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# The $k$-edge connected subgraph problem I: Polytopes and critical extreme points

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**Abstract**

In this paper we consider the linear relaxation of the $k$-edge connected subgraph polytope, $P(G, k)$, given by the trivial and the so-called cut inequalities. We introduce an ordering on the fractional extreme points of $P(G, k)$ and describe some structural properties of the minimal extreme points with respect to that ordering. Using this we give sufficient conditions for $P(G, k)$ to be integral.


*Keywords:* Polytope; Cut; $k$-Edge connected graph; Critical extreme point

## 1. Introduction and notation

A graph $G = (V, E)$ is called *k-edge connected* (where $k$ is a positive integer) if for every pair of nodes $i, j \in V$, there are at least $k$ edge disjoint paths between $i$ and $j$. Given a graph $G = (V, E)$ and a weight function $w$ on $E$ that associates with an edge $e \in E$, the weight $w(e) \in \mathbb{R}$, the *k-edge connected subgraph problem* ($k$ECSP for short) is to find a $k$-edge connected spanning subgraph $H = (V, F)$ of $G$ such that $\sum_{e\in F} w(e)$ is minimum.

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The $k$ECSP arises in the design of reliable communication networks. In fact, with the introduction of fiber optic technology in telecommunication, designing a minimum cost survivable network has become a major objective in telecommunication industry. Survivable networks have to satisfy some connectivity requirements, this means that they are still functional after the failure of certain links. As pointed out in [23], the topology that seems to be very efficient (and needed in practice) is that corresponding to networks that survive after the loss of $k-1$ or less edges, for some $k \geqslant 2$ ($k$ depends on the level of reliability required in the network). These networks remain connected after the removal of $k-1$ or less edges, in other words, $k$-edge connected networks. For more details on the general survivable network design problem see [17–21].

The $k$ECSP is NP-hard for $k \geqslant 2$. Ko and Monma [23] devise heuristics for obtaining near optimal solution for the $k$ECSP. These extend heuristics previously developed by Monma and Shallcross [26] for the 2ECSP to the $k$ECSP. For $k = 1$, the problem reduces to the minimum spanning tree and thus can be solved in polynomial time.

Given a graph $G = (V, E)$ and an edge subset $F \subseteq E$, the 0–1 vector $x^F \in \mathbb{R}^E$ such that $x^F(e) = 1$ if $e \in F$ and $x^F(e) = 0$ if $e \in E \backslash F$ is called the *incidence vector* of $F$. The *convex hull* of the incidence vectors of the edge sets of the $k$-edge connected subgraphs of $G$, denoted by $k$ECSP($G$), is called the *$k$-edge connected subgraph polytope* of $G$.

Let $G = (V, E)$ be a graph. Given $w : E \mapsto \mathbb{R}$ and $F$ a subset of $E$, $w(F)$ will denote $\sum_{e \in F} w(e)$. For $W \subseteq V$, we let $\overline{W} = V \backslash W$. If $W \subset V$ is a node subset of $G$, then the set of edges that have only one node in $W$ is called a *cut* and denoted by $\delta(W)$. We will write $\delta(v)$ for $\delta(\{v\})$. A cut $\delta(v)$, $v \in V$, will be called a *degree cut*. An edge cutset $F \subseteq E$ of $G$ is a set of edges such that $F = \delta(S)$ for some non-empty set $S \subset V$.

If $x^F$ is the incidence vector of the edge set $F$ of a $k$-edge connected spanning subgraph of $G$, then $x^F$ satisfies the following inequalities:

$$x(e) \geqslant 0 \qquad \forall\, e \in E, \tag{1}$$

$$x(e) \leqslant 1 \qquad \forall\, e \in E, \tag{2}$$

$$x(\delta(W)) \geqslant k \quad \forall\, W \subset V,\ W \neq \emptyset. \tag{3}$$

Conversely, any integer solution of the system defined by inequalities (1)–(3) is the incidence vector of the edge set of a $k$-edge connected subgraph of $G$. Constraints (1) and (2) are called *trivial inequalities* and constraints (3) are called *cut inequalities*. We will denote by $P(G, k)$ the polytope given by inequalities (1)–(3).

Using network flows [12,13], one can compute in polynomial time a minimum cut in a weighted undirected graph. Hence the separation problem for inequalities (3) (i.e. the problem that consists of finding whether a given vector $\bar{x} \in \mathbb{R}^E$ satisfies inequalities (3), and if not to find an inequality which is violated by $\bar{x}$) can be solved in polynomial time. This implies by the ellipsoid method [16] that the $k$ECSP can be solved in polynomial time on graphs $G$ for which $k\mathrm{ECSP}(G) = P(G, k)$. For $k = 2$, Mahjoub [25] called these graphs *perfectly 2-edge connected graphs*. In what follows we call a graph *perfectly k-edge connected* (perfectly-$k$EC) if $k\mathrm{ECSP}(G) = P(G, k)$.

In [14], Fonlupt and Mahjoub study the extreme points of $P(G, 2)$. They introduce an ordering on these extreme points and give necessary conditions for a fractional extreme point to be minimal with respect to that ordering. And as a consequence, they obtain a characterization of the perfectly 2-edge connected graphs. This paper extends some of the results of [14] to $k$-edge connected graphs.

The polytope $k\mathrm{ECSP}(G)$ and its linear relaxation $P(G, k)$ have been the subject of extensive research in the past years. Grötschel and Monma [17] and Grötschel et al. [18–21] study the $k\mathrm{ECSP}(G)$ within the framework of a more general model related to the design of telecommunication survivable networks. In particular, Grötschel and Monma describe several basic facets of the polytope associated with that model. And Grötschel et al. [18,20] study further facets and polyhedral aspects of that model, and devised cutting plane algorithms along with some experimental results are discussed [19]. A complete survey of that model can be found in [27]. In [5], Chopra studies the $k$-edge connected subgraph problem for $k$ odd, when multiple copies of an edge may be used. In particular, he characterizes the associated polyhedron for the class of outerplanar graphs (a graph is outerplanar if it can be drawn in the plane as one cycle with noncrossing chords). This polyhedron has been previously studied by Cornuéjols et al. [6]. They showed that when the graph is series–parallel (a graph is series–parallel if it can be created from a single edge by iterative application of two operations: (i) addition of a parallel edge, and (ii) subdivision of an edge) and $k = 2$, the polyhedron is completely described by the nonnegativity and the cut inequalities. In [10], Didi Biha and Mahjoub give a complete description of the $k\mathrm{ECSP}(G)$ for all $k$, on series–parallel graphs. In particular they show that if $G$ is series–parallel and $k$ is even, then $k\mathrm{ECSP}(G) = P(G, k)$, implying that series–parallel graphs are perfectly-$k$EC.

Much work has been done on $2\mathrm{ECSP}(G)$. In [24] Mahjoub shows that if $G$ is series–parallel then $2\mathrm{ECSP}(G)$ is completely described by the trivial and the cut inequalities. This has been generalized by Baïou and Mahjoub [1] to the Steiner 2-edge connected subgraph polytope, and by Didi Biha and Mahjoub [11] to the Steiner $k$-edge connected subgraph polytope for $k$ even. Mahjoub [24] introduced a general class of valid inequalities for $2\mathrm{ECSP}(G)$. Boyd and Hao [4] describe a class of "comb inequalities" which are valid for $2\mathrm{ECSP}(G)$. This class, as well as that introduced by Mahjoub, are special cases of a more general class of inequalities given by Grötschel et al. [20] for the general survivable network polytope. In [2] Barahona and Mahjoub characterise the polytope $2\mathrm{ECSP}(G)$ for the class of Halin graphs. Kerivin et al. [22] describe a general class of valid inequalities for $2\mathrm{ECSP}(G)$ that generalize the so-called $F$-partition inequalities [24], and introduce a Branch&Cut algorithm for 2ECSP based on these inequalities together with the trivial and the cut inequalities. In [3] Bienstock et al. describe structural properties of the optimal solutions of $k$ECSP when the weight function satisfies the triangle inequalities (i.e. $w(e_1) \leqslant w(e_2) + w(e_3)$ for every three edges $e_1, e_2, e_3$ defining a triangle). In particular, they show that every node of minimum $k$-edge connected subgraph has degree $k$ or $k + 1$. In [7] Coullard et al. studied the Steiner 2-node connected subgraph problem. In [8] they devise a linear time algorithm for this problem on special classes of graphs. And in [9], they characterize the dominant of the polytope associated with this problem on the graphs which do not have $W_4$ (the wheel on 4 nodes) as a minor. In [15], Fonlupt and Naddef characterize the class for which the system given by inequalities (1) and (3), when $k = 2$, defines the convex hull of the incidence vectors of the tours of $G$ (a tour is a cycle going at least once through each node).

