Sequential voting rules and multiple elections paradoxes

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Abstract

Multiple election paradoxes arise when voting separately on each issue from a set of related issues results in an obviously undesirable outcome. Several authors have argued that a sufficient condition for avoiding multiple election paradoxes is the assumption that voters have separable preferences. We show that this extremely demanding restriction can be relaxed into the much more reasonable one: there exists a linear order $\mathbf{x}_1 > \ldots > \mathbf{x}_p$ on the set of issues such that for each voter, every issue $\mathbf{x}_i$ is preferentially independent of $\mathbf{x}_{i+1}, \ldots, \mathbf{x}_p$ given $\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$. This leads us to define a family of sequential voting rules, defined as the sequential composition of local voting rules. These rules relate to the setting of conditional preference networks (CP-nets) recently developed in the Artificial Intelligence literature. We study in detail how these sequential rules inherit, or do not inherit, the properties of their local components. We focus on the case of multiple referenda, corresponding to multiple elections with binary issues.

1 Introduction

In many contexts, a group of voters has to make a common decision on several possibly related issues, such as in multiple referenda, or voting for committees (the issues then are the positions to be filled). As soon as voters have preferential dependencies between issues, it is generally a bad idea to decompose a vote problem on p issues into a set of p smaller problems, each one bearing on a single issue: "multiple election paradoxes" (or "paradoxes of multiple referenda") then arise.

Such paradoxes have been studied in several papers, with two slightly different views. In [4, 12], voters can vote only Y or N on each issue; the paradox occurs when the set of propositions that win, when votes are aggregated separately for each proposition received the fewest votes when votes are aggregated by combination: for instance, suppose there are 3 propositions A, B, C and three voters voting respectively for $AB\bar{C}$, $A\bar{B}C$ and $\bar{A}BC$. Propositionwise aggregation leads to ABC, which ABC receives support for not a single voter. The paradox studied in [8] is a little bit different. They show that voting issue by issue is feasible when preferences are separable, and that it generally fails when they are not (a voter's preferences are separable if her preferences on an issue does not depend on the choice to be made for other issues). However, as argued by [3, 8], separability is an extremely strong assumption that is unlikely to be met in practice.

Example 1 *A common decision has to be made about whether or not to build a new swimming pool (S or $\bar{S}$) and a new tennis court (T or $\bar{T}$). Assume that the preferences of voters 1 and 2 are $S\bar{T} \succ \bar{S}T \succ \bar{S}\bar{T} \succ ST$, those of voters 3 and 4 are $\bar{S}T \succ S\bar{T} \succ \bar{S}\bar{T} \succ ST$ and those of voter 5 are $ST \succ S\bar{T} \succ \bar{S}T \succ \bar{S}\bar{T}$.*

The first problem with Example 1 is that voters 1 to 4 feel ill at ease when asked to report their projected preference on $\{S, \bar{S}\}$ and $\{T, \bar{T}\}$. The analysis of the paradox in [8] considers that voters report their preferences optimistically (thus voters 1-2 report a preference for S over $\bar{S}$), but this assumption, even if it has been justified by experimental studies (see [11]), remains arbitrary, and would not necessarily carry on to more complex situations such as a voter with the following preference relation: $ABC \succ \bar{A}\bar{B}\bar{C} \succ \bar{A}\bar{B}C \succ \bar{A}B\bar{C} \succ \bar{A}BC \succ A\bar{B}\bar{C} \succ A\bar{B}C \succ AB\bar{C}$: only a very optimistic voter would report a preference for A (except, of course, if some prior beliefs about the others' preferences make him believe that the common decision about B and C will be BC.)

The second problem (the paradox itself) is that under this assumption that voters report optimistic preferences, the outcome in Example 1 will be ST, which is *the worst outcome for all but one voter*, and a fortiori, is a Condorcet loser. Lacy and Niou [8] give another example, with three issues, leading to an even worse paradox where the outcome is ranked last by everyone.

The main question is now, how can these paradoxes be avoided? Reformulating the question in a more constructive way, how should a vote on related issues be conducted? We argue that we have to choose one of the following two ways, each of which has some specific pitfalls:

The first way consists in giving up decomposing the global vote into local votes and *voting for combinations of values*. This solution is supported by Brams et al. [3, 4]. There is some ambiguity on how the process should be conducted, thus leading to three possible methods:

1. ask voters to report their entire preference relation on the set of alternatives, and then apply an usual voting rule such as Borda.
2. ask voters to report only a small part of their preference relation and apply a voting rule that needs this information only, such as plurality;
3. limit the number of possible combinations that voters may vote for.

From a theoretical point of view, Solution 1 works: each agent specifies his preference relation *in extenso* and then any fixed voting rule is applied to the obtained profile, with no risk of a paradoxical outcome. However, as noticed in [3], this solution is practically unfeasible if the number of issues is more than a small number (say, 3): the exponential number of alternatives makes it unreasonable to ask voters to rank all alternatives explicitly. In other words, implementing such a voting rule on a multi-issue domain needs an *exponential protocol*. Clearly, exponentially long protocols are not acceptable. Therefore, as soon as the number of issues is not very small, this solution is ruled out by *communication complexity* considerations.

Solution 2 requires little communication, but it is its only merit. Voting rules that are implementable by a cheap protocol make use of a very small part of the voters' preferences: if the protocol is required to have a polynomial communication complexity, then the voting rule it implements use at most a logarithmic part of the profile. Such rules do exist: not only plurality and veto, but more generally all rules that require, for instance, the K top candidates of each voter, where K is a fixed integer. However, when the number of issues grows, these rules could give extremely bad results. For instance, using plurality when the number of issues is significant and the number of voters small could well result in a situation where no outcome gets more than one vote, in which case plurality would give an extremely poor result.

Solution 3, sketched in [3], presents the chairperson with a very problematic choice. This may be feasible when issues can clearly be packaged into groups of issues such that two groups are clearly independent, but this favorable situation is far from being a general rule.

The second way, supported by Lacy and Niou [8] for multiple referenda, consists in sticking to a vote issue by issue, the outcome of the vote on one issue being revealed before the vote on other issues. They show that sequential voting (with whichever agenda) allows for escaping the worst versions of the multiple election paradoxes, namely, it avoids a Condorcet loser to be elected. However, this method still has three major drawbacks. First, the voters may still feel ill at ease when reporting their preference on an issue, when this preference depends on the value of issues not decided yet. Second, the study is based on the assumption that voters will behave optimistically, by reporting the projection of their preferred outcome, which is debatable except in some specific cases. Third, even if a sequential vote avoids the final outcome to be a Condorcet loser, the paradox remains to a large extent, as can be seen on the following example:

Example 2 *We have three issues* A, B, C *and* $2M+1$ *voters.*

M voters: $AB\bar{C} \succ \bar{A}\bar{B}\bar{C} \succ \ldots \succ A\bar{B}C \succ ABC$

M voters: $A\bar{B}C \succ \bar{A}\bar{B}\bar{C} \succ \ldots \succ ABC \succ AB\bar{C}$

1 voter: $\bar{A}BC \succ \bar{A}\bar{B}\bar{C} \succ \bar{A}B\bar{C} \succ \bar{A}\bar{B}C \succ ABC \succ AB\bar{C} \succ A\bar{B}\bar{C} \succ A\bar{B}C$

In Example 2, having voters decide first on A, then to B and then to C, and assuming they behave optimistically, will lead to ABC, which is (a) a "nearly-Condorcet loser" (it is Condorcet-dominated by all candidates except one) and (b) Pareto-dominated by half of the outcomes. (More acute paradoxes can be found, but they need more issues and thus more space.) Actually, the reason why the sequential process avoids a Condorcet loser to be elected is only because the *last* vote is made with a full knowledge of the values of other issues, thus this result loses his significance when the number of issues becomes bigger.

There is a well-known restriction on voter preferences that allows for such paradoxes to be avoided, that is, when all voters have *separable* preferences across the outcomes of the issues. Then, a voter's preferences on the values of an issue is independent from the values of other issues, and the elicitation process can be performed safely issue by issue (and even without needing to resort to sequentiality). Under the separability assumption, voting separately on each issue (either sequentially or simultaneously) enjoys good properties, including the election of a Condorcet winners when there is one. However, the separability restriction is very demanding, and unlikely to be met in practice, especially because separable preferences constitute a very tiny proportion of possible preferences on multiple issues (see [6]).

The question is now, can this extreme separability assumption be relaxed without hampering the nice properties of sequential voting? As it stands, the answer is positive, as the method can be safely applied to far many profiles than separable profiles. Unformally, the condition should be that each time a voter is asked to report his preferences on a single issues or a small set of isssues, these preferences do not depend on the values of the issues *that have not been decided yet*.

Formally, this can be expressed as the following condition: there is a linear order $\mathcal{O} = \mathbf{x}_1 > \ldots > \mathbf{x}_p$ on the set of issues such that for every voter v and every j, the preferences of v on $\mathbf{x}_j$ are preferentially independent from $\mathbf{x}_{j+1}, \ldots, \mathbf{x}_p$ given $\mathbf{x}_1, \ldots, \mathbf{x}_{j-1}$. If this property is satisfied, then a simple protocol can be implemented: the voters' preferences about issue $\mathbf{x}_1$ are elicited; then a voting rule is applied so as to make a decision on the value of $\mathbf{x}_1$; then this chosen value of $\mathbf{x}_1$ is communicated to the voters, who then report their preferences on the values of $\mathbf{x}_2$ given the fixed value of $\mathbf{x}_1$, and so on. Such preference profiles are called $\mathcal{O}$*-legal* and abbreviated as *legal* for $\mathcal{O} = \mathbf{x}_1 > \ldots > \mathbf{x}_p$ in this paper. This protocol generalizes to clusters of issues $I_1, \ldots, I_n$ where for each voter and each i, I_i is preferentially independent of $I_{j+1}, \ldots, I_m$ given $I_1, \ldots, I_{i-1}$, where $\{I_1, \ldots, I_m\}$ forms a partition of the set $\mathcal{I}$ of issues.

This domain restriction ($\mathcal{O}$*-legality*) and the resulting sequential voting rules and correspondences that are then applicable are defined in Section 3. In Section 4 we study in detail the properties of these sequential composition by relating them to the corresponding properties of local voting rules to those of its components. It turns out that while many properties expectedly transfer from local rules to their sequential composition, this is not the case for two important properties, namely neutrality and consensus. In Section 5 we focus on the particular case of *multiple referenda*, obtained where all issues are binary. In Section 6 we briefly mention further issues.

2 Preferences on multi-issue domains

Let $\mathcal{I} = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a set of *issues*. For each $\mathbf{x}_i \in \mathcal{I}$, D_i is the finite *value domain* of $\mathbf{x}_i$. An issue $\mathbf{x}_i$ is *binary* if $D_i = \{x_i, \overline{x_i}\}$, or equivalently $\{1_i, 0_i\}$. (Note the difference between the issue $\mathbf{x}_i$ and the value x_i.) If $X = \{\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_m}\} \subseteq \mathcal{I}$, with $i_1 < \ldots < i_p$, then D_X denotes $D_{i_1} \times \ldots \times D_{i_m}$. $\mathcal{X} = D_1 \times \ldots \times D_p$ is the set of all *alternatives* (or *candidates*). Elements of $\mathcal{X}$ are denoted by $\vec{x}, \vec{x}'$ etc. and represented by concatenating the values of the issues: for instance, if $\mathcal{I} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, $x_1\overline{x_2}x_3$ assigns x_1 to $\mathbf{x}_1$, $\overline{x_2}$ to $\mathbf{x}_2$ and x_3 to $\mathbf{x}_3$. We allow concatenations of vectors of values: for instance, let $\mathcal{I} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$, $Y = \{\mathbf{x}_1, \mathbf{x}_2\}$, $Z = \{\mathbf{x}_3, \mathbf{x}_4\}$, $\vec{y} = x_1\overline{x_2}$, $\vec{z} = \overline{x_3}x_4$, then $\vec{y}.\vec{z}.\overline{x_5}$ denotes the alternative $x_1\overline{x_2}\overline{x_3}x_4\overline{x_5}$.

A *preference relation* on $\mathcal{X}$ is a strict order (an irreflexive, asymmetric and transitive binary relation). A *linear* preference relation V is a *complete* strict order, i.e., for any $\vec{x}$ and $\vec{y} \neq \vec{x}$, either $\vec{x} \succ \vec{y}$ or $\vec{y} \succ \vec{x}$ holds. We generally note $\vec{x} \succ_V \vec{x}'$ instead of $V(\vec{x}, \vec{x}')$.

