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SEPARATION OF PARTITION INEQUALITIES

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Given a graph $G = (V, E)$ with nonnegative weights $x(e)$ for each edge e, a partition inequality is of the form $x(\delta(S_1, \ldots, S_p)) \geq ap + b$. Here $\delta(S_1, \ldots, S_p)$ denotes the multicut defined by a partition $S_1, \ldots, S_p$ of V. Partition inequalities arise as valid inequalities for optimization problems related to k-connectivity. We give a polynomial algorithm for the associated separation problem. This is based on an algorithm for finding the minimum of $x(\delta(S_1, \ldots, S_p)) - p$ that reduces to minimizing a symmetric submodular function. This is handled with the recent algorithm of Queyranne. We also survey some applications of partition inequalities.

1. Introduction. Let $G = (V, E)$ be a graph with edge weights $x(e) \geq 0$ for all $e \in E$. Given a partition $S_1, \ldots, S_p$ of the node set V, we denote by $\delta(S_1, \ldots, S_p)$ the set of edges with endnodes in different sets of the partition. We use $\delta(S)$ instead of $\delta(S, V \setminus S)$ and we use $x(T)$ to denote $\sum_{e \in T} x(e)$.

Given a and b, an inequality of the type

$$x(\delta(S_1, \ldots, S_p)) \geq ap + b \tag{1.1}$$

is called a *partition inequality*.

Partition inequalities define the dominant of the spanning tree polytope, with $a = 1$ and $b = -1$; probably this is the most known case. As we shall see later, they arise as valid inequalities or facets for optimization problems related to k-connectivity. In this paper we study the *separation* problem: Given a vector x find a violated inequality (1.1), if there is any. The separation problem is a key ingredient for being able to use these inequalities inside a cutting plane algorithm.

If $a \leq 0$ and there is a partition with $p > 2$ so that (1.1) is violated, we can collapse two sets, then the left-hand side does not increase, and the right-hand side does not decrease. So in this case one should only deal with $p = 2$ and the problem can be solved by finding a minimum cut.

We have to study the case $a > 0$, and without loss of generality we can assume that $a = 1$. The problem with $b \leq -1$ seems to be easier than the case $b > -1$. The former case had been studied by Cunningham (1985), he reduced it to $O(|E|)$ minimum $s - t$ cut problems. Later it was shown in Barahona (1992) that it can be solved in $O(|V|)$ minimum $s - t$ cut problems. In order to solve the latter case we had to study the problem:

$$\text{minimize } x(\delta(S_1, \ldots, S_p)) - p \tag{1.2}$$

among all partitions with $p \geq 2$. This problem is interesting in its own sake; it also gives lower bounds for the k-cut problem that is NP-Hard. We develop a polynomial algorithm for this case. This uses a recent algorithm of Queyranne (1995) for minimizing

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a symmetric submodular function. We show that the case $b > -1$ reduces to $O(|V|^3)$ minimum $s-t$ cut problems. This case uses the algorithm for the first case as a subroutine.

This paper is organized as follows. Sections 2 and 3 are devoted to the case $b = -1$. Section 4 describes Cunningham's approach to this same case. In §5 we show that problem (1.2) reduces to minimizing a certain submodular function. In §6 we show how Queyranne's algorithm for minimizing symmetric submodular functions applies to our problem. In §7 we discuss some applications of these methods. In §8 we survey partition inequalities.

2. The case $b \leq -1$**.** The first separation algorithm for this case was given by Cunningham (1985); he reduced this to $|E|$ minimum cut problems. We describe his approach in §4. A reduction to $|V|$ min-cut problems was given in Barahona (1992); we present it here with slight modifications. We shall see that this problem is equivalent to optimizing a linear function over a certain *extended polymatroid.* These concepts are discussed in Gröstchel et al. (1988) for instance. This problem can be solved with the greedy algorithm used by Edmonds (1970). At each iteration, finding an inequality that becomes tight is equivalent to finding a minimum $s - t$ cut in a certain network. We present this method below, this will be used as a subroutine in the following sections.