The paper is organized as follows. In Section 2 we introduce some reduction operation that preserve perfectly-$k$EC property. In Section 3 we introduce an ordering on the extreme points of $P(G, k)$ and discuss some structural properties of the minimal extreme points with respect to that ordering. In Section 4 we describe sufficient conditions for a graph to be perfectly-$k$EC. In Section 5 we give some concluding remarks.

The rest of this section is devoted to more definition and notation. The graphs we consider are finite, undirected, loopless and connected. A graph is denoted by $G = (V, E)$ when $V$ is the *node set* and $E$ is the *edge set*. If $e \in E$ is an edge with endnodes $u$ and $v$, we also write $uv$ to denote $e$. Given $W$, $W'$ two disjoint subsets of $V$, $[W, W']$ will denote the set of edges of $G$ having one endnode in $W$ and the other one in $W'$. For $F \subseteq E$, $V(F)$ will denote the set of nodes of the edges of $F$. For $W \subset V$, we denote by $E(W)$ the set of edges having both endnodes in $W$, and by $G(W)$ the subgraph induced by $W$. We also denote by $G\backslash W$ the graph obtained by deleting $W$ and the edges incident to the nodes of $W$, and by $G/W$ the graph obtained by contracting the nodes in $W$ to a new node (retaining multiple edges). Given an edge $e = uv \in E$, *contracting* $e$ consists of deleting $e$, identifying $u$ and $v$ and of preserving all the adjacencies. Contracting a set of edges $F \subset E$ consists of contracting all the edges of $F$. If $G$ is a graph and $e \in E$ is an edge of $G$, then $G - e$ will denote the graph obtained from $G$ by removing $e$. Given a solution $\bar{x}$ of $P(G, k)$, an inequality $ax \geqslant \alpha$ is said to be *tight* for $\bar{x}$ if $a\bar{x} = \alpha$.

## 2. Reduction operations

In this section we describe three operations on graphs that preserve the perfectly-$k$EC property. The first one consists of just removing an edge.

**Lemma 2.1.** *Let $G = (V, E)$ be a graph and $f$ an edge of $E$. If $G$ is perfectly-kEC and $G - f$ is k-edge connected, then $G - f$ is perfectly-kEC.*

**Proof.** Suppose that $G - f$ is not perfectly-$k$EC, and let $x$ be an extreme point of $P(G - f, k)$ which is fractional. Let $\bar{x} \in \mathbb{R}^E$ such that

$$\bar{x}(e) = \begin{cases} x(e) & \text{if } e \neq f, \\ 0 & \text{if } e = f. \end{cases}$$

Thus $\bar{x}$ is an extreme point of $P(G, k)$. Since $\bar{x}$ is fractional, this contradicts the fact that $G$ is perfectly-$k$EC. □

**Lemma 2.2.** *Let $G = (V, E)$ be a graph and $W$ a node subset of $V$ such that $G(W)$ is $k$-edge connected. If $G$ is perfectly-kEC, then $G/W$ is perfectly-kEC.*

**Proof.** Suppose that $P(G/W, k)$ has a fractional extreme point, say $\bar{x}$. Let $\bar{x}' \in \mathbb{R}^E$ be the solution given by

$$\bar{x}'(e) = \begin{cases} \bar{x}(e) & \text{if } e \in E\backslash E(W), \\ 1 & \text{if } e \in E(W). \end{cases}$$

Clearly, $\bar{x}' \in P(G, k)$. Moreover, it is not hard to see that $\bar{x}'$ is an extreme point of $P(G, k)$. Since $\bar{x}'$ is fractional, this is a contradiction. □

**Lemma 2.3.** *Let $G = (V, E)$ be a perfectly-kEC graph and $W$ a node subset of $V$ with $|W| \geqslant 2$. If $|\delta(W)| = k + t$ $(t \geqslant 0)$ and $G/\overline{W}$ is $(k + t)$-edge connected, then $G/W$ is perfectly-kEC.*

**Proof.** Suppose, on the contrary, that $G/W$ is not perfectly-$k$EC, and let $\bar{x}$ be a fractional extreme point of $P(G/W, k)$. Thus $\bar{x}$ is the unique solution of a subsystem $S(\bar{x})$ of $P(G/W, k)$, when the inequalities are replaced by equations. Let $\bar{x}' \in \mathbb{R}^E$ be the solution given by

$$\bar{x}'(e) = \begin{cases} \bar{x}(e) & \text{if } e \in E\backslash E(W), \\ 1 & \text{if } e \in E(W). \end{cases}$$

In what follows we are going to show that $\bar{x}'$ is an extreme point of $P(G, k)$. To this end, let us first show that $\bar{x}'$ is a solution of $P(G, k)$. Let $U \subset V$. If either $U \subseteq \overline{W}$ or $W \subseteq U$, then $\bar{x}'(\delta(U)) = \bar{x}(\delta(U)) \geqslant k$. So, let us suppose first that $U \subset W$. We have

$$\bar{x}(\delta(W)) = \bar{x}([U, \overline{W}]) + \bar{x}([W\backslash U, \overline{W}]) \geqslant k. \tag{4}$$

As $|\delta(W)| = k + t$ and $G/\overline{W}$ is $(k + t)$-edge connected, this yields

$$|[U, \overline{W}]| + |[W\backslash U, \overline{W}]| = k + t,$$
$$|[U, \overline{W}]| + |[U, W\backslash U]| \geqslant k + t.$$

Thus $|[U, W\backslash U]| \geqslant |[W\backslash U, \overline{W}]|$. As $\bar{x}'(e) = 1$ for all $e \in E(W)$ and $\bar{x}'(e) \leqslant 1$ for all $e \in [W\backslash U, \overline{W}]$, it follows that

$$\bar{x}'([U, W\backslash U]) \geqslant \bar{x}'([W\backslash U, \overline{W}]). \tag{5}$$

By (4) and (5) we obtain that

$$\begin{aligned}\bar{x}'(\delta(U)) &= \bar{x}'([U, \overline{W}]) + |[U, W\backslash U]| \\ &\geqslant \bar{x}'([U, \overline{W}]) + \bar{x}'([W\backslash U, \overline{W}]) \\ &\geqslant k.\end{aligned}$$

Suppose now that $U_1 = U \cap W \neq \emptyset$, $U_2 = U \cap \overline{W} \neq \emptyset$, $W\backslash U \neq \emptyset$ and $\overline{W}\backslash U \neq \emptyset$.

Let

$$E_1 = \{e \in [U_1, \overline{W}] \mid \bar{x}(e) < 1\},$$
$$E_2 = \{e \in [W\backslash U, \overline{W}] \mid \bar{x}(e) < 1\},$$

and

$$t_i = |E_i|, \quad \alpha_i = \sum_{e \in E_i} \bar{x}(e), \quad i = 1, 2,$$
$$l_1 = |[U_1, \overline{W}]|,$$
$$l_2 = |[W\backslash U, \overline{W}]|.$$

We have

$$l_1 + l_2 = k + t, \tag{6}$$
$$\bar{x}'([U_1, \overline{W}]) = l_1 - t_1 + \alpha_1, \tag{7}$$
$$\bar{x}'([W\backslash U, \overline{W}]) = l_2 - t_2 + \alpha_2. \tag{8}$$

On the other hand, as $G/\overline{W}$ is $(k + t)$-edge connected and $\bar{x}'(e) = 1$ for all $e \in E(W)$, the following hold:

$$\bar{x}'([U_1, \overline{W}]|) + \bar{x}'([U_1, W\backslash U]) \geqslant k + t + \alpha_1 - t_1, \tag{9}$$
$$\bar{x}'([W\backslash U, \overline{W}]) + \bar{x}'([W\backslash U, U_1]) \geqslant k + t + \alpha_2 - t_2. \tag{10}$$

Moreover, since $\bar{x} \in P(G, k)$ and $\bar{x}'(e) = \bar{x}(e)$ for all $e \in E\backslash E(W)$, we have

$$\bar{x}(\delta(U_2)) = \bar{x}'([U_1, U_2]) + \bar{x}'([W\backslash U, U_2]) + \bar{x}'([U_2, \overline{W}\backslash U]) \geqslant k, \tag{11}$$
$$\bar{x}(\delta(\overline{W}\backslash U)) = \bar{x}'([U_1, \overline{W}\backslash U]) + \bar{x}'([W\backslash U, \overline{W}\backslash U]) + \bar{x}'([U_2, \overline{W}\backslash U]) \geqslant k. \tag{12}$$

From (7), (9) and (8), (10) we respectively get

$$\bar{x}'([U_1, W\backslash U]) \geqslant k + t - l_1, \tag{13}$$
$$\bar{x}'([U_1, W\backslash U]) \geqslant k + t - l_2. \tag{14}$$