Let $\{X, Y, Z\}$ be a partition of the set $\mathcal{I}$ and $\succ$ a linear preference relation over $\mathcal{X} = D_{\mathcal{I}}$. X is *(conditionally) preferentially independent* of Y given Z (w.r.t. $\succ$) if and only if for all $\vec{x}_1, \vec{x}_2 \in D_X$, $\vec{y}_1, \vec{y}_2 \in D_Y$, $\vec{z} \in D_Z$,

$$\vec{x}_1.\vec{y}_1.\vec{z} \succ \vec{x}_2.\vec{y}_1.\vec{z} \text{ iff } \vec{x}_1.\vec{y}_2.\vec{z} \succ \vec{x}_2.\vec{y}_2.\vec{z}$$

Conditional preferential independence originates in the literature of multiattribute decision theory [7]. Unlike probabilistic independence, it is a directed notion: X may be independent of Y given Z without Y being independent of X given Z. Note that preferential independence is weaker than utility independence.

Conditional preference networks, or *CP-nets*, are a language for specifying preferences based on the notion of conditional preferential independence. They allow for eliciting preferences, and for storing them, as economically as possible. Formally, a *CP-net* $\mathcal{N}$ [1] over a set of attributes (or issues) $\mathcal{I}$ is a pair consisting of a directed graph G over $\mathcal{I}$ and a collection of conditional preference tables $CPT(\mathbf{x}_i)$ for each $x_i \in \mathcal{I}$. Appendix 1 gives some fairly detailed background on CP-nets.

Let $\mathcal{O} = \mathbf{x}_1 > \ldots > \mathbf{x}_p$ be a linear order on $\mathcal{I}$. We say that $\succ$ follows $\mathcal{O} = \mathbf{x}_1 > \ldots > \mathbf{x}_p$ if for all $i < p$, $\mathbf{x}_i$ is preferentially independent of $\{\mathbf{x}_{i+1}, \ldots, \mathbf{x}_p\}$ given $\{\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}\}$ with respect to $\succ$.

If $\succ$ follows $\mathcal{O}$ then the *projection* of $\succ$ on $\mathbf{x}_i$ given $(x_1, \ldots, x_{i-1}) \in D_1 \times \ldots \times D_{i-1}$, denoted by $\succ^{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}}$, is the linear preference relation on D_i defined by: for all $x_i, x_i' \in D_i$, $x_i \succ^{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}} x_i'$ iff $x_1 \ldots x_{i-1} x_i x_{i+1} \ldots x_p \succ x_1 \ldots x_{i-1} x_i' x_{i+1} \ldots x_p$ holds for all $(x_{i+1}, \ldots, x_p) \in D_{i+1} \times \ldots \times D_p$.

Due to the fact that $\succ$ follows $\mathcal{O}$ and that $\succ$ is a linear order, $\succ^{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}}$ is a well-defined linear order as well. Note also that if $\succ$ follows both $\mathcal{O} = \mathbf{x}_1 > \ldots > \mathbf{x}_p$ and $\mathcal{O}' = \mathbf{x}_{\sigma(1)} > \ldots > \mathbf{x}_{\sigma(k-1)} > \mathbf{x}_i (= \mathbf{x}_{\sigma(k)}) > \ldots > \mathbf{x}_{\sigma(p)}$, then $\succ^{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}}$ and $\succ^{\mathbf{x}_i|\mathbf{x}_{\sigma(1)}=x_{\sigma(1)},\ldots,\mathbf{x}_{\sigma(k-1)}=x_{\sigma(k-1)}}$ coincide. In other words, the local preference relation on $\mathbf{x}_i$ depends only on the values of the issues that precede $\mathbf{x}_i$ in $\mathcal{O}$ *and* in $\mathcal{O}'$.

Let G be an *acyclic* directed graph (DAG) on $\mathcal{I}$. A preference relation $\succ$ is *compatible with* G, denoted by $\succ \sim G$, if $\succ$ follows some order $\mathcal{O} = \mathbf{x}_1 > \ldots > \mathbf{x}_p$ on $\mathcal{I}$ that follows G, that is, such that for every edge $(\mathbf{x}_i, \mathbf{x}_j)$ in G we have $i < j$. For any two preference relations $\succ_1, \succ_2$ and CP-net $\mathcal{N}$, we use the following notations: $\succ_1 \sim \mathcal{N}$ if $\succ_1$ extends $\mathcal{N}$; $\succ_1 \sim \succ_2$ if there exists a CP-net

$\mathcal{N}'$ s.t. $\succ_1 \sim \mathcal{N}'$ and $\succ_2 \sim \mathcal{N}'$; $\succ_1 \sim_{\mathcal{N}} \succ_2$ if $\succ_1 \sim \mathcal{N}$ and $\succ_2 \sim \mathcal{N}$. Lastly, we say $\succ_1$ and $\succ_2$ are G-equivalent, denoted by $\succ_1 \sim_G \succ_2$, if and only if $\succ_1$ and $\succ_2$ are both compatible with G and for any $\mathbf{x} \in V$, for any $\vec{y}, \vec{y}' \in Dom(par(\mathbf{x}))$ we have $\succ_1^{\mathbf{x}|par(\mathbf{x})=\vec{y}} = \succ_2^{\mathbf{x}|par(\mathbf{x})=\vec{y}}$. Note that $\succ_1 \sim_G \succ_2$ if and only if there exists a CP-net $\mathcal{N}$ whose associated graph is G and such that $\succ_1$ and $\succ_2$ both extend $\mathcal{N}$. We frequently use the notation V (for " vote") instead of $\succ$.

Example 3 *Let $\mathcal{I} = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$, all three being binary. and let V and V' be the following votes:*

$$V : xyz \succ xy\bar{z} \succ x\overline{yz} \succ x\bar{y}z \succ \bar{x}y\bar{z} \succ \overline{xyz} \succ \bar{x}yz \succ \overline{xy}z$$
$$V' : xyz \succ xy\bar{z} \succ \bar{x}y\bar{z} \succ x\overline{yz} \succ \bar{x}yz \succ \overline{xyz} \succ x\bar{y}z \succ \overline{xy}z$$

Let G be the graph over $\mathcal{I}$ whose set of edges is $\{(\mathbf{x}, \mathbf{z}), (\mathbf{y}, \mathbf{z})\}$. V and V' are both compatible with G. Moreover, $V \sim_G V'$, since all local preference relations coincide: $x \succ_V^{\mathbf{x}} \bar{x}$ and $x \succ_{V'}^{\mathbf{x}} \bar{x}$; $z \succ_V^{\mathbf{z}|\mathbf{x}=x,\mathbf{y}=y} \bar{z}$ and $z \succ_{V'}^{\mathbf{z}|\mathbf{x}=x,\mathbf{y}=y} \bar{z}$; etc. The CP-net that V and V' both extend is defined by the following preferences tables: $x \succ \bar{x}$; $y \succ \bar{y}$; $xy : z \succ \bar{z}$; $x\bar{y} : \bar{z} \succ z$; $\bar{x}y : \bar{z} \succ z$; $\overline{xy} : \bar{z} \succ z$.

3 Sequential voting rules and correspondences

We start by recalling briefly some necessary background on voting rules and correspondences (for more details see for instance [2]). Let $\mathcal{A} = \{1, ..., N\}$ be a finite set of *voters* and $\mathcal{X}$ a finite set of *candidates*. A *profile* w.r.t. $\mathcal{A}$ and $\mathcal{X}$ is a collection of N individual linear preference relations over $\mathcal{X}$: $P = (V_1, ..., V_N)$. Let $P_{\mathcal{A},\mathcal{X}}$ be the set of all preference profiles for $\mathcal{A}$ and $\mathcal{X}$. A *voting correspondence* $C : P_{\mathcal{A},\mathcal{X}} \to 2^{\mathcal{X}} \setminus \{\emptyset\}$ maps each preference profile P of $P_{\mathcal{A},\mathcal{X}}$ into a nonempty subset $C(P)$ of $\mathcal{X}$. A *voting rule* $r : P_{\mathcal{A},\mathcal{X}} \to \mathcal{X}$ maps each preference profile P of $P_{\mathcal{A},\mathcal{X}}$ into a single candidate $r(P)$. The correspondence that elects the candidates that are ranked first by the largest number of voters is the *plurality* correspondence. When there are only two candidates $\{x, y\}$, the *majority* correspondence maj is defined by $maj(P) = \{x\}$ (resp. $\{y\}$ if more voters in P prefer x to y (resp. y to x), and $maj(P) = \{x, y\}$ in case of tie.

Given a profile P, $x \in \mathcal{X}$ is a *Condorcet winner* (resp. *weak Condorcet winner*) if it is preferred to any other candidate by a strict (resp. non-strict) majority of voters: for all $y \neq x$, $\#\{i : x \succ_i y\} > \frac{N}{2}$ (resp. $\geq \frac{N}{2}$). A *Condorcet-consistent* rule (resp. correspondence) is a voting rule r (resp. correspondence C) such that whenever there exists a Condorcet winner x for the profile P then $r(P) = x$ (resp. $C(P) = \{x\}$).

These definitions of voting rules are not concerned with how the votes are elicited from the voters. As in [5] we distinguish between the voting rule and a *protocol* (which determines which relevant information is elicited, and when, from the voters) that implements it. The deterministic communication complexity of a voting rule r is the worst-case number of bits sent in the best protocol implementing r. See [5] for a communication complexity study of various voting rules.

From now on, we assume that the set of candidates is a multi-issue domain $\mathcal{X} = D_1 \times ... \times D_p$. *Sequential voting* consists in applying "local" voting rules or correspondences on single issues, one after the other, in such an order that the local vote on a given issue can be performed only when the local votes on all its parents in the graph G have been performed. Note that, unlike in [3, 4, 8], we do not assume that issues are binary. We now define our crucial domain restriction:

Definition 1 *Given a linear order $\mathcal{O} = \mathbf{x}_1 > ... > \mathbf{x}_p$ on $\mathcal{I}$, we define $Legal(\mathcal{O})$ as the set of all profiles $P = (V_1, \ldots, V_N)$ such that each V_i follows $\mathcal{O}$.*

We might wonder how strong this restriction is. First of all, note that it is much less demanding than separability. Second, it can be generalized by partitioning the set of issues into subsets $I_1, \ldots, I_q$ such that I_i is preferentially independent of $I_{i+1} \cup \ldots \cup I_q$ given $I_1 \cup \ldots \cup I_{i-1}$. Obviously, all profiles are of this form, the worst case being $q = 1$[1] However, we can assume without loss of

[1]The smaller the size of the subsets, the cheaper the protocol: the communication cost of the protocol for computing a sequential rule using such decomposition into clusters is $\sum_{i=1}^{q} \prod_{\mathbf{x}_j \in I_i} |D_j|$. The protocol is guaranteed to remain cheap (that is, polynomial) if there exists a constant K (independent from the number of issues and voters) such that $|I_i| \leq K$ for every cluster I_i.

generality (and we will do so in the remainder of the paper) that each cluster consist of a single issue (if this were not the case from the beginning, then each cluster I_i can be considered as a new single issue, with domain $D_{I_i} = \prod_{\mathbf{x}_j \in I_i} D_j$.)

Definition 2 *Let $\mathcal{O} = \mathbf{x}_1 > ... > \mathbf{x}_p$ be a linear order on $\mathcal{I}$, and $(r_1, \ldots, r_p)$ a collection of deterministic voting rules (one for each issue $\mathbf{x}_i$). The* sequential voting rule *$Seq(r_1, \ldots, r_p)$ is defined on all profiles following $\mathcal{O}$ as follows: for any $P = (V_1, ..., V_N)$ in $Legal(\mathcal{O})$:*

- $x_1^* = r_1(V_1^{\mathbf{x}_1}, \ldots, V_N^{\mathbf{x}_1})$;
- $x_2^* = r_2(V_1^{\mathbf{x}_2 | \mathbf{x}_1 = x_1^*}, \ldots, V_N^{\mathbf{x}_2 | \mathbf{x}_1 = x_1^*})$;
- ...
- $x_p^* = r_p(V_1^{\mathbf{x}_p | \mathbf{x}_1 = x_1^*, \ldots, \mathbf{x}_{p-1} = x_{p-1}^*}, \ldots, V_N^{\mathbf{x}_p | \mathbf{x}_1 = x_1^*, \ldots, \mathbf{x}_{p-1} = x_{p-1}^*})$

Then $Seq(r_1, \ldots, r_p)(P) = (x_1^*, \ldots, x_p^*)$.

Example 4 *Let $N = 12$, $\mathcal{I} = \{\mathbf{x}, \mathbf{y}\}$ with $D_{\mathbf{x}} = \{x, \bar{x}\}$ and $D_{\mathbf{y}} = \{y, \bar{y}\}$, and $P = (V_1, \ldots, V_{12})$ the following 12-voter profile:*

$$V_1, V_2, V_3, V_4 : xy \succ \bar{x}y \succ x\bar{y} \succ \overline{xy} \quad V_5, V_6, V_7 : x\bar{y} \succ xy \succ \bar{x}y \succ \overline{xy}$$
$$V_8, V_9, V_{10} : \bar{x}y \succ \overline{xy} \succ xy \succ x\bar{y} \quad V_{10}, V_{11} : \bar{x}y \succ \overline{xy} \succ x\bar{y} \succ xy$$

All these linear preference relations follow the order $\mathbf{x} > \mathbf{y}$. Hence, $P \in Legal(\mathbf{x} > \mathbf{y})$.