Given a graph $G = (V, E)$ the spanning tree polytope $T(G)$ is the convex hull of incidence vectors of spanning trees of G, its dominant is the polyhedron $P(G) = T(G) + \mathbb{R}^E_+$ obtained by adding the nonnegative orthant. It has been proved in Nash-Williams (1961) and Tutte (1961) that $P(G)$ is defined by

$$x(\delta(V_1, \dots, V_p)) \geq p - 1 \quad \text{for every partition of } V, \tag{2.1}$$

$$x \geq 0. \tag{2.2}$$

Jünger and Pulleyblank (1995) have given an extended formulation for $P(G)$ as follows. Associate the variables x with the edges and the variables y with the nodes. The system below defines a polyhedron whose projection onto the variables x is $P(G)$. This will be proved at the end of this section. The node r is an arbitrary element of V.

$$x(\delta(S)) + y(S) \geq 2 \quad \text{if } r \notin S,\ S \subset V, \tag{2.3}$$

$$x(\delta(S)) + y(S) \geq 0 \quad \text{if } r \in S,\ S \subset V, \tag{2.4}$$

$$y(V) = 0, \tag{2.5}$$

$$x \geq 0. \tag{2.6}$$

So given a vector $\bar{x} \geq 0$ we can try to find a vector $\bar{y}$ such that $(\bar{x}, \bar{y})$ satisfies (2.3)–(2.6), or prove that $\bar{y}$ does not exist.

Let

$$f(S) = \begin{cases} 2 - \bar{x}(\delta(S)) & \text{if } r \notin S, \\ -\bar{x}(\delta(S)) & \text{if } r \in S, \end{cases}$$

for $\emptyset \neq S \subseteq V$. The function $-f$ is submodular for intersecting pairs, i.e.,

$$f(S) + f(T) \leq f(S \cup T) + f(S \cap T),$$

for $S \neq \emptyset \neq T,\ S, T \subseteq V,\ S \cap T \neq \emptyset$.

We are going to solve

$$\text{(2.7)}\qquad \begin{aligned}&\text{minimize} \quad y(V)\\ &\text{subject to}\\ &y(S) \geq f(S) \quad \text{for } S \subseteq V.\end{aligned}$$

Edmonds (1970) showed that the greedy algorithm solves this linear program. This algorithm, which we present below, produces also an optimal solution of the dual problem. We shall see that this gives a most violated partition inequality, if there is any. The dual problem is

$$\text{(2.8)}\qquad \begin{aligned}&\text{maximize} \quad \sum z_S f(S)\\ &\text{subject to}\\ &\sum \{z_S \mid u \in S\} = 1 \quad \text{for all } u \in V,\\ &z \geq 0.\end{aligned}$$

Given a vector y satisfying (2.7), a set S is called *tight* if $y(S) = f(S)$. The function $y(\cdot) - f(\cdot)$ is nonnegative and submodular for intersecting pairs. So if S and T are tight, and $S \cap T \neq \emptyset$, then $S \cap T$ and $S \cup T$ are also tight.

We start with $y(v_i) = 2\ \forall i$, and decrease the value of each $y(v_i)$ until a set becomes tight. We denote by $\mathscr{F}$ the family of tight sets with a positive dual variable. If we try to add S to $\mathscr{F}$ and there is a set $T \in \mathscr{F}$ with $S \cap T \neq \emptyset$, then we replace S and T by $S \cup T$. This is also tight.

Let $V = \{v_1, \ldots, v_n\}$. The algorithm follows:

Algorithm 1.
Step 0. Set $\bar{y}(v_i) \leftarrow 2$ for $i = 1, \ldots, n$; $k \leftarrow 1$; $\mathscr{F} \leftarrow \emptyset$.
Step 1. If v_k belongs to a set in $\mathscr{F}$ go to step 3, otherwise
set $\alpha \leftarrow f(\bar{S}) - \bar{y}(\bar{S}) = \max\{f(S) - \bar{y}(S) \mid v_k \in S\}$,
$\bar{y}(v_k) \leftarrow \bar{y}(v_k) + \alpha$,
$\mathscr{F} \leftarrow \mathscr{F} \cup \{\bar{S}\}$.
Step 2. While there are two sets S and T in $\mathscr{F}$ with $S \cap T \neq \emptyset$ do
$\mathscr{F} \leftarrow (\mathscr{F} \backslash \{S, T\}) \cup \{T \cup S\}$.
Step 3. Set $k \leftarrow k + 1$, if $k \leq n$ go to Step 1, otherwise stop.