Also from (11) and (12) we obtain that

$$\begin{aligned}2\bar{x}'([U_2, \overline{W}\backslash U]) &\geqslant 2k - \bar{x}'([U_1, U_2]) - \bar{x}'([W\backslash U, U_2]) \\ &\quad -\bar{x}'([U_1, \overline{W}\backslash U]) - \bar{x}'([W\backslash U, \overline{W}\backslash U]) \\ &= 2k - \bar{x}'([U_1, \overline{W}]) - \bar{x}'([W\backslash U, \overline{W}]).\end{aligned}$$

By (7) and (8), this yields

$$2\bar{x}'([U_2, \overline{W}\backslash U]) \geqslant 2k - l_1 - l_2 + t_1 + t_2 - \alpha_1 - \alpha_2. \tag{15}$$

Combining (6) and (13)–(15), we get

$$\bar{x}'([U_1, W\backslash U]) + \bar{x}'([U_2, \overline{W}\backslash U]) \geqslant k + \frac{(t_1 + t_2) - (\alpha_1 + \alpha_2)}{2} \geqslant k.$$

As $\bar{x}'(e) \geqslant 0$ for all $e \in E$, it follows that

$$\bar{x}'(\delta(U)) \geqslant \bar{x}'([U_1, W\backslash U]) + \bar{x}'([U_2, \overline{W}\backslash U]) \geqslant k.$$

Consequently, $\bar{x}' \in P(G, k)$. Moreover, $\bar{x}'$ is an extreme point of $P(G, k)$. In fact, $\bar{x}'$ is the unique solution of the system formed by $S(\bar{x}')$ and the equations $x(e) = 1$ for all $e \in E(W)$. As $\bar{x}'$ is fractional, this contradicts the fact that $G$ is perfectly-$k$EC. □

Let $\theta_1$, $\theta_2$ be the operations described by Lemmas 2.1–2.2, respectively and $\theta_3$ the operation described by Lemma 2.3 when $t = 1$. An immediate consequence of Lemmas 2.1–2.3 is the following.

**Lemma 2.4.** *Let $G$ be a perfectly-kEC graph. If $G'$ is a graph obtained for $G$ by repeated applications of operations $\theta_1, \theta_2, \theta_3$, then $G'$ is perfectly-kEC.*

## 3. Structural properties

In this section, we introduce an ordering on the extreme points of $P(G, k)$ and describe some structural properties of these extreme points with respect to that ordering. These properties will be useful in the sequel to describe sufficient conditions for a graph to be perfectly-$k$EC.

Let $G = (V, E)$ be a graph. A cut $\delta(W)$ of $G$ will be called *proper* if $|W| \geqslant 2$ and $|\overline{W}| \geqslant 2$. If $\bar{x}$ is a solution of $P(G, k)$, we will denote by $E_0(\bar{x})$, $E_1(\bar{x})$, $E_f(\bar{x})$ the sets of edges $e$ such that $\bar{x}(e) = 0$, $\bar{x}(e) = 1$, $0 < \bar{x}(e) < 1$, respectively. We also denote by $C_d(\bar{x})$ the set of degree tight cuts $\delta(v)$ such that $\delta(v) \cap E_f(\bar{x}) \neq \emptyset$, and by $C_p(\bar{x})$ the set of proper tight cuts $\delta(S)$ with $\delta(S) \cap E_f(\bar{x}) \neq \emptyset$. Let $\bar{x}$ be an extreme point of $P(G, k)$. Thus there is a set of cuts $C_p^*(\bar{x}) \subseteq C_p(\bar{x})$ such that $\bar{x}$ is the unique solution of the system

$$S(\bar{x})\begin{cases} x(e)=0 & \forall e\in E_0(\bar{x}),\\ x(e)=1 & \forall e\in E_1(\bar{x}),\\ x(\delta(v))=k & \forall \delta(v)\in C_d(\bar{x}),\\ x(\delta(S))=k & \forall \delta(S)\in C_p^*(\bar{x}).\end{cases}$$

We have the following lemma, its proof is omitted because it is similar to that of a similar result in [6].

**Lemma 3.1.** *Let $\delta(W)$ be a tight proper cut. Then system $S(\bar{x})$ can be chosen so that if $\delta(Z)\in C_p^*(\bar{x})$, then either $Z\subseteq W$ or $Z\subseteq \overline{W}$.*

In what follows we are going to define a ranking function on the extreme points of $P(G,k)$. This function has been introduced by Fonlupt and Mahjoub [14] for the polytope $P(G,2)$.

**Definition 3.1.** Let $x$ and $y$ be two extreme points of $P(G,k)$. We say that $x$ *dominates* $y$ and we write $x\mathscr{D}y$, if either $y=x$ or the following hold:

(1) $E_0(x)\subseteq E_0(y)$,
(2) $E_1(x)\subseteq E_1(y)$,
(3) $E_0(x)\cup E_1(x)\subsetneq E_0(y)\cup E_1(y)$.

The relation '$\mathscr{D}$' defines a partial ordering on the extreme points of $P(G,k)$. The *minimal* elements of this relation (i.e. the extreme points $x$ that do not dominate any other extreme point $y$, $y\neq x$) correspond to the integer extreme points of $P(G,k)$. These extreme points will be called of rank 0. In what follows, we define in a recursive way the rank of any extreme point of $P(G,k)$.

**Definition 3.2.** An extreme point $x$ of $P(G,k)$ will be called of rank $p$, where $p\geqslant 1$ is a fixed integer, if

(i) $x$ dominates only extreme points of rank $\leqslant p-1$, and
(ii) there exists at least one extreme point of $P(G,k)$ of rank $p-1$.

Note that extreme points of rank 1 only dominate integer extreme points.

**Remark 3.1.** Let $x$ be an extreme point of $P(G,k)$ of rank $p$ and $f\in E_f(x)$. Let $x'\in\mathbb{R}^E$ be given by

$$x'(e)=\begin{cases} x(e) & \text{if } e\in E\backslash f,\\ 1 & \text{if } e=f.\end{cases}$$

Then $x'\in P(G,k)$, and hence can be written as a convex combination of extreme points of rank $\leqslant p-1$. In particular, if $x$ is of rank 1, then $x'$ can be written as a convex combination of integer extreme points of $P(G,k)$.

Let $G = (V, E)$ be a graph and $\bar{x}$ a solution of $P(G, k)$. In what follows we are going to describe some operations that preserve rank 1. The two first ones are easy to prove

**Lemma 3.2.** *Let $f \in E$ be an edge such that $\bar{x}(f) = 0$ and let $\bar{x}'$ be the restriction of $\bar{x}$ on $G - f$. Then $\bar{x}$ is an extreme point of $P(G, k)$ of rank 1 if and only if $\bar{x}'$ is an extreme point of $P(G - f, k)$ of rank 1.*

**Lemma 3.3.** *Let $W \subset V$ be a node subset such that $G(W)$ is $k$-edge connected and $\bar{x}(e) = 1$ for all $e \in E(W)$. Let $\bar{x}'$ be the restriction of $\bar{x}$ on $E \backslash E(W)$. Then $\bar{x}$ is an extreme point of $P(G, k)$ of rank 1 if and only if $\bar{x}'$ is an extreme point of $P(G/W, k)$ of rank 1.*

**Lemma 3.4.** *Let $W \subset V$ be a node subset such that $|W| \geqslant 2$, $|\delta(W)| = k$ and $\bar{x}(e) = 1$ for all $e \in E(W)$. Let $\bar{x}'$ be the restriction of $\bar{x}$ on $E \backslash E(W)$. Then $\bar{x}$ is an extreme point of $P(G, k)$ of rank 1 if and only if $\bar{x}'$ is an extreme point of $P(G/W, k)$ of rank 1.*

**Proof.** We first show that $\bar{x}'$ is an extreme point of $P(G/W, k)$. Observe that, as $|\delta(W)| = k$, one should have $\bar{x}(e) = 1$ for all $e \in \delta(W)$. Now, it is easy to see that $\bar{x}' \in P(G/W, k)$. Moreover, by Lemma 3.1, system $S(\bar{x})$ can be chosen so that for every cut $\delta(Z)$ of $C_p^*(\bar{x})$, either $Z \subseteq W$ or $Z \subseteq \overline{W}$. Since $\bar{x}(e) = 1$ for all $e \in E(W) \cup \delta(W)$ it follows that $C_p^*(\bar{x}) \subseteq C_p(\bar{x}')$ and $C_d(\bar{x}) = C_d(\bar{x}')$. Therefore $\bar{x}'$ is the unique solution of a subsystem of $S(\bar{x})$. As all the equations of that subsystem correspond to constraints of $P(G/W, k)$, this implies that $\bar{x}'$ is an extreme point of $P(G/W, k)$.