Take $r_{\mathbf{x}}$ and $r_{\mathbf{y}}$ both equal to the majority rule, together with a tie-breaking mechanism which, in case of a tie between x and $\bar{x}$ (resp. between y and $\bar{y}$), elects x (resp. y). The projection of P on $\mathbf{x}$ is composed of 7 votes for x and 5 for $\bar{x}$, that is, $P_i^{\mathbf{x}}$ is equal to $x \succ \bar{x}$ for $1 \leq i \leq 7$ and to $\bar{x} \succ x$ for $8 \leq i \leq 12$. Therefore $x^* = r_{\mathbf{x}}(P_1^{\mathbf{x}}, \ldots, P_{12}^{\mathbf{x}}) = x$: the $\mathbf{x}$-winner is $x^* = x$. Now, the projection of P on $\mathbf{y}$ given $\mathbf{x} = x$ is composed of 7 votes for y and 5 for $\bar{y}$, therefore $y^* = y$, and the sequential winner is now obtained by combining the $\mathbf{x}$–winner and the conditional $\mathbf{y}$-winner given $\mathbf{x} = x^* = x$, namely $Seq(r_{\mathbf{x}}, r_{\mathbf{y}})(P) = xy$.

In addition to sequential rules, we define *sequential correspondences* in a similar way: if for each i, c_i is a correspondence on D_i, then $Seq(c_1, \ldots, c_p)(P)$ is the set of all $(x_1, \ldots, x_p)$ s.t. $x_1 \in c_1(P_1^{\mathbf{x}_1}, \ldots, P_N^{\mathbf{x}_1})$, and for all $i \geq 2$, $x_i \in c_i(P_i^{\mathbf{x}_i | \mathbf{x}_1 = x_1, \ldots, \mathbf{x}_{i-1} = x_{i-1}}, \ldots, P_N^{\mathbf{x}_i | \mathbf{x}_1 = x_1, \ldots, \mathbf{x}_{i-1} = x_{i-1}})$.

It is important to remark that, in order to compute $Seq(r_1, \ldots, r_p)(P)$, *we do not need to know the linear preference relations V_1, ..., V_N entirely*: everything we need is the local preference relations: for instance, if $\mathcal{I} = \{\mathbf{x}, \mathbf{y}\}$ and G contains the only edge $(\mathbf{x}, \mathbf{y})$, then we need first the unconditional linear preference relations on $\mathbf{x}$ and then the linear preference relations on $\mathbf{y}$ conditioned by the value of $\mathbf{x}$. In other words, if we know the conditional preference tables (for all voters) associated with the graph G, then *we have enough information to determine the sequential winner for this profile*, even though some of the preference relations induced from these tables are incomplete. This is expressed more formally by the following fact (see Observation 4 in [9]): let $\mathcal{I} = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$, G an acyclic graph over $\mathcal{I}$, and $P = (V_1, \ldots, V_N)$, $P' = (V_1', \ldots, V_N')$ two complete preference profiles such that for all $i = 1, \ldots, N$ we have $V_i \sim_G V_i'$. Then, for any collection of local voting rules $(r_1, \ldots, r_p)$, we have $Seq(r_1, \ldots, r_p)(P) = Seq(r_1, \ldots, r_p)(P')$. (A similar result holds for correspondences.) This implies that applying sequential voting to two profiles corresponding to the same collection of CP-nets will give the same result.

We may now wonder whether a Condorcet winner (CW), when there exists one, can be computed sequentially. Sequential Condorcet winners (SCW) are defined similarly as for sequential winners for a given rule: the SCW is the sequential combination of "local" Condorcet winners.

Definition 3 *Let $\mathcal{O} = \mathbf{x}_1 > ... > \mathbf{x}_p$ be a linear order on $\mathcal{I}$, and $P \in Legal(\mathcal{O})$. $(x_1^*, \ldots, x_p^*)$ is a* sequential Condorcet winner *for P if and only if*

- $\forall x_1' \in D_1, \#\{i, x_1^* \succ_i^{\mathbf{x}_1} x_1'\} > \frac{N}{2}$;

- *for every* $k > 1$ *and* $\forall x'_k \in D_k$, $\#\{i, x^*_k \succ_i^{\mathbf{x}_k|\mathbf{x}_1=x^*_1,\ldots,\mathbf{x}_{k-1}=x^*_{k-1}} x'_k\} > \frac{N}{2}$.

Clearly, the existence of a SCW is no more guaranteed than that of a CW, and there cannot be more than one SCW. We have the following positive result in [9] (Proposition 3): if $(x^*_1, x^*_2, \ldots, x^*_p)$ is a Condorcet winner for P, then it is a sequential Condorcet winner for P. (Note that the converse fails). An important corollary of this result is the following:

Theorem 3.1 *If every* r_i *is Condorcet-consistent then* $Seq(r_1, \ldots, r_p)$ *is Condorcet-consistent.*

Therefore, the output of a sequential voting rule will be the Condorcet winner when there exists one, provided that each local rule r_i is Condorcet-consistent. This applies in particular to sequential majority on domains composed of binary issues, which was already known in the particular case when all voters have separable preferences (see [8]). This allows us to claim that *the restriction to legal profiles (with respect to some order* $\mathcal{O}$*) allows for escaping multiple election paradoxes*, at least the version of the paradox that deals with Condorcet winners failing to be elected. For the version of the paradox concerned with electing a Condorcet loser, a sequential voting rule will not elect a Condorcet loser, provided that each of its local rules never elects a Concorcet loser:

Theorem 3.2 *If there exists* $i \leq p$ *s.t.* r_i *never elects a Condorcet loser, then* $Seq(r_1, \ldots, r_p)$ *never elects a Condorcet loser.*

For sequential majority on multiple referenda, we have a slightly more significant result:

Theorem 3.3 *Let* $c_1, \ldots, c_p$ *all equal to the majority correspondence on binary domains. For any* $\mathcal{O}$*-legal profile* P *and any* $\vec{d} \in Seq(c_1, \ldots, c_p)(P)$*, there exist* p *outcomes* $\vec{x}_1, \ldots \vec{x}_p \in \mathcal{X}$ *such that* $\vec{d}$ *weakly Condorcet-dominate* $\vec{x}_i$ *for all* $i \leq p$*.*

This bound p is actually tight (see Example 6.9 for $p = 3$; it can be generalized to $p \geq 3$).

4 Properties of sequential voting rules

We start be recalling a few important properties that voting rules may (or may not) satisfy. A voting rule satisfies

- **anonymity** if it is unsensitive to any permutation of the voters;
- **homogeneity** if for any vote V and any $n \in \mathbb{N}$, $r(V) = r(nV)$.
- **neutrality** if for any profile P and any permutation M on candidates, $r(M(P)) = M(r(P))$.
- **monotonicity** if for any profiles $P = (V_1, \ldots, V_N)$ and $P' = (V'_1, \ldots, V'_N)$ s.t. each V'_i is obtained from V_i by raising only $r(P)$, we have $r(P') = r(P)$.
- **consistency** if for any two disjoint profiles (that is, given, by two disjoint electorates) P_1, P_2 s.t. $r(P_1) = r(P_2)$, then $r(P_1 \cup P_2) = r(P_1) = r(P_2)$.
- **participation** if for any profile P and any vote V, $r(P \cup \{V\}) >_V r(P)$.
- **consensus** if for any profile $P = (V_1, \ldots, V_N)$, there is no candidate c s.t. $c >_{V_i} r(P)$ for all $i \leq N$.

Since sequential voting rules are sequential composition of multiple local rules, we may wonder whether the properties of local rules carry on to their sequential composition, and vice versa. In this paper, we focus on the above properties. We only give results on voting rules, but most of the them can be easily extended to correspondences.

4.1 From sequential rules to local rules

Notice that decomposable voting rules are defined over legal profiles, therefore, when we say a decomposable voting rule satisfies a property involving several profiles, it means that it holds for all *legal* profiles. This applies to neutrality and monotonicity: $Seq(r_1,\dots,r_p)$ is neutral if for any permutation M and any legal profile P, *if $M(P)$ is legal*, then $M(Seq(r_1,\dots,r_p)(P)) = Seq(r_1,\dots,r_p)(M(P))$. (And similarly for monotonicity.)

Theorem 4.1 *If $Seq(r_1,\dots,r_p)$ satisfies anonymity (resp. homogeneity, neutrality, consistency, participation, consensus), then for any $1 \leq i \leq p$, r_i also satisfies anonymity (resp. homogeneity, neutrality, consistency, participation, consensus).*

Monotonicity transfers to the last local rule only. This seemingly strange results is mainly caused by our restriction to legal profiles.

Theorem 4.2 *If $Seq(r_1,\dots,r_p)$ satisfies monotonicity, then r_p also satisfies monotonicity.*

Since the way to obtain a new legal profile P' from P by just raising one candidate can only affect the conditional orders on D_p, we consider now a stronger monotonicity by allowing multiple candidates to be raised simultaneously.

Definition 4.3 *A voting rule r is* **strongly monotonic** *if for any profile P, any $Y \subseteq \mathcal{X}$, and any P' obtained from P by only raising the candidates in Y while keeping their relative position unchanged, we have $r(P') \in r(P) \cup Y$.*

Let $Y = \{r(P)\}$, we immediately know if r is strongly monotonic, then it is also monotonic. The next theorem shows that strong monotonicity can be transfers to every local rule.

Theorem 4.4 *If $Seq(r_1,\dots,r_p)$ satisfies strong monotonicity, then for any $1 \leq i \leq p$, r_i also satisfies strong monotonicity.*

4.2 From local rules to sequential rules

Then we give results on whether the sequential composition of local rules inherit a given property satisfied by all local rules. Here are the positive results:

Theorem 4.5 *If for all $1 \leq i \leq p$, r_i satisfies anonymity (resp. homogeneity, consistency, strong monotonicity), then $Seq(r_1,\dots,r_p)$ also satisfies anonymity (resp. homogeneity, consistency, strong monotonicity).*

The next theorem shows that the converse of Theorem 4.2 also holds.

Theorem 4.6 *If r_p satisfies monotonicity, then $Seq(r_1,\dots,r_p)$ also satisfies monotonicity.*

Neutrality, consensus, and participation are not transferred from local rules to their sequential composition. We first give the following result, about neutrality and consensus.

Theorem 4.7 *Let $r_1,\dots,r_p$, $p \geq 2$ be plurality rules and $|D_i| \geq 2$ for all $i \leq p$. If there exists $i \leq p$ s.t. $|D_i| > 2$, then $Seq(r_1,\dots,r_p)$ does not satisfy neutrality, nor consensus.*

The next example shows that participation cannot be lifted from local rules to their sequential composition.

Example 4.8 *Let $\mathcal{N}_1, \mathcal{N}_2$ be two CP-nets on $\{0_1,1_1,2_1\} \times \{0_2,1_2\}$ s.t. in $\mathcal{N}_1$*

$$0_1 \succ_{\mathcal{N}_1} 1_1 \succ_{\mathcal{N}_1} 2_1, 0_1 : 0_2 \succ_{\mathcal{N}_1} 1_2, 1_1 : 1_2 \succ_{\mathcal{N}_1} 0_2, 2_1 : 1_2 \succ_{\mathcal{N}_1} 0_2,$$

in $\mathcal{N}_2$, $\mathbf{x}_1$ and $\mathbf{x}_2$ are independent, and $1_1 \succ_{\mathcal{N}_2} 2_1 \succ_{\mathcal{N}_2} 0_1, 0_2 \succ_{\mathcal{N}_2} 1_2$.

Clearly $\mathcal{N}_2 \not\models 1_1 1_2 \succ 0_1 0_2$, therefore, there exists a vote V_2 consistent with $\mathcal{N}_2$, and $0_1 0_2 \succ_{V_2} 1_1 1_2$ (cf. Lemma 6.2), for example $V_2 : 2_1 0_2 \succ 2_1 1_2 \succ 1_1 0_2 \succ 0_1 0_2 \succ 1_1 1_2 \succ 0_1 1_2$

Let r_1 be a scoring rule with score vector $(3,2,0)$, r_2 be the plurality rule. Obviously both r_1, r_2 satisfy participation. We consider a profile $P = (V_1, V_3)$ s.t. V_1 and V_3 are consistent with $\mathcal{N}_1$. Then $Seq(r_1,r_2)(P) = 0_1 0_2$ and $Seq(r_1,r_2)(P \cup \{V_2\}) = 1_1 1_2$. But $0_1 0_2 \succ_{V_2} 1_1 1_2$. Hence $Seq(r_1,r_2)$ does not satisfy participation.