The vector $\bar{y}$ is built so it satisfies (2.7), the family $\mathscr{F}$ defines a partition of V and $\bar{y}(S) = f(S)$ for every $S \in \mathscr{F}$. We set $\bar{z}_S = 1$, if $S \in \mathscr{F}$, and $\bar{z}_S = 0$ otherwise. We have

$$\bar{y}(V) = \sum \{\bar{y}(S) \mid S \in \mathscr{F}\} = \sum \{f(S) \mid S \in \mathscr{F}\} = \sum \{f(S)\bar{z}_S \mid S \subseteq V\}.$$

This proves that $\bar{y}$ and $\bar{z}$ are optimal solutions.

If the value of the optimum is 0 then $(\bar{x}, \bar{y})$ satisfies (2.3), (2.4), and (2.5). In this case we can pick any partition of V into $V_1, \ldots, V_p$, add the inequalities in (2.3), (2.4) associated with the sets $\{V_i\}$, and $-y(V) \geq 0$. We obtain a partition inequality. This shows that $\bar{x}$ satisfies all the partition inequalities.

Now assume that the value of the optimum is greater than 0. Let $\bar{z}$ be a 0-1 vector that satisfies the equations of (2.8), the family $\mathscr{G} = \{S \mid \bar{z}_S = 1\} = \{S_1, \ldots, S_p\}$ gives a partition of the set V and

$$\sum f(S)\bar{z}_S = 2(p-1) - 2\bar{x}(\delta(S_1, \ldots, S_p)),$$

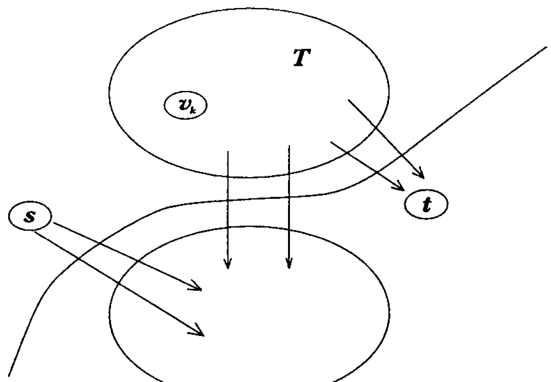

FIGURE 3.2. Structure of a cut

so

$$\frac{1}{2}\sum f(S)\bar{z}_S = (p-1) - \bar{x}(\delta(S_1,\ldots,S_p)).$$

Since $\bar{z}$ is an optimum of (2.8) it gives a most violated partition inequality. Because of this we have a solution of the separation problem not only for $b=-1$, but also for any $b\le -1$.

This procedure shows that (2.8) has an optimal integer solution, this is known as the *total dual integrality* of (2.7). It also shows that $\bar{x}$ satisfies (2.1)–(2.2) if and only if there is a vector $\bar{y}$ such that $(\bar{x},\bar{y})$ satisfies (2.3)–(2.6), so projecting the variables y in (2.3)–(2.6), gives (2.1)–(2.2).

3. Finding tight sets. It remains to show how to compute the number α in Step 1. Construct a directed graph $D=(N,A)$, where $N=V\cup\{s,t\}$ and $A=\{(i,j),(j,i)\mid ij\in E\}\cup\{(s,i),(i,t)\mid i\in V\}$. Define

$$\eta(i) = \bar{y}(i) \quad \text{for } i\in V,\ i\neq r,$$
$$\eta(r) = \bar{y}(r)+2,$$

define capacities

$$c(s,i)=-\eta(i),\ c(i,t)=0 \quad \text{if } \eta(i)<0, \quad i\neq v_k,\ i\in V,$$
$$c(i,t)=\eta(i),\ c(s,i)=0 \quad \text{if } \eta(i)\ge 0, \quad i\neq v_k,\ i\in V,$$
$$c(s,v_k)=\infty, \quad c(v_k,t)=\eta(v_k),$$
$$c(i,j)=c(j,i)=\bar{x}(i,j) \quad \text{for } ij\in E.$$

LEMMA 3.1. *Suppose that $\{s\}\cup T$ induces a cut separating s from t that has capacity λ; see Figure 3.2. Then*

$$\bar{y}(T)+\bar{x}(\delta(T)) = \begin{cases} \lambda+\sum\{\eta(v)\mid \eta(v)<0\}-2 & \text{if } r\in T,\\ \lambda+\sum\{\eta(v)\mid \eta(v)<0\} & \text{if } r\notin T.\end{cases}$$