Now let us suppose, on the contrary, that $\bar{x}'$ is not of rank 1, and that there is a fractional extreme point of $P(G/W, k)$, say $y'$, which dominates $\bar{x}'$. Thus $y'(e) = 1$ for all $e \in \delta(W)$. Let $y \in \mathbb{R}^E$ be the solution such that

$$y(e) = \begin{cases} y'(e) & \text{if } e \in E \backslash E(W), \\ 1 & \text{if } e \in E(W). \end{cases}$$

Obviously, $y \in P(G, k)$. Moreover, $y$ is an extreme point of $P(G, k)$. In fact, $y$ is the unique solution of the system given by system $S(y')$ characterizing $y'$ together with the equations $x(e) = 1$ for all $e \in E(W)$. But this implies that $\bar{x}$ is dominated by $y$. As $y$ is fractional, this contradicts the fact that $\bar{x}$ is of rank 1.

Conversely, suppose that $\bar{x}'$ is an extreme point of $P(G/W, k)$ of rank 1. First, it is clear that $\bar{x}$ is an extreme point of $P(G, k)$. Moreover, if $\bar{x}$ is not of rank 1, then there is an extreme point $y$ of $P(G, k)$ of rank 1 which is dominated by $\bar{x}$. Therefore the restriction $y'$ of $y$ on $E \backslash E(W)$ is a fractional extreme point of $P(G/W, k)$ which is dominated by $\bar{x}'$. This contradicts the fact that $\bar{x}'$ is of rank 1. $\square$

**Lemma 3.5.** *Let $W \subset V$ be a node subset such that $G(W)$ is $\lceil \frac{k}{2} \rceil$-edge connected and $|\delta(W)| = k + 1$. Suppose also that $\bar{x}(e) = 1$ for all $e \in E(W)$. Let $\bar{x}'$ be the restriction of $\bar{x}$ on $E \backslash E(W)$. Then $\bar{x}$ is an extreme point of $P(G, k)$ of rank 1 if and only if $\bar{x}'$ is an extreme point of $P(G/W, k)$ of rank 1.*

**Proof.** Suppose that $\bar{x}$ is an extreme point of $P(G, k)$ of rank 1. It is clear that $\bar{x}'$ is a solution of $P(G/W, k)$. Now to show that $\bar{x}'$ is an extreme point of $P(G/W, k)$, it suffices to show that $C_p^*(\bar{x})$ can be chosen so that if $\delta(Z) \in C_p^*(\bar{x})$, then $Z \subseteq \overline{W}$. Assume that there is $\delta(Z) \in C_p^*(\bar{x})$ such that $Z \not\subset \overline{W}$ and $\overline{Z} \not\subset \overline{W}$. We shall consider two cases.

*Case 1: $Z \subset W$.*

As $\delta(Z) \in C_p^*(\bar{x})$ and $\bar{x}(e) = 1$ for all $e \in [Z, W \backslash Z]$, it follows that $[Z, \overline{W}] \cap E_f(\bar{x}) \neq \emptyset$ and $\bar{x}([Z, \overline{W}]) \leqslant |[Z, \overline{W}]| - 1$. Thus

$$\begin{aligned} k \leqslant \bar{x}(\delta(W)) &= \bar{x}([Z, \overline{W}]) + \bar{x}([W \backslash Z, \overline{W}]) \\ &\leqslant |[Z, \overline{W}]| - 1 + |[W \backslash Z, \overline{W}] \\ &= k, \end{aligned}$$

where the last equality comes from the fact that $|\delta(W)| = k + 1$. Thus the above inequalities are all satisfied with equality. This implies that $\bar{x}(e) = 1$ for all $e \in [W \backslash Z, \overline{W}]$. And, in consequence, the two equations $x(\delta(Z)) = k$ and $x(\delta(W)) = k$ are equivalent in system $S(\bar{x})$.

*Case 2: $Z \not\subset W$, $\overline{Z} \not\subset W$.*

Let $Z_1 = W \cap Z$, $Z_2 = \overline{W} \cap Z$. We have that $Z_1 \neq \emptyset$, $Z_2 \neq \emptyset$, $W \backslash Z \neq \emptyset$, $\overline{W} \backslash Z \neq \emptyset$ (see Fig. 1).

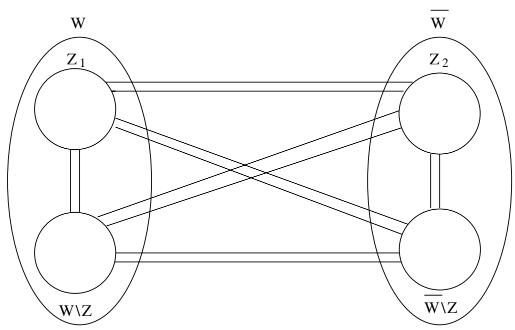


Fig. 1.

As $|\delta(W)| = k + 1$, it follows that $\min\{|[W, Z_2]|, |[W, \overline{W}\backslash Z]|\} \leqslant \lceil \frac{k}{2} \rceil$. Hence, $\bar{x}([Z_2, \overline{W}\backslash Z]) \geqslant \lceil \frac{k-1}{2} \rceil$, for otherwise, we would have either $\bar{x}(\delta(Z_2)) < k$ or $\bar{x}(\delta(\overline{W}\backslash Z)) < k$, a contradiction. As $G(W)$ is $\lceil \frac{k}{2} \rceil$-edge connected this yields

$$\begin{aligned}
k &= \bar{x}(\delta(Z)) \\
&\geqslant |[Z_1, W\backslash Z]| + \bar{x}([Z_2, \overline{W}\backslash Z]) \\
&\geqslant \left\lceil \frac{k}{2} \right\rceil + \left\lceil \frac{k-1}{2} \right\rceil \\
&= k.
\end{aligned}$$

Thus all the inequalities above are satisfied with equality. Moreover, as a consequence, we have

$$\begin{aligned}
&\bar{x}([Z_1, W\backslash Z]) = |[Z_1, W\backslash Z]| = \left\lceil \frac{k}{2} \right\rceil, \\
&\bar{x}([Z_2, \overline{W}\backslash Z]) = \left\lceil \frac{k-1}{2} \right\rceil, \\
&\bar{x}([Z_1, \overline{W}\backslash Z]) = \bar{x}([W\backslash Z, Z_2]) = 0.
\end{aligned}$$

As $\bar{x}(\delta(Z_2)) \geqslant k$ and $\bar{x}(\delta(\overline{W}\backslash Z)) \geqslant k$, it follows that

$$\begin{aligned}
&\bar{x}([Z_1, Z_2]) \geqslant \frac{k}{2}, \\
&\bar{x}([W\backslash Z, \overline{W}\backslash Z]) \geqslant \frac{k}{2}.
\end{aligned}$$

Since $|\delta(W)| = k + 1$, and $\bar{x}(e) \leqslant 1$ for all $e \in E$, this implies that either $\bar{x}([Z_1, Z_2]) = |[Z_1, Z_2]| = \frac{k}{2}$ or $\bar{x}([W\backslash Z, \overline{W}\backslash Z]) = |[W\backslash Z, \overline{W}\backslash Z]| = \frac{k}{2}$.

Suppose, w.l.o.g., that $\bar{x}([Z_1, Z_2]) = |[Z_1, Z_2]| = \frac{k}{2}$. Hence $\bar{x}(e) = 1$ for all $e \in [Z_1, Z_2]$ and $\delta(Z_2)$ is tight for $\bar{x}$. Consequently the equation $x(\delta(Z)) = k$ is redundant with respect to the equations $x(\delta(Z_2)) = k$ and $x(e) = 1$ for all $e \in E_1(\bar{x})$. Thus it can be replaced by $x(\delta(Z_2)) = k$ in system $S(\bar{x})$.

Consequently, $\bar{x}'$ is an extreme point of $P(G/W, k)$. We can also show along the same line as in Lemma 3.4 that $\bar{x}'$ is of rank 1.

The necessary condition can also be shown in a similar way as in Lemma 3.4. □

Let us denote by $\theta'_1, \ldots, \theta'_4$ the operations described by Lemmas 3.2–3.5 respectively. That is

$\theta'_1$: Delete an edge $e$ with $x(e) = 0$.
$\theta'_2$: Contract a node subset $W \subset V$ such that $G(W)$ is $k$-edge connected and $x(e) = 1$ for all $e \in E(W)$.

$\theta_3'$: Contract a node subset $W \subset V$ such that $|W| \geqslant 2$, $|\delta(W)| = k$ and $x(e) = 1$ for all $e \in E(W)$.

$\theta_4'$: Contract a node subset $W \subset V$ such that $G(W)$ is $\lceil \frac{k}{2} \rceil$-edge connected, $|\delta(W)| = k + 1$ and $x(e) = 1$ for all $e \in E(W)$.

An immediate consequence of Lemmas 3.2–3.5 is the following.