5 Multiple referenda

In this section, we focus on the case where all issues are binary (i.e., multiple referenda). Clearly, if $Seq(r_1,\ldots,r_p)$ is "reasonable" to some extent to be defined, then each r_i should be the majority rule. We give below a characterization of sequential majority that generalizes May's theorem [10] to multi-issue domains. It is more natural to consider the sequential composition majorities as a correspondence, namely $Seq(c_1,\ldots,c_p)$, where each c_i is the majority correspondence for two candidates. Notice if the number of voters is odd, then sequential majority outputs a single winner, which obviously is not necessarily the case where the number of voters is even: for instance, let us consider 2 voters, with respective preference orders $xy \succ x\overline{y} \succ \overline{x}y \succ \overline{xy}$ and $\overline{x}y \succ \overline{xy} \succ x\overline{y} \succ xy$. The profile is legal for $\mathbf{x} > \mathbf{y}$, and the outcome of sequential majority consists here of the set of three alternatives $\{xy, x\overline{y}, \overline{x}y\}$.

First we make an observation on the neutrality of each c_i. Our aim is to find a necessary and sufficient condition for each c_i to be neutral, based on some observations on $Seq(c_1,\ldots,c_p)$. Recall in Theorem 4.1 it has been proved that if $Seq(c_1,\ldots,c_p)$ is neutral then c_i is neutral. But this is not a necessary condition (and we will prove that if $p \geq 3$ then the sequential majority is not neutral, see Theorem 5.6). Fortunately, for multiple referenda, we can find a suitable condition. Denote M_R the permutation on $\mathcal{X}$ that exchanges $(d_1,\ldots,d_p)$ to $(\overline{d_1},\ldots,\overline{d_p})$, for example $M_R(0_1 1_2 0_3) = 1_1 0_2 1_3$. We say that $Seq(c_1,\ldots,c_p)$ is *insensitive to* M_R if for any legal profile P, $M_R(Seq(c_1,\ldots,c_p)(P)) = Seq(c_1,\ldots,c_p)(M_R(P))$. The next theorem says that a decomposable voting correspondence is insensitive to M_R iff its local correspondences are neutral.

Theorem 5.1 *c_i is neutral for all $i \leq p$ if and only if $Seq(c_1,\ldots,c_p)$ is insensitive to M_R.*

The next theorem characterizes sequential composition of majority correspondences.

Theorem 5.2

1. *On the domain of all profiles that consists of odd number of votes, a decomposable voting correspondence $Seq(c_1,\ldots,c_p)$ is the sequential majority correspondence if and only if $Seq(c_1,\ldots,c_p)$ satisfies anonymity, strong monotonicity, and is insensitive to M_R.*

2. *A decomposable correspondence $C = Seq(c_1,\ldots,c_p)$ is the sequential majority correspondence if and only if it satisfies anonymity, strong monotonicity, consistency, and insensitivity to M_R, and if whenever $|C(P)| \geq 2$ for some profile P, then $|P|$ is even.*

Remark that the sets of properties in 1. and 2. are minimal (for instance, in 1., all three properties are required).

Recall that Theorem 4.7 says that if some $|D_i| > 2$, then the sequential composition of rules that satisfies neutrality (resp. consensus) might not satisfy neutrality (resp. consensus). We may wonder how about if $|D_i| = 2$ for all i. Notice first that plurality and majority coincides on binary domains. We observe that when $p = 2$, sequential majority is neutral.

Theorem 5.3 *Let c_1, c_2 be equal to the majority correspondence on binary domains. Then $Seq(c_1,c_2)$ is a neutral correspondence.*

Then we consider $p \geq 2$ and we give an important result for multiple referenda, namely, an impossibility theorem that can be used to prove that several common voting rules are not decomposable. This theorem says that if a voting rule satisfies decomposability and consensus, then any candidate that is not regarded the first by any voter cannot be the winner of the voting process, even if every voter thinks he is the second best. Here we only state the theorem, an example of its application will be presented in the Appendix 3.

Theorem 5.4 *If a sequential voting rule $Seq(r_1,\ldots,r_p)$ on a domain consisting of binary issues satisfies consensus or neutrality, then for any preference profile $P = \{V_1,\ldots,V_N\}$ following $\mathcal{O}$, $\vec{x} = Seq(r_1,\ldots,r_p)(P)$ must be top ranked in at least one of $\{V_1,\ldots,V_N\}$.*

By this theorem, we can easily prove that many voting rules are not decomposable for domains consisting of binary issues, such as *Bucklin, Maximin, Copeland, Ranked pairs*; see Example 6.8.

With Theorem 5.4 and Example 6.8 we are able to prove that when $p > 2$, the sequential composition of majority (plurality) is not neutral.

Theorem 5.5 *If $p \geq 3$, then the sequential composition of majority on p binary domains does not satisfy neutrality, nor consensus.*

Together with Theorem 4.7 and Theorem 5.3, we know that the only neutral sequential plurality rule is the one on a 2×2 domain.

Theorem 5.6 *A sequential composition of plurality rule is neutral iff $p = 2$ and $|D_1| = |D_1| = 2$.*

For neutrality of non-binary subdomains, we have proved that for $p \geq 3$, if a decomposable voting rule satisfies neutrality or consensus, then it is not Condorcet-consistent. Since this paper mainly discusses multiple referenda, we do not present it here.

We end this section with some considerations on manipulability. We know that the majority rule for 2 candidates is not manipulable. What about sequential majority? We know from [8] that if *all* voters have separable preferences, then sequential majority is non-manipulable. Does this extend to legal profiles in which some voters have non-separable preferences? Unfortunately, it does not:

Theorem 5.7 *Sequential majority is manipulable.*

This is easily seen on this counterexample with two binary issues **x** and **y**: voter 1 has the preference relation $xy \succ \bar{x}y \succ x\bar{y} \succ \bar{x}\bar{y}$, voter 2 has $x\bar{y} \succ xy \succ \bar{x}y \succ \bar{x}\bar{y}$ and voter 3 has $\bar{x}y \succ \bar{x}\bar{y} \succ x\bar{y} > xy$. The profile is in $Legal(x > y)$. If 1 knows the preferences of 2 and 3 then he has no interest to vote sincerely on issue **x**, *even though his preference relation is separable*: if he votes sincerely, then he votes x and then the outcome is $x\bar{y}$. If he votes for $\bar{x}$ instead, then the outcome is $\bar{x}y$, which is better to him.

As a corollary of this result, strategyproofness does not transfer from the local level to the global level.

6 Discussion

We have shown that the sequential composition of local voting rules allows for escaping usual multiple election paradoxes, under a domain restriction much weaker than separability. Moreover, these sequential rules have a cheap communication complexity. We have established many results concerning the transfer (or the failure of transfer) of important properties from local rules to/from their sequential composition.

Interestingly, our work has benefited from several previous streams of work that were almost unrelated: on the one hand, social choice, and on the other hand, conditional preferential independence, initially developed in the literature of multiattribute decision making and now widely used in artificial intelligence (with CP-nets). The initial motivation of our work was also inspired by the notion of cheap protocol, as defined in the literature on communication complexity.

An important aspect of multiple election paradoxes that would deserve more attention is the role of *knowledge*. What makes our protocols interesting is the conjunction of two properties: they are *cheap* (in terms of communication complexity) and *epistemically safe*: our domain restriction ensures that each time an elicitation query is asked to the voters, the voters *know* the answer, that is, they have all the necessary information needed to give the answer. Multiple election paradoxes, where voters experience regret after voting for a given issue when learning the outcome of other issues, is to a large extent due to the fact that voters are asked to cast a vote about a given issue whereas they *don't know* their true preference, the latter depending on the value of some other issues. This, of course, is guaranteed with separability, but this assumption is far too demanding. We believe that our restriction to legal profiles constitutes a reasonable sufficient condition for the existence of a cheap and epistemically safe protocol. However it is not *necessary*, because we may consider sequential rules where the order in which the issues are considered depends on the value of some previously decided issue; these rules would work for a more general class of profiles. Looking for a sufficient and necessary condition is left for further study, as well as a formalization of epistemically safe protocols within epistemic logic.

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Appendix 1: conditional preference networks (CP-nets)

Let $\mathcal{I}$ ba a finite set of variables, and for each $\mathbf{x}_i \in \mathcal{I}$, let D_i be a finite value domain. Let $\mathcal{X} = \prod_{\mathbf{x}_i \in \mathcal{I}} D_i$.

A CP-net over $\mathcal{I}$ is a pair $\mathcal{N} = \langle G, CPT \rangle$, where G is a directed graph over $\mathcal{I}$ and CPT is a set of conditional preference tables $\{CPT(\mathbf{x}_i) : \mathbf{x}_i \in \mathcal{I}\}$. Each conditional preference table $CPT(\mathbf{x}_i)$ associates a total order $\succ^i_{\vec{u}}$ over D_i, with each instantiation $\vec{u}$ of $\mathbf{x}_i$'s parents $Par(\mathbf{x}_i) = U$, where $Par(\mathbf{x}_i)$ denote the parents of $\mathbf{x}_i$ in G.

For instance, let $\mathcal{I} = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$, all three being binary, and assume that preference of a given agent over $2^{\mathcal{I}}$ can be defined by a CP-net whose structural part is the directed acyclic graph $G = \{(x,y),(y,z),(x,z)\}$; this means that the agent's preference over the values of $\mathbf{x}$ is unconditional, preference over the values of $\mathbf{y}$ (resp. $\mathbf{z}$) is fully determined given the value of $\mathbf{x}$ (resp. the values of $\mathbf{x}$ and $\mathbf{y}$).

Example 5 *Let $N = 12$, $\mathcal{I} = \{\mathbf{x}, \mathbf{y}\}$ with $Dom(\mathbf{x}) = \{x, \bar{x}\}$ and $Dom(\mathbf{y}) = \{y, \bar{y}\}$, and $P = (V_1, \ldots, V_{12})$ the following 12-voter profile:*

V_1, V_2, V_3, V_4 : $xy \succ \bar{x}y \succ x\bar{y} \succ \overline{xy}$
V_5, V_6, V_7: $x\bar{y} \succ xy \succ \bar{x}y \succ \overline{xy}$
V_8, V_9, V_{10}: $\bar{x}y \succ \overline{xy} \succ xy \succ x\bar{y}$
V_{11}, V_{12}: $\bar{x}y \succ \overline{xy} \succ x\bar{y} \succ xy$

All these linear preference relations are compatible with the graph G over $\{\mathbf{x}, \mathbf{y}\}$ whose single edge is $(\mathbf{x}, \mathbf{y})$; equivalently, they follow the order $\mathbf{x} > \mathbf{y}$: for all voters, the preference on $\mathbf{x}$ is unconditional and the preference on $\mathbf{y}$ may depend on the value of $\mathbf{x}$.

The corresponding conditional preference tables are:

voters 1,2,3,4	*voters 5,6,7*	*voters 8,9,10*	*voters 11,12*
$x \succ \bar{x}$	$x \succ \bar{x}$	$\bar{x} \succ x$	$\bar{x} \succ x$
$x : y \succ \bar{y}$	$x : \bar{y} \succ y$	$x : y \succ \bar{y}$	$x : \bar{y} \succ y$
$\bar{x} : y \succ \bar{y}$	$\bar{x} : y \succ \bar{y}$	$\bar{x} : y \succ \bar{y}$	$\bar{x} : y \succ \bar{y}$

The conditional preference statements contained in these tables are written with the following usual notation: for instance, in a CP-net $\mathcal{N}$, $x_1\overline{x_2} : x_3 \succ \overline{x_3}$ means that when $\mathbf{x}_1$ is true and $\mathbf{x}_2$ is false then $\mathbf{x}_3 = x_3$ is preferred to $\mathbf{x}_3 = \overline{x_3}$ *ceteris paribus*, that is, for any fixed values of the other variables $\mathbf{x}_4, \ldots, x_p$.

Formally in CP-net $\mathcal{N}$, for any $\mathbf{x}_i \in \mathcal{I}$, the conditional independence in CP-net leads to the following preference relations. Define first

$$\succ^{\mathbf{x}_i} = \{\vec{u}\vec{z}x \succ_{\mathcal{N}} \vec{u}\vec{z}y : x \succ^i_{\vec{u}} y, \vec{z} \in \prod_{\mathbf{x}_j \notin Par(\mathbf{x}_i)} D_j\}.$$

Write $\succ^{\mathcal{N}} = \bigcup_{\mathbf{x}_i} \succ^{\mathbf{x}_i}$ the union of all relations $\succ^{\mathbf{x}_i}$ encoded in $CPT(\mathbf{x}_i)$. Notice we require $\succ_{\mathcal{N}}$ be a linear order, so $\succ_{\mathcal{N}}$ is transitive. Therefore the full preferential information encoded in $\mathcal{N}$ is the *transitive closure* of $\succ^{\mathcal{N}}$, namely $\succ_{\mathcal{N}} = \overline{\succ^{\mathcal{N}}}$. It has been proved [1] that if G is acyclic, then $\succ_{\mathcal{N}}$ is consistent, namely for any $\vec{x}, \vec{y}$, at most one of $\vec{x} \succ_{\mathcal{N}} \vec{y}$ and $\vec{y} \succ_{\mathcal{N}} \vec{x}$ holds.

