "Automatic Diagnoses for Properly Stratified Knowledge Bases", Proc. of the International Conference on Tools with Artificial Intelligence (ICTAI'96), 392-399, 1996.

[BKMS 92] C. Baral, S. Kraus, J. Minker, V. S. Subrahmanian, "Combining Knowledge Bases Consisting in First-Order Theories", *Computational Intelligence* 8, 45-71, 1992.

[BCDLP 93] S. Benferhat, C. Cayrol, D. Dubois, J. Lang, H. Prade, "Inconsistency Management and Prioritized Syntax-Based Entailment", Proc. of the 13th Int. Joint Conf. on Artificial Intelligence (IJCAI'93), Chambéry (France), 640-645, 1993.

[BDP 92] S. Benferhat, D. Dubois, H. Prade, "Representing Default Rules in Possibilistic Logic", Proc. of the 3rd Conf. on Principles of Knowledge Representation and Reasoning (KR'92), 673-684, 1992.

[BDP 97] S. Benferhat, D. Dubois, H. Prade, "Syntactic Combination of Uncertain Information: A Possibilistic Approach", Proc. of the Int. Conf. on Qualitative and Quantitative Practical Reasoning, Lectures Notes in Artificial Intelligence 1244, Springer-Verlag, 30-42, 1997.

[BDLPSS 98] S. Benferhat, D. Dubois, J. Lang, H. Prade, A. Saffiotti, P. Smets, "A General Approach for Inconsistency Handling and Merging Information in Prioritized Knowledge Bases", Proc. of the 6th Conf. on Principles of Knowledge Representation and Reasoning (KR'98), Morgan Kaufmann, 466-477, 1998.

[BL 94a] Ph. Besnard, E. Laenens, "A Knowledge Representation Perspective: Logics for Paraconsistent Reasoning", *International Journal of Intelligent Systems* 9, 153-168, 1994.

[BL 94b] Ph. Besnard, J. Lang, "Possibility and Necessity Functions over Non-Classical Logics", Proc. of the 10th Int. Conf. on Uncertainty in Artificial Intelligence (UAI'94), Morgan Kaufmann, 69-76, 1994.

[BCST 97] P. Bison, G. Chemello, C. Sossai, G. Trainito, "A Syntactical Approach to Data Fusion", Proc. of the Int. Conf. on Qualitative and Quantitative Practical Reasoning, Lectures Notes in Artificial Intelligence 1244, Springer-Verlag, 58-70, 1997.

[Bou 92] C. Boutilier, "Modal Logics for Qualitative Possibility and Beliefs", Proc. of the 8th Int. Conf. on Uncertainty in Artificial Intelligence (UAI'92), 17-23, 1992.

[Bre 89] G. Brewka, "Preferred Subtheories: An Extended Logical Framework for Default Reasoning", Proc. of the 11th Int. Joint Conf. on Artificial Intelligence (IJCAI'89), Detroit, 1043-1048, 1989.

[CL 92] W. A. Carnielli, M. Lima-Marques, "Reasoning under Inconsistent Knowledge", *Applied Non-Classical Logics* 2, 49-79, 1992.

[Cho 94] L. Cholvy, "Fusion de sources d'information ordonnées en fonction des thèmes", Proc. of RFIA'94 (in French), Paris, 487-494, 1994.

[Cho 95] L. Cholvy, "Automated Reasoning with Merged Information whose Reliability Depends on Topics", Proc. of the 3rd Conf. on Symbolic and Quantitative Approaches to Reasoning under Uncertainty, Lectures Notes in Artificial Intelligence 946, Springer-Verlag, 125-132, 1995.

[daC 74] N. C. A. da Costa, "On the Theory of Inconsistent Formal Systems", *Notre Dame Journal of Formal Logic* 15, 497-510, 1974.

[CA 77] N. C. A. da Costa, E. H. Alves, "A Semantical Analysis of the Calculi $C_n$", *Notre Dame Journal of Formal Logic* 18, 621-630, 1977.

[Dub 86] D. Dubois, "Belief Structures, Possibility Theory, Decomposable Confidence Measures on Finite Sets", *Computers and Artificial Intelligence* 5, 403-417, 1986.

[DP 86] D. Dubois, H. Prade, "The Principle of Minimum Specificity as a Basis for Evidential Reasoning", in *Uncertainty in Knowledge Bases* (B. Bouchon-Meunier, R. Yager, eds.), 75-84, 1986.

[DP 88] D. Dubois, H. Prade, "An Introduction to Possibilistic and Fuzzy Logics (with discussions). In *Non-Standard Logics for Automated Reasoning* (P. Smets et al., eds.), 287-315. Academic Press, 1988.

[DP 91a] D. Dubois, H. Prade, "Epistemic Entrenchment and Possibilistic Logic", *Artificial Intelligence* 50, 223-239, 1991.

[DP 91b] D. Dubois, H. Prade, "Possibilistic Logic, Preferential Models, Non-monotonicity and Related Issues", Proc. of the 12th Int. Joint Conf. on Artificial Intelligence (IJCAI'91), Sydney, 419-424, 1991.

[DLP 94] D. Dubois, J. Lang, H. Prade, "Possibilistic Logic", in: *Handbook of Logic in Artificial Intelligence and Logic Programming* (D. M. Gabbay, C. Hogger, J. A. Robinson, eds.), Oxford University Press, 439-513.

[FHL 94] L. Fariñas del Cerro, A. Herzig, J. Lang, "From Ordering-Based Non-monotonic Reasoning to Conditional Logics", *Artificial Intelligence* 66, 375-393, 1994.

[GH 91] D. Gabbay, A. Hunter, "Making Inconsistency Respectable: A Logical Framework for Inconsistency in Reasoning, Part I", *Fundamental of Artificial In-*

*telligence Research* (Ph. Jorrand, J. Kelemen, eds.), Lectures Notes in Artificial Intelligence 535, Springer-Verlag, 19-32, 1991.

[GM 88] P. Gärdenfors, D. Makinson, "Revision of Knowledge Systems using Epistemic Entrenchment", Proc. of the 2nd Conf. on Theoretical Aspects of Reasoning about Knowledge (TARK'88), 83-95, 1988.

[KLM 90] S. Kraus, D. Lehmann, M. Magidor, "Nonmonotonic Reasoning, Preferential Models and Cumulative Logics", *Artificial Intelligence* 44, 167-207, 1990.

[KP 98] S. Konieczny, R. Pino Perez, "On the Logic of Merging", Proc. of the 6th Conf. on Principles of Knowledge Representation and Reasoning (KR'98), Morgan Kaufmann, 488-498, 1998.

[Leh 89] D. Lehmann, "What does a Conditional Knowledge Base Entail?", Proc. of the 1st Conf. on Principles of Knowledge Representation and Reasoning (KR'89), Toronto, 357-367, 1989.

[Neb 91] B. Nebel, "Belief Revision and Default Reasoning: Syntax-Based Approaches", Proc. of the 2nd Conf. on Principles of Knowledge Representation and Reasoning (KR'91), Cambridge (MA), 417-428, 1991.

[Pea 90] J. Pearl, "System Z: A Natural Ordering of Defaults with Tractable Applications to Nonmonotonic Reasoning", Proc. of the 3rd Conf. on Theoretical Aspects of Reasoning about Knowledge (TARK'90), 121-135, 1990.

[Zad 78] L. A. Zadeh, "Fuzzy Sets as a Basis for a Theory of Possibility", *Fuzzy Sets and Systems* 1, 3-28, 1978.