**Lemma 3.6.** *Let $G = (V, E)$ be a graph and $\bar{x}$ a solution of $P(G, k)$. Let $G' = (V', E')$ be a graph obtained from $G$ by repeated applications of the operations $\theta_1', \theta_2', \theta_3', \theta_4'$. Let $\bar{x}'$ be the restriction of $\bar{x}$ on $E'$. Then $\bar{x}$ is an extreme point of $P(G, k)$ of rank 1 if and only if $\bar{x}'$ is an extreme point of $P(G', k)$ of rank 1.*

**Definition 3.3.** An extreme point $\bar{x}$ of $P(G, k)$ will be called *critical* if

(i) $\bar{x}$ is of rank 1 and
(ii) none of the operation $\theta_1', \ldots, \theta_4'$ can be applied to it.

In what follows we are going to describe some properties of the critical extreme points of $P(G, k)$.

Let $G = (V, E)$ be a $k$-edge connected graph and $\bar{x}$ a critical extreme point of $P(G, k)$. We have the following lemmas. The two first ones will be given without proof, they are direct consequences of Definition 3.3.

**Lemma 3.7.** $\bar{x}(e) > 0$ *for all* $e \in E$.

**Lemma 3.8.** *Let $W \subseteq V$ such that $|W| \geqslant 2$. If $G(W)$ is $k$-edge connected, then $E(W) \cap E_f(\bar{x}) \neq \emptyset$.*

**Lemma 3.9.** *If $W \subseteq V$ such that $|\delta(W)| = k$, then either $|W| = 1$ or $|\overline{W}| = 1$.*

**Proof.** Suppose that $|W| \geqslant 2$ and $|\overline{W}| \geqslant 2$. As $|\delta(W)| = k$, it follows that $\bar{x}(e) = 1$ for all $e \in \delta(W)$ and $\bar{x}(\delta(W)) = k$. Thus by Lemma 3.1 we may suppose that the set of cuts $C_p^*(\bar{x})$ in system $S(\bar{x})$ is such that for all $\delta(Z) \in C_p^*(\bar{x})$, either $Z \subseteq W$ or $Z \subseteq \overline{W}$. Let $\bar{x}_1$ (resp. $\bar{x}_2$) be the restriction of $\bar{x}$ on the graph $\overline{G}_1$ (resp. $\overline{G}_2$) obtained from $G$ by contracting $W$ (resp. $\overline{W}$). Note that both $\bar{x}_1$ and $\bar{x}_2$ are fractional (otherwise, operation $\theta_3'$ could be applied to $\bar{x}$, contradicting the fact that $\bar{x}$ is critical). Now let $\bar{x}_1'$ and $\bar{x}_2'$ be the solutions of $\mathbb{R}^E$ defined as

$$\bar{x}_1'(e) = \begin{cases} \bar{x}_1(e) & \text{if } e \in E(\overline{W}) \cup \delta(W), \\ 1 & \text{if } e \in E(W), \end{cases}$$

and

$$\bar{x}_2'(e) = \begin{cases} \bar{x}_2(e) & \text{if } e \in E(W) \cup \delta(W), \\ 1 & \text{if } e \in E(\overline{W}). \end{cases}$$

It is clear that $\bar{x}_1'$ and $\bar{x}_2'$ both belong to $P(G,k)$. As $\bar{x}$ is critical and thus of rank 1, by Remark 3.1 both $\bar{x}_1'$ and $\bar{x}_2'$ can be written as convex combinations of integer extreme points of $P(G,k)$. Let $y_1$ and $y_2$ be two points of these convex combinations, related to $\bar{x}_1'$ and $\bar{x}_2'$, respectively. We note that every constraint of $P(G,k)$ that is tight for $\bar{x}_1'$ (resp. $\bar{x}_2'$) is also tight for $y_1$ (resp. $y_2$). In particular, one should have $y_1(e) = y_2(e) = 1$ for all $e \in \delta(W)$. Let $y \in \mathbb{R}^E$ be given by

$$y(e) = \begin{cases} y_1(e) & \text{if } e \in E(\overline{W}), \\ y_2(e) & \text{if } e \in E(W), \\ 1 & \text{if } e \in \delta(W). \end{cases}$$

We claim that $y$ is a solution of system $S(\bar{x})$. In fact, first it is clear that $y(e) = 1$ for all $e \in E_1(\bar{x})$. Now let $\delta(Z)$ be a cut of system $S(\bar{x})$ ($\delta(Z)$ may be either a cut of $C_d(\bar{x})$ or a cut of $C_p^*(\bar{x})$). If $Z \subseteq W$, then $y(\delta(Z)) = y_2(\delta(Z)) = \bar{x}_2'(\delta(Z)) = \bar{x}(\delta(Z)) = k$. If $Z \subseteq \overline{W}$, then $y(\delta(Z)) = y_1(\delta(Z)) = \bar{x}_1'(\delta(Z)) = \bar{x}(\delta(Z)) = k$. Consequently, $y$ is a solution of system $S(\bar{x})$. As $y \neq \bar{x}$, this is a contradiction with the fact that $\bar{x}$ is the unique solution of that system. □

**Lemma 3.10.** *Let $\delta(W) \in C_p(\bar{x})$ be a tight cut with $|\delta(W)| = k+1$. Then either $|W| = 1$ or $|\overline{W}| = 1$.*

**Proof.** We first show that both $G(W)$ and $G(\overline{W})$ are $\lceil \frac{k}{2} \rceil$-edge connected. Let us suppose for instance that $G(W)$ is not $\lceil \frac{k}{2} \rceil$-edge connected. Then there is a node subset $W_1 \subset W$ such that $|[W_1, W \backslash W_1]| < \lceil \frac{k}{2} \rceil$. Hence $\bar{x}([W_1, W\backslash W_1]) \leqslant \lceil \frac{k}{2} \rceil - 1$. As $\bar{x}(\delta(W_1)) \geqslant k$ and $\bar{x}(\delta(W \backslash W_1)) \geqslant k$, it follows that $\bar{x}([W_1, \overline{W}]) \geqslant \lceil \frac{k-1}{2} \rceil + 1$ and $\bar{x}([W\backslash W_1, \overline{W}]) \geqslant \lceil \frac{k-1}{2} \rceil + 1$. But this implies that $\bar{x}(\delta(W)) \geqslant k+1$, which contradicts the fact that $\delta(W)$ is tight.

Thus both $G(W)$ and $G(\overline{W})$ are $\lceil \frac{k}{2} \rceil$-edge connected. Now suppose the statement does not hold, that is $|W| \geqslant 2$ and $|\overline{W}| \geqslant 2$. Also suppose that $|W|$ is minimum, that is if $Z \subset W$ such that $\delta(Z) \in C_p^*$ and $|\delta(Z)| = k+1$, then $|Z| = 1$. Since $\bar{x}$ is critical and hence, cannot be reduced by operation $\theta_4'$, there must exist two edges $f_1 \in E(W)$ and $f_2 \in E(\overline{W})$ such that $0 < \bar{x}(f_1) < 1$ and $0 < \bar{x}(f_2) < 1$. Since $|\delta(W)| = k+1$ and $\bar{x}(\delta(W)) = k$, there must also exist an edge $e_1 \in \delta(W)$ such that $0 < \bar{x}(e_1) < 1$. Let $\bar{x}_1$ and $\bar{x}_2$ be the solutions given by

$$\bar{x}_1(e) = \begin{cases} \bar{x}(e) & \text{if } e \in E(W) \cup \delta(W), \\ 1 & \text{if } e \in E(\overline{W}), \end{cases}$$

and

$$\bar{x}_2(e) = \begin{cases} \bar{x}(e) & \text{if } e \in E(\overline{W}) \cup \delta(W), \\ 1 & \text{if } e \in E(W). \end{cases}$$

As $\bar{x}_1$ and $\bar{x}_2$ belong to $P(G,k)$, and $\bar{x}$ is critical, by Remark 3.1, $\bar{x}_1$ and $\bar{x}_2$ can be written as convex combinations of integer extreme points of $P(G,k)$. Let $y_1$ and $y_2$ be two points of these convex combinations. As $\bar{x}_1(e_1) = \bar{x}_2(e_1) < 1$; $y_1$ and

$y_2$ can be chosen so that $y_1(e_1) = y_2(e_1) = 0$. As $|\delta(W)| = k + 1$, this implies that $y_1(e) = y_2(e) = 1$ for all $e \in \delta(W)\backslash\{e_1\}$. Let $y \in \mathbb{R}^{|E|}$ such that

$$y(e) = \begin{cases} y_1(e) & \text{if } e \in E(W), \\ y_2(e) & \text{if } e \in E(\overline{W}), \\ 1 & \text{if } e \in \delta(W)\backslash\{e_1\}, \\ 0 & \text{if } e = e_1. \end{cases}$$