In the paper we make the classical assumption that G is **acyclic**. A CP-net $\mathcal{N}$ induces a preference ranking on $\mathcal{X}$: $\mathcal{N} \models \vec{x} \succ \vec{y}$ iff $\vec{x} \succ_{\mathcal{N}} \vec{y}$. Notice for any $\vec{x} \succ^{\mathcal{N}} \vec{y}$, $\vec{x}$ and $\vec{y}$ differs only in one issue, and $\vec{z} \succ_{\mathcal{N}} \vec{w}$ is obtained through a transitive sequence of relations $\vec{z} \succ^{\mathcal{N}} \vec{x}_1, \vec{x}_1 \succ^{\mathcal{N}} \vec{x}_2, \ldots, \vec{x}_{m-1} \succ^{\mathcal{N}} \vec{x}_m, \vec{x}_m \succ^{\mathcal{N}} \vec{w}$. So $\mathcal{N} \models \vec{x} \succ \vec{y}$ is thus equivalent to: There is a sequence of improving flips from $\vec{y}$ to $\vec{x}$, where an improving flip is the flip of a single issue $\vec{x}_i$ "respecting" the preference table $CPT(\mathbf{x}_i)$ (see [1]). Note that the preference relation induced from a CP-net is generally not complete, as seen on the following example.

Example 6.1 *Consider the example depicted in Figure 1.*

$\succ^{\mathbf{x}}$: $xyz \succ \bar{x}yz$, $xy\bar{z} \succ \bar{x}y\bar{z}$, $x\bar{y}z \succ \overline{xy}z$, $x\overline{yz} \succ \overline{xyz}$
$\succ^{\mathbf{y}}$: $xyz \succ x\bar{y}z$, $xy\bar{z} \succ x\overline{yz}$, $\overline{xy}z \succ \bar{x}yz$, $\overline{xyz} \succ \bar{x}y\bar{z}$
$\succ^{\mathbf{z}}$: $xyz \succ xy\bar{z}$, $x\bar{y}z \succ x\overline{yz}$, $\bar{x}yz \succ \bar{x}y\bar{z}$, $\overline{xyz} \succ \overline{xy}z$, *illustrated as*

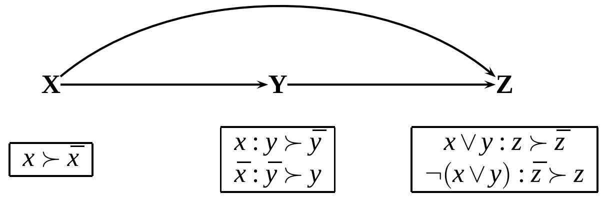
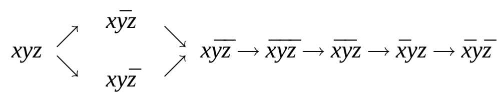


Figure 1: A CP-net.

Now, $\succ_{\mathcal{N}}$ is the transitive closure of $\succ^{\mathbf{x}} \cup \succ^{\mathbf{y}} \cup \succ^{\mathbf{z}}$, illustrated b y the followign diagram:



To see how to generate $xyz \succ \overline{xyz}$, we consider a three-step increasing flip: in the first step, $\mathbf{x}$ is flipped according to $x \succ \bar{x}$, thus $x\overline{yz}$ is obtained; then in the second step, $\mathbf{z}$ is flipped according to $x\bar{y} : z \succ \bar{z}$, thus leads to $x\bar{y}z$; finally $\mathbf{y}$ is flipped according to $x : y \succ \bar{y}$, reaching xyz.

An important property of such sequential voting rules and correspondences is that the outcome does not depend on $\mathcal{O}$, provided that G follows $\mathcal{O}$. This can be expressed formally:

Observation 1 *Let $\mathcal{O} = (\mathbf{x}_1 > \ldots > \mathbf{x}_p)$ and $\mathcal{O}' = (\mathbf{x}_{\sigma(1)} > \ldots > \mathbf{x}_{\sigma(p)})$ be two linear orders on V such that G follows both $\mathcal{O}$ and $\mathcal{O}'$. Then*

$$Seq(r_1, \ldots, r_p)(P) = Seq(r_{\sigma(1)}, \ldots, r_{\sigma(p)})(P)$$

and similarly for voting correspondences.

Example 6 *Everything is as in Example 5, except that we don't know the voters' complete preference relations, but only their corresponding conditional preference tables. These conditional preferences contain strictly less information than P, because some of the preference relations they induce are not complete: for instance, the induced preference relation for the first 4 voters is $xy \succ \bar{x}y \succ \overline{xy}$, $xy \succ x\bar{y} \succ \overline{xy}$, with $x\bar{y}$ and $\bar{x}y$ being incomparable. However, we have enough information to determine the sequential winner for this profile, even though some of the preference relations are incomplete. For instance, taking again the majority rule for $r_{\mathbf{x}}$ and $r_{\mathbf{y}}$, the sequential winner is xy for any complete profile $P' = (V_1', \ldots, V_{12}')$ extending the incomplete preference relations induced by the 12 conditional preference tables above.*

Appendix 2: Proofs

The following Lemma will be used frequently in the following sections. The proof is easy so we omit it.

Lemma 6.2 *Given a CP-net $\mathcal{N}$, if $\mathcal{N} \not\models \beta \succ \alpha$, then there exists a linear preference V extending $\mathcal{N}$ s.t.*

1. $\alpha \succ_V \beta$.
2. *α and β are adjacent, i.e. there is no γ s.t. $\alpha \succ_V \gamma \succ_V \beta$.*

If $\mathcal{N} \not\models \alpha \succ \beta$, then V also satisfies:

3. *Denote V' obtained from V by exchanging α and β, then $V' \sim_{\mathcal{N}} V$.*

Proof of Theorem 3.2: If r_i never elects a Condorcet loser and for a profile P, $Seq(r_1, \ldots, r_p)(P) = (d_1, \ldots, d_p)$ is the Condorcet loser of P, then for any $d_i' \in D_i$, $(d_1, \ldots, d_{i-1}, d_i', d_{i+1}, \ldots, d_p)$ Condorcet-dominates $(d_1, \ldots, d_p)$. Therefore in $P^{\mathbf{x}_i|\mathbf{x}_1=d_1 \ldots \mathbf{x}_{i-1}=d_{i-1}}$, d_i' Condorcet-dominates d_i, which means d_i is the Condorcet loser in $P^{\mathbf{x}_i|\mathbf{x}_1=d_1 \ldots \mathbf{x}_{i-1}=d_{i-1}}$. Since r_i would not select a Condorcet loser, $r_i(P^{\mathbf{x}_i|\mathbf{x}_1=d_1 \ldots \mathbf{x}_{i-1}=d_{i-1}}) \neq d_i$. This contradicts with $Seq(r_1, \ldots, r_p)(P) = (d_1, \ldots, d_p)$. So $Seq(r_1, \ldots, r_p)(P)$ is not the Condorcet loser of P. □

Theorem 3.3 Denote $\vec{d} = (d_1, \ldots, d_p)$, $\vec{x}_i = (d_1, \ldots, d_{i-1}, \overline{d_i}, d_{i+1}, \ldots, d_p)$. We claim $\vec{d}$ weak Condorcet-dominate $\vec{x}_i$. This is because r_i elects d_i, so

$$\#\{V : V \in P, d_i \succ_{V^{\mathbf{x}_i|\mathbf{x}_1=d_1 \ldots \mathbf{x}_{i-1}=d_{i-1}}} \overline{d_i}\} \geq \frac{N}{2}.$$

Therefore $\#\{V : V \in P, \vec{d} \succ_V \vec{x}_i\} \geq \frac{N}{2}$. The theorem is thus proved. □

Proof of Theorem 4.1: Anonymity and homogeneity are obvious.

• Neutrality: If for some $i \leq N$, r_i is not neutral, then there exists a permutation M^i on D_i and a profile P^i on D_i s.t.

$$M^i(r_i(P_i)) \neq r_i(M^i(P^i)),$$

then we construct a profile P on D as follows:

1. Define G to be the graph in which there is no edge.
2. For any $V_j^i \in P^i$, construct a vote $V_j \sim G$, and $V_j^{\mathbf{x}_i} = V_j^i$.

Then define a permutation M on D s.t.

$$M(d_1,\ldots,d_{i-1},d_i,d_{i+1}\ldots,d_p) = (d_1,\ldots,d_{i-1},M^i(d_i),d_{i+1}\ldots,d_p).$$

Suppose $Seq(r_1,\ldots,r_p)(P) = (d_1,\ldots,d_p)$, then

$$Seq(r_1,\ldots,r_p)(M(P)) = (d_1,\ldots,d_{i-1},r_p(M^i(P^i)),d_{i+1},\ldots,d_p).$$

But from neutrality we have

$$\begin{aligned}
&Seq(r_1,\ldots,r_p)(M(P))\\
=&M(Seq(r_1,\ldots,r_p)(P))\\
=&(d_1,\ldots,d_{i-1},M^i(r_p(P^i)),d_{i+1},\ldots,d_p)\\
\neq&(d_1,\ldots,d_{i-1},M^i(r_p(P^i)),d_{i+1},\ldots,d_p)
\end{aligned}$$

Contradiction. So for all $i \leq p$, r_i must be neutral.

• Participation: Suppose $Seq(r_1,\ldots,r_p)$ satisfies participation, then we need to check that for any r_i, any profile $P^i = \{V_1^i,\ldots,V_N^i\}$ and any vote V_{N+1}^i on D_i

$$r_i(P^i \cup \{V_{N+1}^i\}) \geq_{V_{N+1}^i} r_i(P^i).$$

We prove this by constructing a profile $P = V_1,\ldots,V_N$ and a vote V_{N+1} on D s.t. for any $r,l \leq N+1$,

1. V_r is consistent with the CP-net in which all issues are independent.
2. For any $j \leq p, j \neq i$, the preference of V_r restricted on D_j is the same as that of V_l.
3. The preference of V_r restricted on D_i is V_r^i.

Since $Seq(r_1,\ldots,r_p)$ satisfies homogeneity, each r_j also satisfies homogeneity. Denote d_j the first ranked candidate in D_j by each V_j for all $j \neq i$, if

$$r_i(P^i \cup \{V_{N+1}^i\}) <_{V_{N+1}^i} r_i(P^i)$$

then

$$\begin{aligned}
&Seq(r_1,\ldots,r_p)(P)\\
=&(d_1,\ldots,d_{i-1},r_i(P^i),d_{i+1},\ldots,d_p)\\
>_{V_{N+1}}&(d_1,\ldots,d_{i-1},r_i(P^i \cup \{V_{N+1}^i\}),d_{i+1},\ldots,d_p)\\
=&Seq(r_1,\ldots,r_p)(P \cup \{V_{N+1}\})
\end{aligned}$$

which contradicts with that $Seq(r_1,\ldots,r_p)$ satisfies participation.

• Consensus: Similarly if r_i does not satisfy consensus principle, then there exists a profile $P^i = (V_1^i,\ldots,V_N^i)$ on D_i and $d_i \in D_i$ s.t. for all $V^i \in P^i$, $d_i >_{V^i} r_i(P^i)$. Construct $P = (V_1,\ldots,V_N)$ on D as follows:

1. V_j follows the DAG that all issues are independent.
2. $V_j^{\mathbf{x}_i} = V_j^i$.
3. V_j is a conditional lexicographical order.

Clearly

$$Seq(r_1,\ldots,r_p)(P)=(d_1,\ldots,d_{i-1},r_i(P^i),d_{i+1},\ldots,d_p).$$

But for any $V_j \in P$,

$$(d_1,\ldots,d_{i-1},d_i,d_{i+1},\ldots,d_p) >_{V_j} (d_1,\ldots,d_{i-1},r_i(P^i),d_{i+1},\ldots,d_p),$$

which contradicts with that $Seq(r_1,\ldots,r_p)$ satisfies consensus.

• Consistency: Similar with the proof for neutrality, if there exists $i \leq p$ s.t. r_i is not consistent, then there exist two profiles

$$P^i_1=(V^i_{11},\ldots,V^i_{1N_1}), P^i_2=(V^i_{21},\ldots,V^i_{2N_2})$$

on D_i s.t. $r_i(P^i_1)=r_i(P^i_2)$ and $r_i(P^i_1\cup P^i_2)\neq r_i(P^i_1)$. We construct two profiles

$$P_1=(V_{11},\ldots,V_{1N_1}), P_2=(V_{21},\ldots,V_{2N_2})$$

s.t.

1. Any vote in P_1 or P_2 is consistent with the graph in which all issues are independent.
2. For any vote V in P_1 or P_2 and any $j\neq i$ we have $V^{\mathbf{x}_j}=\{0_j>1_j>\ldots(|D_j|-1)_j\}$.
3. For all $j\leq N_1$, $V^{\mathbf{x}_i}_{1j}=V^i_{1j}$ and for all $j\leq N_2$, $V^{\mathbf{x}_i}_{2j}=V^i_{2j}$.