PROOF. Suppose that $r\in T$. Then

$$\lambda = \sum\{-\eta(i)\mid i\notin T,\ \eta(i)<0\} + \bar{x}(\delta(T)) + \sum\{\eta(i)\mid i\in T,\ \eta(i)\ge 0\},$$

and

$$\begin{aligned}\lambda + \sum \{\eta(v) \mid \eta(v) < 0\} - 2 \\ &= \sum \{\eta(i) \mid i \in T,\ \eta(i) < 0\} + \bar{x}(\delta(T)) + \sum \{\eta(i) \mid i \in T,\ \eta(i) \geq 0\} - 2 \\ &= \bar{y}(T) + \bar{x}(\delta(T)).\end{aligned}$$

The case $r \notin T$ is analogous. □

Therefore, if β is the minimum capacity of a cut separating s from t, then the value α is

$$2 - \beta - \sum \{\eta(v) \mid \eta(v) < 0\}.$$

Suppose now that we have solved problem (2.7), and that we add a new vertex to the graph. We are going to show that resolving (2.7) takes just one min-cut calculation. This will be used in the later sections.

Lemma 3.3. *After solving problem* (2.7), *if we add a new vertex*, *it will take one minimum cut calculation to resolve* (2.7).

Proof. Suppose that $\bar{y}$ is the solution of (2.7), for the graph $G = (V, E)$. Suppose that we add the vertex w. Define

$$\bar{\bar{y}}(u) = \bar{y}(v) - \bar{x}(wv),$$

for all $v \in V$, and $\bar{\bar{y}}(w) = 2$. It is easy to see that $\bar{\bar{y}}$ satisfies the inequalities of (2.7), and every set that was tight before will remain tight. Thus the only component that can be modified by the greedy algorithm is $y(w)$, this takes one minimum cut problem. □

4. Cunningham's approach. Cunningham (1985) used the method below for the case $b \leq -1$. We present his method because it will be used in §5 to prove submodularity. Let f be the rank function of the graphic matroid. It satisfies

$$f(\emptyset) = 0, \tag{4.1}$$

$$f(A) \leq f(B) \quad \text{if } A \subseteq B \subseteq E, \tag{4.2}$$

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B) \quad \text{if } A, B \subseteq E. \tag{4.3}$$

Then he proposes to solve

$$\begin{aligned} &\text{maximize } y(E) \\ &\text{subject to} \\ &y(R) \leq f(R) \quad \text{if } R \subseteq E, \\ &0 \leq y \leq x. \end{aligned} \tag{4.4}$$

For $A \subseteq E$, let $\bar{A} = E \backslash A$. We have

$$y(E) = y(A) + y(\bar{A}) \leq x(A) + f(\bar{A}).$$

We apply the greedy algorithm to (4.4), cf. Edmonds (1970). Starting with a feasible vector $\tilde{y}$, (for instance $\tilde{y} = 0$), we increase one component until one inequality becomes tight, then we continue with another component, and so on. Let $\bar{y}$ be the final vector. Then for any $e \in E$ either $y(e) = x(e)$ or e is in a tight set. An edge set is called *tight* if its corresponding inequality in (4.4) is tight. Let $\bar{A}$ be the union of the tight sets. This is also tight, by submodularity. Notice that this time the tight sets do not have to intersect as was required in §2. Thus

$$\bar{y}(E) = \bar{y}(A) + \bar{y}(\bar{A}) = x(A) + f(\bar{A}).$$

This shows that the solution of (4.4) gives the minimum of

$$x(A) + f(\bar{A}).$$

By taking $A = \emptyset$, we have that the minimum is at most $n-1$. So it is equivalent to look for the minimum of

$$x(A) + f(\bar{A}) - (n-1) = x(E) - (n-1) + f(\bar{A}) - x(\bar{A})$$

that is nonpositive.

For $B \subseteq V$, let $\gamma(B)$ be the set of edges with both endnodes in B. We need the following lemma.

Lemma 4.5. *It is enough to take sets* $R = \gamma(B)$ *in* (4.4).