Since $\delta(W)$ is tight for $\bar{x}$, by Lemma 3.1 the set $C_p^*(\bar{x})$ can be supposed consisting of tight cuts $\delta(Z)$ with either $Z \subseteq W$ or $Z \subseteq \overline{W}$. Now it easily follows as in Lemma 3.9 that $y$ is a solution of $C_p^*(\bar{x})$. Since $y \neq \bar{x}$ this is a contradiction. □

**Lemma 3.11.** *Let $\delta(W)$ be a tight cut with $|\delta(W) \cap E_1(\bar{x})| = k - 1$. Then exactly one of the following statements holds*:

(i) *either $|W| = 1$ or $|\overline{W}| = 1$,*
(ii) *either $\bar{x}(e) = 1$ for all $e \in E(W)$ or $\bar{x}(e) = 1$ for all $e \in E(\overline{W})$.*

**Proof.** Suppose that (i) does not hold, that is $|W| \geqslant 2$ and $|\overline{W}| \geqslant 2$. We will show that (ii) necessarily holds. For this let us assume, on the contrary, that both $E(W)$ and $E(\overline{W})$ contain fractional edges. Also suppose that $|W|$ is minimum, that is if for $Z \subset W$, $\delta(Z)$ is tight for $\bar{x}$ and $|\delta(Z) \cap E_1(\bar{x})| = k - 1$, then either $|Z| = 1$ or $\bar{x}(e) = 1$ for all $e \in E(Z)$. By Lemma 3.1, we may also suppose that for every cut $\delta(S)$ of $C_p^*(\bar{x})$, either $S \subseteq W$ or $S \subseteq \overline{W}$. Let $e_1, \ldots, e_{k-1} \in \delta(W)$ with $\bar{x}(e_i) = 1$ for $i = 1, \ldots, k - 1$. Let $G_1 = (V_1, E_1)$ (resp. $G_2 = (V_2, E_2)$) be the graph obtained from $G$ by contracting $W$ (resp. $\overline{W}$). Let $\bar{x}_1$ (resp. $\bar{x}_2$) be the restriction of $\bar{x}$ on $G_1$ (resp. $G_2$). Obviously, $\bar{x}_i$ is a fractional solution of $P(G_i, k)$ for $i = 1, 2$. We claim that $\bar{x}_1$ is not an extreme point of $P(G_1, k)$. Suppose that this is not the case. Then let $y_1 \in \mathbb{R}^E$ be given by

$$y_1(e) = \begin{cases} \bar{x}_1(e) & \text{if } e \in E_1, \\ 1 & \text{if } e \in E(W). \end{cases}$$

Obviously $y_1 \in P(G, k)$. Moreover $y_1$ is an extreme point of $P(G, k)$. This would follow from the fact that $y_1$ is the unique solution of the system given by the system defining $\bar{x}_1$ and the equations $x(e) = 1$ for all $e \in E(W)$. As $y_1$ is fractional and dominated by $\bar{x}$, this contradicts the fact that $\bar{x}$ is of rank 1.

Now, since $\bar{x}_1$ is not an extreme point of $P(G_1, k)$, it can be then written as a convex combination of $t$ extreme points $y_1^1, \ldots, y_t^1$ of $P(G_1, k)$. That is

$$\bar{x}_1 = \sum_{i=1}^{t} \alpha_i y_i^1$$

with $\alpha_i > 0$ for $i = 1, \ldots, t$ and $\sum_{i=1}^{t} \alpha_i = 1$. Note that every constraint of $P(G_1, k)$ that is tight for $\bar{x}_1$ is at the same time tight for $y_i^1$, $i = 1, \ldots, t$. In particular $y_i^1(e) = 1$ for $e \in \{e_1, \ldots, e_{k-1}\}$ and $i = 1, \ldots, t$. We are going to show that $y_i^1$ is integer for

$i = 1, \ldots, t$. Indeed, suppose that, for instance, $y_1^1$ is fractional. Let $z \in \mathbb{R}^E$ be the solution given by

$$z(e) = \begin{cases} y_1^1(e) & \text{if } e \in E_1, \\ 1 & \text{if } e \in E(W). \end{cases}$$

We claim that $z \in P(G, k)$. To prove this we first show that $G(W)$ is $\lceil \frac{k}{2} \rceil$-edge connected. Indeed, suppose there is a subset $W_1$ of $W$ such that $|[W_1, W \backslash W_1]| < \lceil \frac{k}{2} \rceil$. Also suppose, w.l.o.g., that $\bar{x}([W_1, \overline{W}]) \leqslant \bar{x}([W \backslash W_1, \overline{W}])$. Thus

$$\begin{aligned} \bar{x}(\delta(W_1)) &= \bar{x}([W_1, W \backslash W_1]) + \bar{x}([W_1, \overline{W}]) \\ &\leqslant |[W_1, W \backslash W_1]| + \frac{k}{2} \\ &\leqslant \left\lceil \frac{k}{2} \right\rceil - 1 + \frac{k}{2} \\ &< k, \end{aligned}$$

a contradiction. Now it is clear that $z$ satisfies the trivial inequalities and the inequalities corresponding to cuts $\delta(S)$ with $W \subseteq S$. So consider a cut $\delta(S)$ such that $W \neq S \cap W \neq \emptyset$. Suppose first that $S \subset W$. Also suppose, w.l.o.g., that $[S, \overline{W}] \cap \{e_1, \ldots, e_{k-1}\} = \{e_1, \ldots, e_s\}$, $s \leqslant k - 1$. Hence $z([S, \overline{W}]) \geqslant s$ and $\lambda = \bar{x}([S, \overline{W}]) - s \leqslant 1$. As $\bar{x}(\delta(S)) \geqslant k$ and $\lambda \leqslant 1$, one should have $\bar{x}([S, W \backslash S]) + s \geqslant k - 1$. Hence $|[S, W \backslash S]| \geqslant k - 1 - s$. If $|[S, W \backslash S]| \geqslant k - s$, then

$$\begin{aligned} z(\delta(S)) &= z([S, W \backslash S]) + z([S, \overline{W}]) \\ &\geqslant |[S, W \backslash S]| + s \\ &\geqslant k. \end{aligned}$$

If $|[S, W \backslash S]| < k - s$, then $|[S, W \backslash S]| = k - s - 1$. This implies that $\bar{x}(e) = 1$ for all $e \in [S, W \backslash S]$ and $\lambda = 1$. Moreover, as $\delta(W)$ is tight and $|\delta(W) \cap E_1(\bar{x})| = k - 1$, one should have $|[W \backslash S, \overline{W}]| = k - s - 1$ and $\bar{x}(e) = 1$ for all $e \in [W \backslash S, \overline{W}]$. It thus follows that $z([S, \overline{W}]) = y_1^1([S, \overline{W}]) = s + 1$, and hence $z(\delta(W)) = z([S, W \backslash S]) + z([S, \overline{W}]) = k$.

Now suppose that $S \not\subset W$, $S \not\subset \overline{W}$, $\overline{S} \not\subset W$ and $\overline{S} \not\subset \overline{W}$. Let $S_1 = S \cap W$ and $S_2 = S \cap \overline{W}$. Then all the sets $S_1$, $S_2$, $W \backslash S_1$, $\overline{W} \backslash S_2$ are nonempty. We have that $z(\delta(S_2)) \geqslant k$, $z(\delta(\overline{W} \backslash S_2)) \geqslant k$ and $z(\delta(W)) = k$. This implies that $z([S_2, \overline{W}]) \geqslant \frac{k}{2}$. As $|[S_1, W \backslash S_1]| \geqslant \lceil \frac{k}{2} \rceil$ and $z(e) = 1$ for all $e \in E(W)$, it follows that

$$\begin{aligned} z(\delta(S)) &\geqslant z([S_1, W \backslash S_1]) + z([S_2, \overline{W} \backslash S_2]) \\ &\geqslant \left\lceil \frac{k}{2} \right\rceil + \frac{k}{2} \\ &\geqslant k. \end{aligned}$$

Consequently, $z \in P(G, k)$. Moreover it is easy to see that $z$ is an extreme point of $P(G, k)$. Since $z$ is fractional and dominated by $\bar{x}$, this is a contradiction. Thus, $y_1^1, \ldots, y_t^1$ are all integer. Let $e_0 \in \delta(W)\backslash\{e_1, \ldots, e_{k-1}\}$. As $\bar{x}(e_0) > 0$, w.l.o.g., we may suppose that $y_1^1(e_0) = 1$. As $y_1^1(e_i) = 1$ for $i = 1, \ldots, k-1$, it then follows that $y_1^1(e) = 0$ for all $e \in \delta(W)\backslash\{e_0, e_1, \ldots, e_{k-1}\}$.