Since $Seq(r_1,\ldots,r_p)$ satisfies consistency, for any profile p, we have $Seq(r_1,\ldots,r_p)(P)=Seq(r_1,\ldots,r_p)(2P)=Seq(r_1,\ldots,r_p)(P\cup 2P)=\ldots=Seq(r_1,\ldots,r_p)(P)(nP)$. So it also satisfies homogeneity. By Theorem 4.1 each r_j satisfies homogeneity. So

$$\begin{aligned}&Seq(r_1,\ldots,r_p)(P_1)=Seq(r_1,\ldots,r_p)(P_2)\\ =&(0_1,\ldots,0_{i-1},r_i(P^i_1),0_{i+1},\ldots,0_p)\\ \neq&(0_1,\ldots,0_{i-1},r_i(P^i_1\cup P^i_2),0_{i+1},\ldots,0_p)\\ =&Seq(r_1,\ldots,r_p)(P_1\cup P_2)\end{aligned}$$

This contradicts with the consistency of $Seq(r_1,\ldots,r_p)$, so r_i must satisfies consistency. □

Proof of Theorem 4.2: If $Seq(r_1,\ldots,r_p)$ satisfies monotonicity but r_p does not, then there exists a profile $P^p=(V^p_1,\ldots,V^p_N)$ on D_p and another profile $P'^p=(V'^p_1,\ldots,V'^p_N)$ s.t. V'^p_i is obtained from V^p_i by lifting $r_p(P^p)$ and keep other relative order unchanged, and $r_p(P'^p)\neq r_p(P^p)$.

Now consider CP-net $\mathcal{N}_1,\ldots,\mathcal{N}_N$ s.t.

(a) All issues in $\mathcal{N}_i$ are independent.

(b) For all $j,k\leq N$, $i\leq p-1$, $\mathcal{N}^{\mathbf{x}_i}_j=\mathcal{N}^{\mathbf{x}_i}_k$.

(c) For all $j\leq N$ $\mathcal{N}^{\mathbf{x}_p}_j=V^p_j$.

Then consider the conditional lexicographical order V_j for $\mathcal{N}_j$, respectively. Denote d_i the top ranked candidate of $\mathcal{N}^{\mathbf{x}_i}_1$. We then obtain V'_i by raising only $(d_1,\ldots,d_{p-1},r_p(P^p))$ exactly as we raise $r_p(P^p)$ from V^p_i to V'^p_i. Since V_i are all conditional lexicographical order, all V'_i are legal, and $V'^{\mathbf{x}_p|d_1\ldots d_{p-1}}_i=V'^p_i$. Then combined with the homogeneity of $r_1,\ldots,r_{p-1}$, it is easy to check

$$Seq(r_1,\ldots,r_p)(V'_1,\ldots,V'_N)=(d_1,\ldots,d_{p-1},r_p(P'^p)),$$

$$Seq(r_1,\ldots,r_p)(V_1,\ldots,V_N)=(d_1,\ldots,d_{p-1},r_p(P^p)).$$

Since $r_p(P'^p)\neq r_p(P^p)$, we have

$$Seq(r_1,\ldots,r_p)(V'_1,\ldots,V'_N)\neq Seq(r_1,\ldots,r_p)(V_1,\ldots,V_N),$$

which contradicts with the monotonicity of $Seq(r_1,\ldots,r_p)$. Theorem is thus proved. □

Proof of Theorem 4.4: If not, then there exists $i\leq p$ s.t. r_i does not satisfy strong monotonicity but $Seq(r_1,\ldots,r_p)$ does. This means there exists profiles P^i,P'^i on D_i s.t. P'^i is obtained from P^i by raising candidates in $Y\subseteq D_i$ and $r_i(P'^i)\notin\{r_i(P_i)\cup Y\}$. Let $P^i=(V^i_1,\ldots,V^i_N)$, we construct two sets of CP-nets $\{\mathcal{N}_1,\ldots,\mathcal{N}_N\}$, $\{\mathcal{N}'_1,\ldots,\mathcal{N}'_N\}$ by lifting each linear order in P^i and P'^i to a linear order on $\mathcal{X}$ similarly with the proof of Theorem 4.2, and consider the conditional lexicographical orders of them. Denote P

and P' the resulting profiles over $\mathcal{X}$. Then it is easy to see P' is obtained from P by raising candidates in $Y_{\mathcal{X}} = D_1 \times \dots D_{i-1} \times Y \times D_{i+1} \times \dots \times D_p$. Since $r_i(P'^i) \notin \{r_i(P_i) \cup Y\}$, we know

$$Seq(r_1, \dots, r_p)(P') \notin \{Seq(r_1, \dots, r_p)(P)\} \cup Y_{\mathcal{X}}.$$

This contradicts with the assumption that $Seq(r_1, \dots, r_p)$ satisfies monotonicity. □

Proof of Theorem 4.5: Anonymity and homogeneity are obvious.

• Consistency: Let P_1 and P_2 be two profiles on D s.t.

$$Seq(r_1, \dots, r_p)(P_1) = Seq(r_1, \dots, r_p)(P_2) = (d_1, \dots, d_p).$$

Since r_1 satisfies consistency, and $d_1 = r_1(P_1^{\mathbf{x}_1}) = r_1(P_2^{\mathbf{x}_1})$, we have $d_1 = r_1(P_1^{\mathbf{x}_1} \cup P_2^{\mathbf{x}_1}) = r_1((P_1 \cup P_2)^{\mathbf{x}_1})$.

Suppose after the first i steps r_j selected d_j from $P_1 \cup P_2$, then

$$d_{i+1} = r_{i+1}(P_1^{\mathbf{x}_{i+1}|d_1 \dots d_i}) = r_{i+1}(P_2^{\mathbf{x}_{i+1}|d_1 \dots d_i}).$$

From the consistency of r_{i+1} we have

$$d_{i+1} = r_{i+1}(P_1^{\mathbf{x}_{i+1}|d_1 \dots d_i} \cup P_2^{\mathbf{x}_{i+1}|d_1 \dots d_i}) = r_{i+1}((P_1 \cup P_2)^{\mathbf{x}_{i+1}|d_1 \dots d_i}).$$

So r_{i+1} would select d_{i+1} from $P_1 \cup P_2$. Therefore $Seq(r_1, \dots, r_p)(P_1 \cup P_2) = Seq(r_1, \dots, r_p)(P_1)$, $Seq(r_1, \dots, r_p)$ satisfies consistency.

• Strong monotonicity: For any $Y \subseteq \mathcal{X}$, we write

$$Y^{\mathbf{x}_i|d_1 \dots d_{i-1}} = \{x_i : \vec{x} \in Y, x_j = d_j \text{ for all } j \le i-1\}.$$

Suppose $r_1, \dots, r_p$ satisfy strong monotonicity, first we prove for any profile P and P', if P' is obtained from P by raising candidates in Y, then

$$(Seq(r_1, \dots, r_p)(P'))_1 \in \{(Seq(r_1, \dots, r_p)(P'))_1\} \cup Y^{\mathbf{x}_1}. \tag{1}$$

To prove this, we only need to check that $P'^{\mathbf{x}_1}$ is obtained from $P^{\mathbf{x}_1}$ by raising $Y^{\mathbf{x}_1}$. It suffices to check for any $V \in P$ and its counterpart $V' \in P'$, for any $x \in Y^{\mathbf{x}_1}$, $y \in D_1$

$$x \succ_V y \Rightarrow x \succ_{V'} y.$$

If not, suppose $x \succ_V y$ but $y \succ_{V'} x$, and $(x, \vec{d}_2) \in Y$ for some $\vec{d}_2 \in D_2 \times \dots \times D_p$. Then we know $(x, \vec{d}_2) \succ_V (y, \vec{d}_2)$ and $(y, \vec{d}_2) \succ_{V'} (x, \vec{d}_2)$. Since V' is obtained from V by raising candidates in Y, for any $\vec{d} \in Y$ we have

$$\{\vec{x} : \vec{x} \succ_{V'} \vec{d}\} \subseteq \{\vec{x} : \vec{x} \succ_V \vec{d}\}.$$

But $(y, \vec{d}_2) \in \{\vec{x} : \vec{x} \succ_{V'} \vec{d}\}$, and $(y, \vec{d}_2) \notin \{\vec{x} : \vec{x} \succ_V \vec{d}\}$, contradiction.

So we know Equation (1) holds. Denote $w_1 = r_1(P'^{\mathbf{x}_1})$. Now there are two cases: $w_1 \neq r_1(P^{\mathbf{x}_1})$ and $w_1 = r_1(P^{\mathbf{x}_1})$. For the first case there must exist $V \in P$ s.t. the rank of w_1 in $V'^{\mathbf{x}_1}$ is higher than in $V^{\mathbf{x}_1}$. If not, then $V'^{\mathbf{x}_1}$ is obtained from $V^{\mathbf{x}_1}$ by raising candidates in $Y^{\mathbf{x}} \backslash \{w_1\}$ for all $V \in P$, so by strong monotonicity of r_1, $w_1 \in (\{r_1(P^{\mathbf{x}_1})\} \cup Y^{\mathbf{x}_1}) \backslash \{w_1\}$, contradiction. Suppose there exists $V \in P$ and $y \in D_1$ s.t. $y \succ_V w_1$ and $w_1 \succ_{V'} y$. Then we know for all $\vec{d}_2 \in D_2 \times \dots D_p$, $(w_1, \vec{d}_2) \succ_{V'} (y, \vec{d}_2)$ and $(y, \vec{d}_2) \succ_V (w_1, \vec{d}_2)$. Therefore in V', $(w_1, \vec{d}_2)$ must be raised, which means $\{w_2\} \times D_2 \times \dots \times D_p \subseteq Y$. So $Seq(r_1, \dots, r_p)(P') \in Y$.

For the second case, we can move to the second step of sequential voting process, considering fixing $\mathbf{x}_1 = w_1$. Then following the same proof we know $Seq(r_1, \dots, r_p)(P') \in Y$ or $r_2(P^{\mathbf{x}_2|w_1}) = r_2(P'^{\mathbf{x}_2|w_1})$. Repeat this process recursively, finally we can prove $Seq(r_1, \dots, r_p)(P') \in Y$ or $Seq(r_1, \dots, r_p)(P') = Seq(r_1, \dots, r_p)(P)$. This completes the proof of the theorem. □

Proof of Theorem 4.6: We first present and proof a lemma.

Lemma 6.3 *Suppose $V \sim \mathcal{N}$, and V' is obtained from V by only raising $\vec{d} \in D$. If V' is also legal and $V' \sim \mathcal{N}'$ and $\mathcal{N}' \neq \mathcal{N}$, then $\mathcal{N}'$ differs from $\mathcal{N}$ only on $\mathbf{x}_p : d_1 \dots d_{p-1}$, and the conditional order of $\mathcal{N}'^{\mathbf{x}_p : d_1 \dots d_{p-1}}$ is obtained from $\mathcal{N}^{\mathbf{x}_p : d_1 \dots d_{p-1}}$ by raising only d_p.*

Lemma 6.3 *Proof.* We first prove that $\mathcal{N}'$ differs from $\mathcal{N}$ only on the conditional order $\mathbf{x}_p : d_1 \dots d_{p-1}$. If not, then there exists $i < p$ and $s_j \in D_j, j < i$ s.t.

$$\mathcal{N}'^{\mathbf{x}_i:s_1\dots s_{i-1}} \neq \mathcal{N}^{\mathbf{x}_i:s_1\dots s_{i-1}}.$$

Then $\exists s_i, s_i' \in D_i$ s.t. $s_1 \dots s_{i-1} : s_i >_{\mathcal{N}} s_i'$ but $s_1 \dots s_{i-1} : s_i <_{\mathcal{N}'} s_i'$. Choose any $\vec{v_1}, \vec{v_2} \in D_{i+1} \times \dots \times D_p$ s.t. $\vec{v_1} \neq \vec{v_2}$, then the relative position of two pairs: $(s_1, \dots, s_i, \vec{v_1})$ and $(s_1, \dots, s_i', \vec{v_1})$, $(s_1, \dots, s_i, \vec{v_2})$ and $(s_1, \dots, s_i', \vec{v_2})$, are exchanged. Since we are raising only $\vec{d}$, the relative order exchanging pair must contain $\vec{d}$, but the four candidates in the two pairs are all different from one another, contradiction.

From the observation that each exchanging pair must contain $\vec{d}$ we can similarly prove the remaining part of this lemma. □

Now given any profile $P = (V_1, \dots, V_N)$, a legal profile $P' = (V_1', \dots, V_N')$ is obtained by raising only $Seq(r_1, \dots, r_p)(P) = (d_1, \dots, d_p)$. From Lemma 6.3 we know that

1. $V_j'^{\mathbf{x}_i|\vec{s_i}} = V_j^{\mathbf{x}_i|\vec{s_i}}$ for all $i \leq N, i \leq p-1, \vec{s_i} \in D_1 \times \dots \times D_{i-1}$
2. $V_j'^{\mathbf{x}_p|d_1\dots d_{p-1}}$ is obtained by $V_j^{\mathbf{x}_p|d_1\dots d_{p-1}}$ by raising d_p.