Proof. Let R be a set of edges and consider

$$y(R) \le f(R). \tag{4.6}$$

Let r be the number of nodes covered by R and p the number of connected components defined by R. Then $f(R) = r - p$. If $p > 1$ then (4.6) can be obtained as the sum of inequalities associated with the connected components. We can assume that $p = 1$. Let B be the set of nodes covered by R. If $R \neq \gamma(B)$, then (4.6) is implied by $y(\gamma(B)) \le f(\gamma(B)) = f(R)$ and $y \ge 0$. □

This lemma shows that $\bar{A}$ is the union of sets of type $\gamma(B)$. We are going to obtain the minimum of

$$\sum_{i=1}^{p} [(|S_i| - 1) - x(\gamma(S_i))] + x(E) - (n-1),$$

where p is the number of connected components in $\bar{A}$.

Let us renumber the nodes so that $A = \{v_1, v_2, \ldots\}$. By taking $S_{p+i} = \{v_i\}$ for $v_i \notin S_j$, $1 \le j \le p$, we have

$$x(\delta(S_1, \ldots, S_k)) - (k-1),$$

where $k = p + n - \sum_{i=1}^{p} |S_i|$.

Applying the greedy algorithm to (4.4) would require $|E|$ iterations. Each time finding a tight set reduces to solving a min-cut problem, so this method requires $|E|$ min-cut problems, cf. Cunningham (1985).

5. The case $b > -1$**.** Now we study the general case. We shall see that it reduces to minimizing a symmetric submodular function. The evaluation of this submodular function will require the algorithm of §2.

We could not find a simple extension of the methods in the preceding sections to the case $b > -1$. When we tried to modify them we would lose submodularity. Finally we decide to study the problem

$$\text{minimize } x(\delta(S_1, \ldots, S_p)) - p, \tag{5.1}$$

with $p \ge 2$. We call this the *multicut problem*.

This is equivalent to minimize

$$g(S) = x(\delta(S)) - 2 + \min\{x(\delta_S(T_1, \dots, T_k)) - (k-1)\}, \tag{5.2}$$

where $S \subset V$ and $\{T_i\}$ is a partition of S. Notice that $\{T_i\}$ could be the trivial partition, i.e., $k = 1$. In this section we use $\delta_S(T_1, \dots, T_k)$ to denote the set of edges between different sets T_i. The resulting partition is $\bar{S}, T_1, \dots, T_k$.

We want to prove that g is submodular. For that all we need is to prove that

$$h(S) = \min\{x(\delta_S(T_1, \dots, T_k)) - (k-1)\} \tag{5.3}$$

is submodular (as a function of S). As we saw in §4, this is $1 - |S|$ plus the value of the optimum of (4.4), where $E = \gamma(S)$.

So we have to consider (4.4) for $E = \gamma(S)$, $E = \gamma(T)$, $E = \gamma(S \cup T)$, and $E = \gamma(S \cap T)$. Suppose that $\bar{y}$ was obtained after applying the greedy algorithm for $E = \gamma(S \cap T)$. We can extend $\bar{y}$ to a solution for $E = \gamma(S \cup T)$, denote this by $\bar{\bar{y}}$. Now denote by $\bar{\bar{y}}_S$ the vector obtained from $\bar{\bar{y}}$ by setting to zero all components not in $\gamma(S)$. We can extend $\bar{\bar{y}}_S$ to a solution for $E = \gamma(S)$. We can proceed similarly for $E = \gamma(T)$. This shows that

$$h(S \cup T) + h(S \cap T) \leq h(S) + h(T).$$

The submodularity of h also follows from observing that h is one plus the *Dilworth truncation* of $x(\delta(S)) - 1$. See Lovász (1983) for a definition of this concept.

We have proved that the minimum of (5.1) can be obtained by minimizing the submodular function g. Queyranne (1995) gave an algorithm that takes $O(n^3)$ evaluations of f when this is submodular and symmetric. This means that $f(S) = f(\bar{S})$ for all $S \subseteq V$. Since g is not symmetric, we define $g'(S) = h(S) + 1/2x(\delta(S))$ and look for the minimum of $g''(S) = g'(S) + g'(\bar{S})$, for $\emptyset \neq S \subset V$. It is clear that g'' is symmetric, and submodular because it is the sum of submodular functions.