Similarly, there exists an integer solution say $y_1^2$ of $P(G_2, k)$ such that $y_1^2(e) = 1$ for all $e \in \{e_0, e_1, \ldots, e_{k-1}\}$ and $y_1^2(e) = 0$ for all $e \in \delta(W)\backslash\{e_0, e_1, \ldots, e_{k-1}\}$. Let $y \in \mathbb{R}^E$ be the solution defined as

$$y(e) = \begin{cases} y_1^1(e) & \text{if } e \in E(\overline{W}), \\ y_1^2(e) & \text{if } e \in E(W), \\ 1 & \text{if } e \in \{e_0, e_1, \ldots, e_{k-1}\}, \\ 0 & \text{if } e \in \delta(W)\backslash\{e_0, e_1, \ldots, e_{k-1}\}. \end{cases}$$

Along a similar way as we did in Lemma 3.6, we can show that $y$ is a solution of system $S(\bar{x})$. As $y \neq \bar{x}$, this is a contradiction. □

## 4. Classes of perfectly-*k*EC graphs

As it has been mentioned before, series–parallel graphs have been shown to be perfectly-$k$EC for $k$ even. However, as pointed out in [10] this is no longer true if $k$ is odd. To the best of our knowledge no nontrivial classes of perfectly-$k$EC have been characterized for $k$ odd.

Using the previous results, we shall introduce further classes of perfectly-$k$EC graphs for arbitrary $k$. To this end, we first give the following lemma.

**Lemma 4.1.** *Let $G = (V, E)$ be a graph and $\bar{x}$ an extreme point of $P(G, k)$ of rank 1. Suppose that $C_p^*(\bar{x}) = \emptyset$. Then the graph induced by $E_f(\bar{x})$, $G_f(\bar{x})$ is an odd cycle $C$ such that*

(i) $\bar{x}(e) = \frac{1}{2}$ *for all* $e \in C$;
(ii) $\bar{x}(\delta(v)) = k$ *for all* $v \in V(C)$.

**Proof.** The proof will be a consequence of the following claims.

**Claim 1.** *Every edge $f$ of $E_f(\bar{x})$ belongs to at least two tight cuts of $S(\bar{x})$.*

**Proof.** It is clear that $f$ must belong to at least one tight cut of $S(\bar{x})$. Otherwise, one can increase $x(f)$ and obtain a solution still satisfying system $S(\bar{x})$, which is impossible. Now let us suppose that $f$ belongs to exactly one tight cut $\delta(W)$ of $S(\bar{x})$. Let $S(\bar{x})'$ be the system obtained from $S(\bar{x})$ by deleting the equation associated with $\delta(W)$. Thus $S(\bar{x})'$ is a nonsingular system. Let $x' \in \mathbb{R}^E$ be the solution given by

$$x'(e) = \begin{cases} x(e) & \text{if } e \in E\backslash\{f\}, \\ 1 & \text{if } e = f. \end{cases}$$

We have that $x' \in P(G, k)$. Furthermore, $x'$ is the unique solution of the system

$$\begin{cases} S(\bar{x})', \\ x(f) = 1. \end{cases}$$

Thus $x'$ is an extreme point of $P(G, k)$. Since $\delta(W)$ is tight for $\bar{x}$, there must exist at least one more fractional edge in $\delta(W)$ and thus $x'$ is fractional. This implies that $x'$ dominates $\bar{x}$, which contradicts the fact that $\bar{x}$ is of rank 1. □

**Claim 2.** *$G_f(\bar{x})$ does not contain a pendant node.*

**Proof.** Suppose that $G_f(\bar{x})$ contains a pendant node, say $v_0$. Let $f_0$ be the edge of $G_f(\bar{x})$ adjacent to $v_0$. By Claim 1, we have that $x(\delta(v_0)) = k$. But $v_0$ must be adjacent to at least $k$ edges of $E_1(\bar{x})$ (otherwise, one would have $x(\delta(v_0)) < k$). Since $\bar{x}(f_0) > 0$, this yields $x(\delta(v_0)) > k$, a contradiction. □

**Claim 3.** *$G_f(\bar{x})$ does not contain an even (simple or not) cycle.*

**Proof.** If $G_f(\bar{x})$ contains an even cycle, say, $(f_1, f_2, \ldots, f_{2l})$, $l \geqslant 1$, then let $\bar{x}'$ be the solution given by

$$\bar{x}'(e) = \begin{cases} \bar{x}(e) + \varepsilon & \text{for } e \in \{f_1, f_3, \ldots, f_{2l-1}\}, \\ \bar{x}(e) - \varepsilon & \text{for } e \in \{f_2, f_4, \ldots, f_{2l}\}, \\ \bar{x}(e) & \text{otherwise}, \end{cases}$$

where $\varepsilon$ is a positive scalar sufficiently small. Since $C_p^*(\bar{x}) = \emptyset$, $\bar{x}'$ satisfies system $S(\bar{x})$. As $\bar{x}' \neq \bar{x}$, this is a contradiction. □

**Claim 4.** *$G_f(\bar{x})$ is connected.*

**Proof.** Suppose that this is not the case. By Claims 2 and 3, there are two odd cycles $C_1$ and $C'$ of $G_f(\bar{x})$ such that $C_1 \cap C' = \emptyset$. Consider the solution $\tilde{x}$ defined as

$$\tilde{x}(e) = \begin{cases} \frac{1}{2} & \text{if } e \in C_1, \\ 1 & \text{if } e \in E\backslash C_1. \end{cases}$$

Obviously, $\tilde{x} \in P(G, k)$. Moreover $\tilde{x}$ is an extreme point of $P(G, k)$ which is dominated by $\bar{x}$. Since $\tilde{x}$ is fractional, this is a contradiction. □

By Claims 2–4, it follows that $G_f(\bar{x})$ contains an odd cycle, say $C$. Suppose that $E_f(\bar{x})\backslash C \neq \emptyset$. Then by Claims 2–4, there is at least one more simple odd cycle, say $C'$ such that $C$ and $C'$ are joined by a path, say $P$. W.l.o.g., we may suppose that $P$ is odd (see Fig. 2). Let

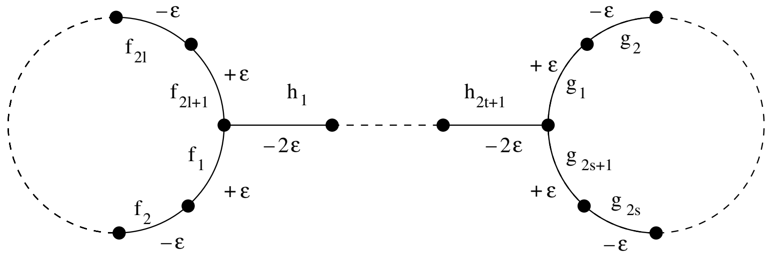


Fig. 2.

$$C = (f_1, \dots, f_{2l+1}),$$
$$C' = (g_1, \dots, g_{2s+1}),$$
$$P = (h_1, \dots, h_{2t+1}).$$

Consider the solution $x'$ defined as

$$x'(e) = \begin{cases} x(e) & \text{if } e \in E\backslash(C \cup C' \cup P), \\ x(e) + \varepsilon & \text{if } e \in \{f_1, f_3, \dots, f_{2l+1}; g_1, g_3, \dots, g_{2s+1}\}, \\ x(e) - \varepsilon & \text{if } e \in \{f_2, f_4, \dots, f_{2l}; g_2, g_4, \dots, g_{2s}\}, \\ x(e) - 2\varepsilon & \text{if } e \in \{h_1, h_3, \dots, h_{2t+1}\}, \\ x(e) + 2\varepsilon & \text{if } e \in \{h_2, h_4, \dots, h_{2t}\}. \end{cases}$$

where $\varepsilon$ is a positive scalar sufficiently small (see Fig. 2). Since $C_p^*(\bar{x}) = \emptyset$, $x'$ satisfies system $S(\bar{x})$. As $x' \neq \bar{x}$, we have a contradiction. □

Consequently, $G_f(\bar{x})$ consists of only one odd cycle namely $C$. Moreover we have that $\bar{x}$ is the solution of the system

$$\begin{cases} x(e) = 1 & \text{for all } e \in E_1(\bar{x}), \\ x(e) = 0 & \text{for all } e \in E_0(\bar{x}), \\ x(f_1) + x(f_2) = 1, & \\ x(f_2) + x(f_3) = 1, & \\ \vdots & \\ x(f_{2l}) + x(f_{2l+1}) = 1, & \\ x(f_{2l+1}) + x(f_1) = 1. & \end{cases}$$

This yields $\bar{x}(e) = \frac{1}{2}$ for all $e \in C = E_f(\bar{x})$, which finishes the proof of our lemma. □

Let $\Gamma$ be the class of graphs $G = (V_1 \cup V_2, E)$, $V_1 \cap V_2 = \emptyset$, such that:

(1) $|V_1| = 3$ and $E(V\backslash V_1) = \emptyset$,
(2) $|V_2| \geqslant 3$ and if $|V_2| = 3$, then $E(V_1) = \emptyset$,
(3) $|[v_1, v_2]| \leqslant \left\lfloor \frac{k}{2} \right\rfloor$ for all nodes $v_1$ and $v_2$ such that $v_1 \in V_1$ and $v_2 \in V_2$.