So from the definition of $Seq(r_1, \dots, r_p)$, we know that r_i would select d_i from P' for all $i \leq p-1$, r_p would select $r_p(V_1'^{\mathbf{x}_p|d_1\dots d_{p-1}}, \dots, V_N'^{\mathbf{x}_p|d_1\dots d_{p-1}})$. Since r_p satisfies monotonicity, we have

$$r_p(V_1'^{\mathbf{x}_p|d_1\dots d_{p-1}}, \dots, V_N'^{\mathbf{x}_p|d_1\dots d_{p-1}}) = d_p.$$

Hence $Seq(r_1, \dots, r_p)(P') = (d_1, \dots, d_p) = Seq(r_1, \dots, r_p)(P)$. This is exactly the monotonicity of $Seq(r_1, \dots, r_p)$. □

Proof of Theorem 4.7: Take $\mathcal{X} = \{0_1, 1_1, 2_1\} \times \{0_2, 1_2\}$ as an example (the proof is similar in other cases). Let $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ be three CP-nets s.t. $\mathbf{x}_1$ and $\mathbf{x}_2$ are independent, and

$$\begin{array}{ll} 0_1 \succ_{\mathcal{N}_1} 1_1 \succ_{\mathcal{N}_1} 2_1, & 1_2 \succ_{\mathcal{N}_1} 0_2; \\ 1_1 \succ_{\mathcal{N}_2} 0_1 \succ_{\mathcal{N}_2} 2_1, & 0_2 \succ_{\mathcal{N}_2} 1_2; \\ 2_1 \succ_{\mathcal{N}_3} 1_1 \succ_{\mathcal{N}_3} 0_1, & 0_2 \succ_{\mathcal{N}_3} 1_2; \end{array}$$

It is easy to verify that $\mathcal{N}_i \not\models 0_1 0_2 \succ 1_1 1_2, \mathcal{N}_i \not\models 1_1 1_2 \succ 0_1 0_2$ for $i = 1, 2, 3$. Denote M the permutation on D that only exchange $0_1 0_2$ and $1_1 1_2$. Then by Lemma 6.2, there exist three votes V_1, V_2, V_3 s.t. for $i = 1, 2, 3$

1. V_i and $M(V_i)$ are consistent with $\mathcal{N}_i$, and
2. $1_1 1_2 \succ_{V_i} 0_1 0_2$.

Consider a profile consisting of three V_1, two V_2 and two V_3. Clearly $Seq(r_1, r_2)(3V_1 + 2V_2 + 2V_3) = 0_1 0_2$, but we know $1_1 1_2 \succ_{V_i} 0_1 0_2$, which means that $Seq(r_1, r_2)$ cannot satisfies consensus. Also from $M(Seq(r_1, r_2)(3V_1 + 2V_2 + 2V_3)) = 1_1 1_2$ and $Seq(r_1, r_2)(M(3V_1 + 2V_2 + 2V_3)) = 0_1 0_2$ we know that $Seq(r_1, r_2)$ is not neutral. The proof is complete. □

Proof of Theorem 5.1: First we prove if $Seq(c_1, \dots, c_p)$ is insensitive to M_R, then each c_i is neutral. If there exists i s.t. c_i is not neutral, denote M_i the non-identity permutation on D_i, then there exists $P^i = \{V_1^i, \dots, V_N^i\}$ s.t. $M_i(r_i(P^i)) \neq r_i(M_i(P^i))$, namely $r_i(P^i) = r_i(M_i(P^i))$. Without loss of generality we assume $r_i(P^i) = 1_i$. Construct a profile $P = \{V_1, \dots, V_N\}$ on D s.t.

1. For any $V \in P$, V follows a DAG in which any issue is independent from others.
2. For any $V \in P$, and $j \neq i$, $V^{\mathbf{x}_j} = 1_j > 0_j$.
3. For all $j \leq N$, $V_j^{\mathbf{x}_i} = V_j^i$.

Then the i-th component of $Seq(c_1, \dots, c_p)(P)$ and $Seq(c_1, \dots, c_p)(M_R(P))$ are $r_i(P^i) = 1_i$, which means $M_R(Seq(c_1, \dots, c_p)) \neq Seq(c_1, \dots, c_p)(M_R(P))$, contradiction.

Then notice that for any vote V on $\mathcal{X}$, $d_1 \dots d_i : x_{i+1} >_V \overline{x_{i+1}}$ iff $\overline{d_1} \dots \overline{d_i} : \overline{x_{i+1}} >_{M_R(V)} x_{i+1}$. Obviously if each c_i is neutral, then $Seq(c_1, \dots, c_p)$ is insensitive to M_R. □

Proof of Theorem 5.2:

1. The necessity is easy to check. We only prove sufficiency. By Theorem 5.1 we know that $Seq(c_1, \dots, c_p)$ is insensitive to M_R is equivalent to each c_i is neutral. By Theorem 4.1 we know each c_i satisfies strong monotonicity. Denote $V_i = 0_i \succ 1_i, V_i' = 1_i \succ 0_i$. Then we claim $c_i(nV_i + mV_i') = 0_i$ iff $n > m$. If not, there exists

$n > m$ s.t. $n+m$ is odd and $c_i(nV_i + mV_i') = 1_i$. We raise 1_i in $(n-m)V_i$, then by strong monotonicity, $c_i(mV_i + nV_i') = 1_i$, but by neutrality $c_i(mV_i + nV_i') = 0_i$, contradiction. Therefore we know c_i is the majority correspondence.

2. Again we only prove sufficiency. Since whenever $|C(P)| \geq 2$, $|P|$ is even, we know on the domain of all legal profiles consisting of odd votes, $Seq(c_1, \ldots, c_p)$ satisfies anonymity, strong monotonicity, and is insensitive to M_R. So by part 1 of this theorem, we know if $|P|$ is odd, then C is the sequential majority correspondence. Now for each c_i consider $c_i(nV_i + mV_i')$, where $m+n$ can be divided by 2. By neutrality we can assume $n \geq m$ without loss of generality. Then $c_i((n-1)V_i + mV_i') = c_i(V_i) = 0_i$, so by consistency $c_i(nV_i + mV_i') = 0_i$, which means c_i is the majority correspondence.

To prove the property set in 2. is minimal, we present examples for removing each condition. Anonymity is obvious. For strong monotonicity, let each c_i be the correspondence that select a minority. Then c_i is consistent and neutral, so C satisfies consistency by Theorem 4.5, C is insensitive to M_R by Theorem 5.1. Clearly when $|P|$ is odd, $|C(P)| = 1$. So we know C satisfies the other four conditions, and is not the sequential composition of majority correspondences.

For consistency, let c_i be majority correspondence if $|P|$ is odd, otherwise it is trivial (always outputs D_i). Since majority and trivial correspondence are both neutral and strong monotonic, we know C satisfies strong monotonicity and is insensitive to M_R by Theorem 4.5. Notice when $|P|$ is odd, C is the sequential composition of majority correspondence, we know C satisfies the four properties other than consistency.

For insensitive to M_R, we simply let $c_i(P) = 0_i$ for all P, it is easy to check C satisfies other four properties.

For $|C(P)| \geq 2 \Rightarrow |P|$ is odd, we consider the trivial correspondence $C(P) = \mathcal{X}$ for all P. By simple calculation we know all the other four properties holds.

So the property set in 2. is minimal. Similar examples show the property set in 1. is also minimal. $\square$

Proof of Theorem 5.3: Let $D_1 = \{0_1, 1_1\}, D_2 = \{0_2, 1_2\}$. We would in fact prove that the $Seq(c_1, c_2)$ is *strong decomposable*, which is more specific than decomposability. We need to prove that for each permutation M on $D_1 \times D_2$ and $P = (P_1, \ldots, P_N)$ following $\mathcal{O} = \mathbf{x}_1 > \mathbf{x}_2$, if $M(P)$ is also a legal preference profile, then

$$M(Seq(c_1, c_2)(P)) = Seq(c_1, c_2)(M(P)).$$

Because M^{-1} is also a permutation, it suffices to prove that for each permutation M on $D_1 \times D_2$ and $P = (P_1, \ldots, P_N)$ following $\mathcal{O} = \mathbf{x}_1 > \mathbf{x}_2$, if $M(P)$ is a legal preference profile, then

$$M(Seq(c_1, c_2)(P)) \subseteq Seq(c_1, c_2)(M(P)),$$

namely

$$(x_1, x_2) \in Seq(c_1, c_2)(P) \Rightarrow M(x_1, x_2) \in Seq(c_1, c_2)(M(P)) \tag{2}$$

To prove this, for any given permutation M, we find all votes P following $\mathcal{O}$ s.t. $M(P)$ is legal. Recall that the CP-nets determine the voting results, we only need to know the effects of M on CP-nets. For example, if we know that M can transform a vote P_1 following $\mathcal{N}_1$ to another vote P_2 following $\mathcal{N}_2$, then the result of the sequential voting process is indifferent with what P_1, P_2 exactly are.

We wrote a program to calculate all possible transformations of CP-nets for any permutation. Without loss of generality, we only consider preference profiles that follow $\mathbf{x}_1 > \mathbf{x}_2$ before permutation. For example, the following is part of the outcome, we will explain the meaning of the symbols right after presenting them.

0132

000—>001 001—>000 001—> r000 010—>011 010—> r100 011—>010 100—>101 110—>111 110—> r111 101—>100 101—> r011 111—>110

We encode each candidate in $D_1 \times D_2$ as a number—$0_1 0_2$ as 0, $0_1 1_2$ as 1, $1_1 0_2$ as 2, $1_1 1_2$ as 4. In the output, the first 4 numbers 0132 encode the permutation M s.t. $M(0_1 0_2) = 0_1 0_2, M(0_1 1_2) = 0_1 1_2, M(1_1 0_2) = 1_1 1_2, M(1_1 1_2) = 1_1 0_2$.

A CP-net that follows $\mathbf{x}_1 > \mathbf{x}_2$ is encoded as 3 numbers abc, where

$$a = \begin{cases} 0 & \text{iff } 0_1 \succ 1_1 \\ 1 & \text{iff } 1_1 \succ 0_1 \end{cases}, b = \begin{cases} 0 & \text{iff } 0_1 : 0_2 \succ 1_2 \\ 1 & \text{iff } 0_1 : 1_2 \succ 0_2 \end{cases}, c = \begin{cases} 0 & \text{iff } 1_1 : 0_2 \succ 1_2 \\ 1 & \text{iff } 1_1 : 1_2 \succ 0_2 \end{cases}.$$

For a CP-net following $\mathbf{x}_2 > \mathbf{x}_1$, the 3 numbers are defined similarly with them firstly considering the preference on D_2, then add "r" before the numbers.

A transformation $\mathcal{N} \text{—>} \mathcal{N}'$ means that $\mathcal{N}$ is M-transformable to $\mathcal{N}'$. To prove neutrality, we need to verify for each permutation and any legal preference profile, Equation (2) holds. We verify $M = 0132$ in this paper as an example.

From the output related to $M = 0132$ listed above, there are 8 possible transformations from a CP-net following $\mathbf{x}_1 > \mathbf{x}_2$ to another CP-net following the same order, and 4 possible transformations from a CP-net following $\mathbf{x}_1 > \mathbf{x}_2$ to another CP-net following $\mathbf{x}_2 > \mathbf{x}_1$. By anonymity, we assume that there are a,b,c,d,e,f,g,h votes for $000 \text{---}> 001, 001 \text{---}> 000, 010 \text{---}> 011, 011 \text{---}> 010, 100 \text{---}> 101, 110 \text{---}> 111, 101 \text{---}> 100, 111 \text{---}> 110$ respectively. If $1_1 0_2$ is selected by $Seq(c_1, c_2)$ before permutation, then since c_1, c_2 are both plurality rules, we have

$$e + f + g + h \geq a + b + c + d$$

and

$$a + c + e + f \geq b + d + g + h.$$

After permutation, from $e + f + g + h \geq a + b + c + d$ we know that c_1 would select 1_1 (of course, it may also select 0_1) and from $a + c + e + f \geq b + d + g + h$ we know that after c_1 selected 1_1, c_2 would select 1_2. So $1_1 1_2$ must be selected after permutation.

Suppose there are a,b,c,d votes for $001 \text{---}> r000$, $010 \text{---}> r100$, $110 \text{---}> r111$, $101 \text{---}> r011$ respectively. If $(1_1, 0_2)$ is selected by $Seq(c_1, c_2)$ before permutation, then $c + d \geq a + b, b + c \geq a + d$. Since the order is reversed after permutation, we would firstly consider c_2. From $b + c \geq a + d$ we know c_2 would select 1_2 and from $c + d \geq a + b$ we know that after 1_2 is selected, 1_1 is selected by c_1. So $1_1 1_2$ must be selected after permutation, Equation (2) holds for $1_1 0_2$ case.