We have that

$$g'(S) = \frac{1}{2}x(\delta(S)) + x(\delta_S(T_1, \dots, T_k)) - (k-1)$$

for some partition of S, and

$$g'(\bar{S}) = \frac{1}{2}x(\delta(\bar{S})) + x(\delta_{\bar{S}}(U_1, \dots, U_l)) - (l-1)$$

for some partition of $\bar{S}$. Then

$$g''(S) = x(\delta(T_1, \dots, T_k, U_1, \dots, U_l)) - (k+l) + 2.$$

Thus the minimum in (5.1) is equal to the minimum of g'' minus 2.

6. Queyranne's algorithm. For a symmetric submodular function f, Queyranne (1995) gave the algorithm below. It generalizes the minimum cut algorithm of Nagamochi and Ibaraki (1992) as simplified by Stoer and Wagner (1994) and Frank (1994). Following Queyranne's notation we use $S + u$ to denote $S \cup \{u\}$.

Algorithm A

Step 0. Start with $W_0 = \emptyset$, $i = 0$.

Step 1. For all $u \notin W_i$ set $k(u) = f(W_i + u) - f(u)$. Let $k(u_{i+1}) = \min\{k(u)\}$.

Step 2. Set $W_{i+1} \leftarrow W_i + u_{i+1}$, set $i \leftarrow i + 1$. If $i = n$ stop, otherwise go to Step 1.

He proved that

$$f(u_n) = \min\{f(S) \mid S \text{ separates } u_n \text{ and } u_{n-1}\}, \tag{6.1}$$

The next step is to identify u_n and u_{n-1}, apply Algorithm A, and continue until we are left with two elements.

The proof of (6.1) relies on the lemma below.

LEMMA 6.2. Queyranne (1995). *For* $1 \le i \le n-1$, $y \in V \backslash W_i$ *and* $X \subseteq W_{i-1}$,

$$f(W_i) + f(y) \le f(W_i \backslash X) + f(X + y).$$

If we apply this algorithm with a function f that is not symmetric, we obtain the minimum of $f(S) + f(\bar{S})$; this follows from Lemma 6.2. In our case we should use the function g' defined in the last section.

Each application of Algorithm A requires $O(n^2)$ evaluations of f. Since Algorithm A is used $n-1$ times, we need $O(n^3)$ evaluations of f. In our case, one evaluation of the function g' would take $O(n)$ min-cut problems, thus the straightforward implementation of this method would require $O(n^4)$ min-cut problems. However each evaluation in Step 1 is of the type $f(W_i + u)$ where $f(W_i)$ is already known. We have seen at the end of §3 that this takes only one min-cut calculation. Thus the entire algorithm requires $O(n^3)$ min-cut problems.

7. Related problems. Let us turn our attention to ratio objectives. The problem

$$\text{minimize } \frac{x(\delta(S_1, \ldots, S_p))}{p-1},$$

was studied in Cunningham (1985) and Cheng and Cunningham (1994). Chvátal (1973) showed that the minimum of

$$\frac{x(\delta(S_1, \ldots, S_p))}{p}$$

is given by a minimum cut. Now consider

$$\text{minimize } \frac{x(\delta(S_1, \ldots, S_p)) + a}{p+b},$$

where $p + b > 0$, for $p > 1$. Cunningham (1985) pointed out that his method could be extended to the case $b \le a - 1$. The general case can be solved using Dinkelbach's algorithm' see Dinkelbach (1967), and problem (5.1) as follows. Given a partition $\{T_i\}$, compute

$$\lambda = \frac{x(\delta(T_1, \ldots, T_p)) + a}{p+b},$$

and look for the minimum of

$$x(\delta(S_1, \ldots, S_p)) + a - \lambda(p+b).$$

If this minimum is negative replace $\{T_i\}$ by the new partition, otherwise stop. This procedure takes at most n iterations cf. Cunningham (1985).

Now consider the *k-cut problem*. We want to minimize

$$x(\delta(S_1, \ldots, S_k))$$

for a fixed value of k. Goldschmidt and Hochbaum (1994) gave an $O(n^{k^2})$ algorithm, and they showed that it is NP-Hard if k is part of the input. Karger and Stein (1996) gave an algorithm with expected running time $O(n^{2k} \log^2 n)$.

For any number $\lambda > 0$ we can obtain a lower bound

$$f(\lambda) = \min\{x(\delta(S_1, \dots, S_p)) - \lambda(p - k)\}, \tag{7.1}$$

and then look for the maximum of f. This is a piecewise linear concave function with at most n breakpoints, so this maximum is easy to find. Then for any value of λ such that the solution of (7.1) has $p \geq k$, we can obtain an upper bound by shrinking some sets in the partition, notice that for large values of λ we should have $p \geq k$. The details of this and some computational experiments appear in Barahona (1996b).