We checked all the 4 candidates in $D_1 \times D_2$ for all permutations in a similar way, and found that Equation (2) always hold. The lengthy verifications are omitted here. Therefore $Seq(c_1, c_2)$ is neutral. □

Proof of Theorem 5.4: To prove this theorem, we need some definition and lemmas.

Definition 6.4 *For any $\mathcal{O} = \mathbf{x}_{i_1} > \ldots > \mathbf{x}_{i_p}$ with $i_1, \ldots, i_p$ a permutation of $\{1, \ldots, p\}$, and a linear preference relation P following $\mathcal{O}$, define a mapping Con_P from $\mathcal{X}$ to a subset of all CPTs items in CP-net of $\mathcal{O}$ s.t.*

$$Con_P(\vec{x}) = \{\succ_P^{\mathbf{x}_1}, \succ_P^{\mathbf{x}_2|\mathbf{x}_1 = x_1}, \ldots, \succ_P^{\mathbf{x}_p|\mathbf{x}_1 = x_1, \mathbf{x}_2 = x_2, \ldots, \mathbf{x}_{p-1} = x_{p-1}}\}$$

For example, let $p = 3$, $P = 1_1 1_2 1_3 \succ 1_1 1_2 0_3 \succ 1_1 0_2 0_3 \succ 1_1 0_2 1_3 \succ 0_1 1_2 0_3 \succ 0_1 0_2 0_3 \succ 0_1 1_2 1_3 \succ 0_1 0_2 1_3$, then $Con_P(1_1 0_2 1_3) = Con_P(1_1 0_2 0_3) = \{1_1 \succ 0_1, 1_1 : 1_2 \succ 0_2, 1_1 0_2 : 0_3 \succ 1_3\}$. The subscription P is sometimes omitted when there is no confusion.

Lemma 6.5 *Let $V = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$, $P = \{P_1, \ldots, P_N\}$ following $\mathcal{O} = \mathbf{x}_1 > \ldots > \mathbf{x}_p$ and*

$$Seq(r_1, \ldots, r_p)(P_1, \ldots, P_N) = \vec{x}.$$

If another linear preference profile $P' = \{P'_1, \ldots, P'_N\}$ satisfies:

(a) *P'_i follows $\mathcal{O}$,*

(b) *$(\forall 1 \leq i \leq N)\big(P'_i$ satisfies $Con_{P_i}(\vec{x})\big)$,*

then

$$Seq(r_1, \ldots, r_n)(P'_1, \ldots, P'_N) = \vec{x}.$$

Proof. This is obvious, since that $x_i = x_i^* = x_i'^*$ by the assumption of P'. □

This lemma says that if we keep some conditional order(i.e. the CPTs related to the result of the sequential voting process in the representing CP-net) unchanged, the sequential voting result would not change as well.

Lemma 6.6 *Let P be a linear preference relation following $\mathcal{O}$, its first ranked vector is $\vec{x}$. For any $\vec{x}' \neq \vec{x}$, not all of $\{x'_{i_1} \succ_P \overline{x'_{i_1}}, x'_{i_1} : x'_{i_2} \succ_P \overline{x'_{i_2}}, \ldots, x'_{i_1} \ldots x'_{i_{p-1}} : x'_p \succ_P \overline{x'_p}\}$ hold in P.*

Proof. On the contrary, suppose all of them hold. Without loss of generality let $\mathcal{O} = \mathbf{x}_1 > \ldots > \mathbf{x}_p$.

Let i be the smallest index that $x_i \neq x'_i$. Since $x'_1 \ldots x'_{i-1} : x'_i \succ \overline{x'_i}$ holds, it must be the case that

$$(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_p) \succ (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p) = \vec{x},$$

which contradicts the assumption that $\vec{x}$ is the first ranked in P. This completes the proof. □

Lemma 6.7 *Let V be a linear preference relation consistent with $\mathcal{O}$, and its first ranked vector is $\vec{x}$. For any $\vec{x}' \neq \vec{x}$, if $\overline{\vec{x}'} \neq \vec{x}$, then there exists V' also consistent with $\mathcal{O}$, s.t.*

a. *$\overline{\vec{x}'} \succ_{V'} \vec{x}'$, where $\overline{\vec{x}'}$ is the complement of $\vec{x}'$;*

b. *$\overline{\vec{x}'}$ and $\vec{x}'$ are adjacent in V'.*

c. *Let M be the permutation on D that only exchanges $\vec{x}'$ and $\overline{\vec{x}'}$, then $V' \sim M(V')$.*

d. *V' also satisfies $Con(\vec{x}')$.*

Proof. Without loss of generality, we can assume that $\mathcal{O} = \mathbf{x}_1 > \ldots > \mathbf{x}_p$, $\vec{x} = (1_1, d_2, \ldots, d_p)$ s.t. $\exists 2 \le i \le p, d_i = 0$, and $\vec{x}' = (1_1, \ldots, 1_p)$. Thus $\overline{\vec{x}'} = (0_1, 0_2, \ldots, 0_p)$. Denote $\mathcal{N}$ the corresponding CP-net of V. Our first goal is to construct a CP-net $\mathcal{N}'$ s.t. all of $Con(\vec{x}')$ hold and

$$\mathcal{N}' \not\models (1_1, \ldots, 1_p) \succ (0_1, \ldots, 0_p).$$

By Lemma 6.6, we know that at least one of $\{1_1 \succ_V 0_1, 1_1 : 1_2 \succ_V 0_2, \ldots, 1_1 \ldots 1_{p-1} : 1_p \succ_V 0_p\}$ does not hold. Let i be the smallest number that $1_1 \ldots 1_{i-1} : 0_i \succ_V 1_i$ and $1_1 \ldots 1_{i-2} : 1_{i-1} \succ_V 0_{i-1}$. In $\mathcal{N}'$, we firstly put all $Con_P(1_1, \ldots, 1_p)$ to $\mathcal{N}'$'s CPTs, then for all $1 \le j \le i$, we put

$$\begin{aligned}
&1_1 \ldots 1_{j-1} 0_j : 0_{j+1} \succ 1_{j+1},\\
&1_1 \ldots 1_{j-1} 0_j 0_{j+1} : 0_{j+2} \succ 1_{j+2},\\
&\vdots\\
&1_1 \ldots 1_{j-1} 0_j \ldots 0_{p-1} : 0_p \succ 1_p,
\end{aligned}$$

denoted as C_j, into the CPTs of $\mathcal{N}'$. The other part of the CPTs of $\mathcal{N}'$ can be chosen arbitrarily, which would not affect the claim of this lemma. Since $Con_P(1_1, \ldots, 1_p)$ and C_js are contained in the CPTs of $\mathcal{N}'$, we know that the only parent of $(0_1, \ldots, 0_p)$ in the induced graph of $\mathcal{N}'$ is $(1_1, 0_2, \ldots, 0_p)$; similarly the only parent of $(1_1, 0_2, \ldots, 0_p)$ is $(1_1, 1_2, 0_3, \ldots, 0_p)$, etc.

In the end, $(1_1, \ldots, 1_{i-1}, 0_i, \ldots, 0_p)$ does not have any parent. Since $i \le p$, $(1_1, \ldots, 1_p)$ is not a parent of $(0_1, \ldots, 0_p)$, which means $\mathcal{N}' \not\models (1_1, \ldots, 1_p) \succ (0_1, \ldots, 0_p)$. On the other hand, since $1_1 >_V 0_1$, we also know $\mathcal{N}' \not\models (0_1, \ldots, 0_p) \succ (1_1, \ldots, 1_p)$. Therefore by Lemma 6.2 there exists a linear preference ordering V' consistent with $\mathcal{N}'$ satisfying a,b,c. The lemma is thus proved. □

Now we can prove the theorem. If the theorem does not hold, by Lemma 6.7 there exists $V'_1, \ldots, V'_N$ s.t. for every $1 \le i \le N$

(a) $\overline{\vec{x}'} \succ_{V'} \vec{x}'$, where $\overline{\vec{x}'}$ is the complement of $\vec{x}'$;

(b) $\overline{\vec{x}'}$ and $\vec{x}'$ are adjacent in V'.

(c) Let M be the permutation on D that only exchanges $\vec{x}'$ and $\overline{\vec{x}'}$, then $V' \sim M(V')$.

(d) V' also satisfies $Con(\vec{x}')$.

Then by Lemma 6.5, $Seq(r_1, \ldots, r_p)(P'_1, \ldots, P'_N) = \vec{x}$, which obviously contradicts the consensus. (c) obviously contradicts with neutrality. The proof is complete. □

Proof of Theorem 5.5: Example 6.8 also shows that the sequential plurality on multiple referenda is not neutral if $p \ge 3$. Consider the profile $P = (V_1, V_2, V_3)$, the sequential plurality rule would select $1_1 1_2 1_3$, which is not ranked first in any V_i. So by Theorem 5.4 the sequential plurality does not satisfy either neutrality or consensus. □

Proof of Theorem 5.7:

This is easily shown on the following counterexample with two binary issues **x** and **y**:

voter 1: $(x,y) \succ (\bar{x},y) \succ (\bar{x},\bar{y}) \succ (x,\bar{y}$

voter 2: $(x,\bar{y}) \succ (x,y) \succ (\bar{x},y) \succ (\bar{x},\bar{y})$

voter 3: $(\bar{x},y) \succ (\bar{x},\bar{y}) \succ (x,\bar{y}) \succ (x,y)$

The profile is legal, the order being $x \succ y$ (note that voter 1's preference order is separable). If voter 1 knows the preferences of voters 2 and 3 then he does not want to vote sincerely on x: if he does, then he votes x and then the outcome is $(x,\bar{y})$. If he votes for $\bar{x}$ instead then then outcome is $\bar{x},y)$, which is better to him. □

Appendix 3: Examples

Example 6.8 *Copeland satisfies consensus, because if $c_1 \succ c_2$ in all votes, then c_1 would gain more points than c_2 in pairwise comparison. Consider $\mathcal{X} = \{0_1, 1_1\} \times \{0_2, 1_2\} \times \{0_3, 1_3\}$ and a profile P consisting of the following three votes*

$$\begin{aligned}
&V_1 : 0_1 1_2 1_3 \succ 1_1 1_2 1_3 \succ 0_1 1_2 0_3 \succ 1_1 1_2 0_3 \succ 0_1 0_2 0_3 \succ 1_1 0_2 0_3 \succ 0_1 0_2 1_3 \succ 1_1 0_2 1_3;\\
&V_2 : 1_1 0_2 1_3 \succ 1_1 1_2 1_3 \succ 1_1 0_2 0_3 \succ 1_1 1_2 0_3 \succ 0_1 0_2 1_3 \succ 0_1 1_2 1_3 \succ 0_1 0_2 0_3 \succ 0_1 1_2 0_3;\\
&V_3 : 1_1 1_2 0_3 \succ 1_1 1_2 1_3 \succ 1_1 0_2 0_3 \succ 1_1 0_2 1_3 \succ 0_1 1_2 0_3 \succ 0_1 1_2 1_3 \succ 0_1 0_2 0_3 \succ 0_1 0_2 1_3,
\end{aligned}$$

Obviously they follow $\mathbf{x}_1 > \mathbf{x}_2 > \mathbf{x}_3$, and the Copeland rule would select $1_1 1_2 1_3$, which is not ranked first in any V_i. If Copeland is decomposable, then by Theorem 5.4, any candidate that is not ranked first in any vote should not be the winner. This contradicts with the profile defined above. So we know Copeland could not be decomposable on a domain of binary composition.

Example 6.9 *Consider three votes*

$$V_1 : 0_1 1_2 1_3 \succ 0_1 1_2 0_3 \succ 0_1 0_2 0_3 \succ 0_1 0_2 1_3 \succ 1_1 1_2 1_3 \succ 1_1 0_2 1_3 \succ 1_1 1_2 0_3 \succ 1_1 0_2 0_3;$$
$$V_2 : 1_1 0_2 1_3 \succ 1_1 0_2 0_3 \succ 0_1 0_2 0_3 \succ 0_1 0_2 1_3 \succ 1_1 1_2 1_3 \succ 0_1 1_2 1_3 \succ 1_1 1_2 0_3 \succ 0_1 1_2 0_3;$$
$$V_3 : 1_1 1_2 0_3 \succ 1_1 0_2 0_3 \succ 0_1 0_2 0_3 \succ 0_1 1_2 0_3 \succ 1_1 1_2 1_3 \succ 1_1 0_2 1_3 \succ 0_1 0_2 1_3 \succ 0_1 1_2 1_3.$$

Let $P = (V_1, V_2, V_3)$, then the sequential majority elects $1_1 1_2 1_3$ from P, which only weak Condorcet-dominates three candidates — $1_1 0_2 1_3, 1_1 1_2 0_3$, and $0_1 1_2 1_3$.