8. Applications of partition inequalities. In this section we mention some of the optimization problems that involve partition inequalities. In general they are useful to deal with connectivity constraints. As mentioned before, the dominant of the spanning tree polytope is defined by nonnegativity and

$$x(\delta(V_1, \dots, V_p)) \geq p - 1,$$

for every partition of V', see Nash-Williams (1961) and Tutte (1961). These inequalities have been used inside an algorithm for the *Network Loading Problem* in Barahona (1996a).

Consider the problem of finding a minimum 2-edge connected spanning subgraph. The inequalities below are valid for the associated polytope.

$$x(\delta(S)) \geq 2 \quad \text{for } S \subset V, \tag{8.1}$$

$$0 \leq x(e) \leq 1 \quad \text{for all } e. \tag{8.2}$$

In Mahjoub (1994) another family of valid inequalities was introduced as follows. Consider a partition of V into $V_0, V_1, \dots, V_p$, and let $F \subseteq \delta(V_0)$ with $|F| = 2k + 1$. If we add the inequalities

$$x(\delta(V_i)) \geq 2, \quad 1 \leq i \leq p,$$

$$-x(e) \geq -1, \quad e \in F,$$

$$x(e) \geq 0, \quad e \in \delta(V_0) \backslash F,$$

we obtain

$$2x(\Delta) \geq 2p - 2k - 1,$$

where $\Delta = \delta(V_0, \dots, V_p) \backslash F$. Dividing by 2 and rounding up the right-hand side we obtain

$$x(\Delta) \geq p - k. \tag{8.3}$$

These are called *F-partition inequalities.* Notice that if $|F| = 2k$, then inequality (8.3) is implied by (8.1) and (8.2).

If the sets $V_i, 1 \leq i \leq p$ are singletons, then these are blossom inequalities for b-matching and the separation problem can be solved with the algorithm of Padberg and Rao (1982). We do not have a polynomial algorithm for the separation problem associated with F-partition inequalities: However if F is fixed, we can delete this set of edges and the problem reduces to the case studied in §§2, 3, and 4.

For 2-connected subgraph problems, we are in the process of implementing a cutting plane algorithm based on F-partitions. The results of this will be reported elsewhere. Since the Travelling Salesman Polytope is a face of the 2-connected subgraph polytope, inequalities (8.3) are also valid for the TSP. We found it interesting that we could solve to optimality some small Travelling Salesman problems using only (8.1), (8.2), and (8.3).

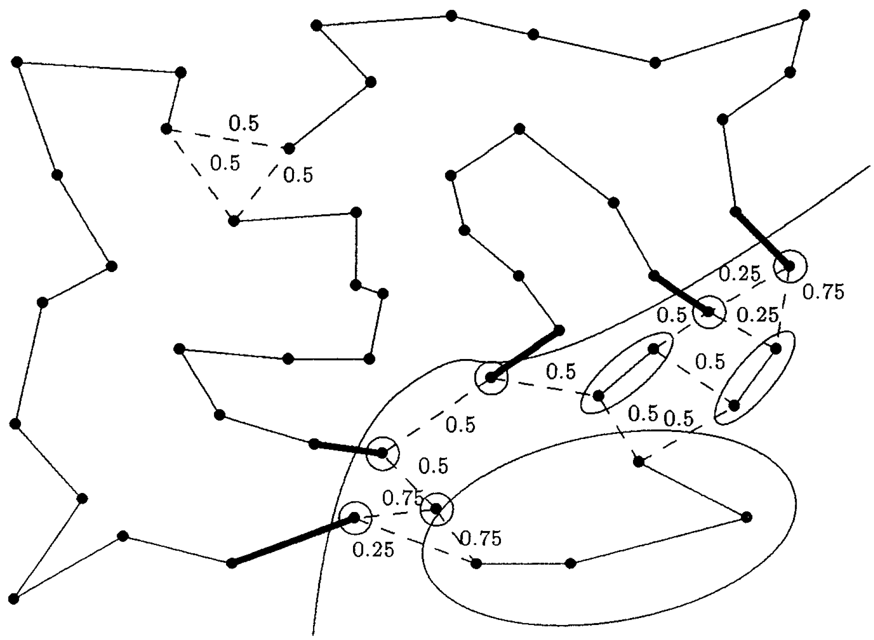

FIGURE 8.4. A fractional TSP solution

In Figure 8.4 we display a fractional vector obtained when solving a TSP with 51 nodes from TSPLIB; cf. Reinelt (1991). We chose a set F among the edges having the value 1 and found a violated partition inequality described below. The dashed lines indicate fractional values, the solid lines indicate the value 1. We draw with bold lines the edges in F. The nodes in the upper half belong to V_0, the other sets V_i are drawn with ellipses and circles. In this case the constraint is $x(\Delta) \geq 7$, however the present vector gives 6.5 for the left-hand side. This fractional vector could also be cut with some of the well-known comb inequalities. Whether F-partitions are useful for larger TSPs remains the subject of further research.

In Grötschel et al. (1992b) they consider inequalities of the same type for a more general model. Inequalities (8.3) have also been studied in Barahona and Mahjoub (1995). Fonlupt and Mahjoub (1999) study the extreme points of the polytope $P(G)$ defined by

$$x(\delta(S)) \geq 2 \quad \text{for } S \subset V, \tag{8.5}$$

$$1 \geq x \geq 0. \tag{8.6}$$

They introduce the notion of *critical* noninteger extreme points of $P(G)$. Roughly speaking, a noninteger extreme point of $P(G)$ is critical if the set of edges with fractional values does not strictly contain the set of fractional edges of another extreme point of $P(G)$. They showed that any critical noninteger extreme point violates an inequality of type (8.3), and this can be found in polynomial time.

Now consider minimum k-edge connected spanning subgraphs, for $k > 1$. Grötschel et al. (1992) introduced the constraints

$$x(\delta(V_1, \ldots, V_p)) \geq \left\lceil \frac{kp}{2} \right\rceil. \tag{8.7}$$

These are redundant if kp is even. Also in Grötschel et al. (1992b) they showed that the separation problem for a more general type of partition inequalities is NP-Hard. Cutting plane algorithms along with computational results with these inequalities have been presented in Grötschel et al. (1992b) and Grötschel et al. (1995). They used heuristics for the separation problem, and concluded that partition inequalities are very useful in this context. They also pointed out the need for better separation routines. An approximation for this is to minimize

$$x(\delta(V_1, \ldots, V_p)) - \frac{kp}{2}.$$

This can be done as in §5.

Chopra (1994) considers the minimum k-edge connected spanning subgraphs when multiple copies of an edge may be used, and k is odd. An obvious system of inequalities is

$$x(\delta(S)) \geq k \quad \text{for } S \subset V, \tag{8.8}$$

$$x \geq 0. \tag{8.9}$$

Then he introduces the following partition inequalities:

$$x(\delta(V_1, \ldots, V_p)) \geq \left\lceil \frac{k}{2} \right\rceil p - 1 \tag{8.10}$$

for all partition $V_1, \ldots, V_p$ of V such that $G(V_i)$ is connected for $i = 1, \ldots, p$. These constraints are valid for the associated polyhedron $P_k(G)$ when the graph G is outerplanar and k is odd. He showed that in this case, $P_k(G)$ is defined by (8.8), (8.9), and (8.10). He also conjectured that this result remains true in the more general class of series-parallel graphs. Recently Didi Biha and Mahjoub (1996) proved this conjecture as a consequence of a more general result. They showed that (8.8), (8.9), and (8.10) define the k-edge connected subgraph polytope for series-parallel graphs, for all k. The conjecture was also proved independently by Chopra and Stoer (1996). Inequalities (8.10) can be separated with the method of §5.

For k-node connected subgraphs the inequalities below have been presented in Grötschel et al. (1992b). Let $Z \subset V$ be a node set with $|Z| = t \leq k - 1$. Let $V_1, \ldots, V_p$ be a partition of $V \backslash Z$. Then the following inequalities are valid:

$$x(\delta(V_1, \ldots, V_p)) \geq \begin{cases} p - 1 & \text{if } k - t = 1, \\ \left\lceil \dfrac{p(k-t)}{2} \right\rceil & \text{if } k - t \geq 2. \end{cases}$$

If $k - t = 1$ they can be separated with the method of §2. If $k - t$ is positive and even, then they can be separated with the method of §5.

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