Note that the graphs of $\Gamma$ can be recognized in polynomial time and may be non series–parallel. The following theorem generalizes a result in [25].

**Theorem 4.2.** *If G is a graph of $\Gamma$, then G is perfectly-kEC.*

**Proof.** Let $G = (V, E)$ be a graph of $\Gamma$. Let $V_1 = \{s_1, s_2, s_3\}$ and $V_2 = \{u_1, \ldots, u_t\}$, $t \geqslant 3$. Let $Q(G, k)$ be the polytope given by the trivial constraints together with the degree constraints, i.e.

$$Q(G,k) = \begin{cases} 0 \leqslant x(e) \leqslant 1 & \forall\, e \in E, \\ x(\delta(v)) \geqslant k & \forall\, v \in V. \end{cases}$$

To show the theorem, we first prove the following.

**Claim.** $Q(G, k) = P(G, k)$.

**Proof.** Clearly, $P(G, k) \subset Q(G, k)$. Now consider a point $x$ of $Q(G, k)$. We shall show that $x$ is also a point of $P(G, k)$. For this we have to show that it satisfies all the proper cut constraints. Let $\delta(W)$ be a proper cut of $G$. Consider first the case when either $V_1 \subseteq W$ or $V_1 \subseteq \overline{W}$. And suppose for instance that $V_1 \subseteq W$. Then $x(\delta(W)) = \sum_{u \in V_2 \cap \overline{W}} x(\delta(u)) \geqslant k$.

Now suppose that $W \cap V_1 \neq \emptyset \neq \overline{W} \cap V_1$. W.l.o.g., we may suppose that $W \cap V_1 = \{s_1\}$ and hence $\overline{W} \cap V_1 = \{s_2, s_3\}$. We consider two cases.

*Case 1:* $|W \cap V_2| = 1$.

Let $\{v_1\} = W \cap V_2$. As, by definition of $\Gamma$, $|[v_1, s_1] \leqslant \lfloor \frac{k}{2} \rfloor$, and $x(e) \leqslant 1$ for all $e \in E$, it follows that $x(\delta(v_1) \cap \delta(W)) \geqslant \lceil \frac{k}{2} \rceil$ and $x(\delta(s_1) \cap \delta(W)) \geqslant \lceil \frac{k}{2} \rceil$. Therefore $x(\delta(W)) \geqslant k$.

*Case 2:* $|W \cap V_2| \geqslant 2$.

Suppose for instance that $u_1, u_2 \in W \cap V_2$. We then have that $x(\delta(W)) \geqslant x(\delta(u_1) \cap \delta(W)) + x(\delta(u_2) \cap \delta(W)) \geqslant 2\lceil \frac{k}{2} \rceil \geqslant k$.

In both cases we have $x(\delta(W)) \geqslant k$. In consequence, $x \in P(G, k)$ and therefore $P(G, k) = Q(G, k)$. □

Now suppose that $G$ is not perfectly-$k$EC, and in consequence, $P(G, k)$ contains a fractional extreme point. This implies that there is an extreme point, say $\bar{x}$, of rank 1 of $P(G, k)$. By the claim above, $\bar{x}$ is also an extreme point of $Q(G, k)$, and hence $S(\bar{x})$ can be chosen so that $C_p^*(\bar{x}) = \emptyset$. From Lemma 4.1 it follows that $G_f(\bar{x})$ is an odd cycle, say $C$. Therefore $G$ is not bipartite, and in consequence, by the definition of $\Gamma$, $t \geqslant 4$. Moreover, as $E(V_2) = \emptyset$, $C$ contains at least one edge of $E(V_1)$. Thus there are two nodes of $V_1$, say $s_1$ and $s_2$ such that $s_1 s_2 \in C$. By Lemma 4.1, we have that $\bar{x}(s_1 s_2) = \frac{1}{2}$ and $\bar{x}(\delta(s_1)) = \bar{x}(\delta(s_2)) = k$. Since there are at most $\lfloor \frac{k}{2} \rfloor$ edges between every two nodes $v_1 \in V_1$ and $v_2 \in V_2$, it follows that

$$\bar{x}([\{s_1, s_2\}, \{u_i\}]) \geqslant \left\lceil \frac{k}{2} \right\rceil \quad \text{for } i = 1, \ldots, t.$$

Hence

$$
\begin{aligned}
2k &= \bar{x}(\delta(s_1)) + \bar{x}(\delta(s_2)) \\
&\geqslant \bar{x}([s_1, s_2]) + \sum_{i=1}^{t} \bar{x}([\{s_1, s_2\}, u_i]) \\
&\geqslant \frac{1}{2} + \sum_{i=1}^{t} \bar{x}([\{s_1, s_2\}, u_i]) \\
&\geqslant \frac{1}{2} + 4\left\lceil \frac{k}{2} \right\rceil \\
&\geqslant \frac{1}{2} + 2k,
\end{aligned}
$$

a contradiction. □

Before introducing our second class of perfectly-$k$EC graphs we give the following lemma.

**Lemma 4.3.** *Let $G = (V, E)$ be a graph and $\bar{x}$ an extreme point of $P(G, k)$. If $\delta(W)$ is a proper cut which is tight for $\bar{x}$, then $G(W)$ and $G(\overline{W})$ are both $\lceil \frac{k}{2} \rceil$-edge connected.*

**Proof.** Suppose not, then there is a partition $W_1$, $W_2$ of $W$ such that $[W_1, W_2] \leqslant \lceil \frac{k}{2} \rceil - 1$. W.l.o.g., we may suppose that $\bar{x}[W_1, \overline{W}] \leqslant \bar{x}[W_2, \overline{W}]$. Thus $\bar{x}[W_1, \overline{W}] \leqslant \frac{k}{2}$. Therefore

$$
\begin{aligned}
\bar{x}(\delta(W_1)) &= \bar{x}[W_1, \overline{W}] + \bar{x}[W_1, W_2] \\
&\leqslant \frac{k}{2} + \left\lceil \frac{k}{2} \right\rceil - 1 \\
&< k,
\end{aligned}
$$

which is impossible. □

**Theorem 4.4.** *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph without multiple edges, with $|V_1 \cup V_2| \leqslant 4k - 1$. Then $G$ is perfectly-$k$EC.*

**Proof.** Suppose that there is an extreme point $\bar{x}$ of $P(G, k)$ which is fractional. W.l.o.g., we may suppose that $\bar{x}$ is of rank 1. Since $G$ is bipartite, from Lemma 4.1, it follows that $C_p^*(\bar{x}) \neq \emptyset$. Let $\delta(W)$ be a cut of $C_p^*(\bar{x})$. As $|V_1 \cup V_2| \leqslant 4k - 1$, we may suppose that $|W| \leqslant 2k - 1$. Since $\delta(W)$ is proper, by Lemma 4.3, $G(W)$ and $G(\overline{W})$ are both $\lceil \frac{k}{2} \rceil$-edge connected. In addition, since $G$ is bipartite without multiple edges, we get

$$m_i = |W \cap V_i| \geqslant \left\lceil \frac{k}{2} \right\rceil, \quad i = 1, 2.$$

It then follows that $k \leqslant |W|$. We may w.l.o.g., suppose that $m_1 \leqslant m_2$. As $|W| \leqslant 2k - 1$, $m_1 \leqslant k - 1$. Moreover, we have

$$\begin{aligned} k = \bar{x}(\delta(W)) &\geqslant \bar{x}([W \cap V_2, \overline{W} \cap V_1]) \\ &\geqslant m_2. \end{aligned}$$

The two last inequalities come from the fact that $m_1 \leqslant k-1$, and hence $\bar{x}([\{v\}, \overline{W}]) \geqslant 1$ for all $v \in W \cap V_2$. Thus $m_2 \leqslant k$. We shall consider two cases

*Case 1:* $m_2 = k$.

Then

$$k = \bar{x}(\delta(W)) \geqslant \bar{x}[W \cap V_2, \overline{W} \cap V_1] \geqslant k.$$

As $\bar{x}(e) \geqslant 0$ for all $e \in E$, this implies that $\bar{x}([W \cap V_1, \overline{W} \cap V_2]) = 0$. And hence

$$\begin{aligned} &m_1 = k - 1, \\ &\bar{x}(e) = 0 \qquad \forall\, e \in [W \cap V_1, \overline{W} \cap V_2], \\ &\bar{x}(\delta(v)) = k \qquad \forall\, v \in W. \end{aligned}$$

Moreover $x(\delta(W)) = k$ is redundant with respect to the degree equations and $x(e) = 1$, $e \in E_1(\bar{x})$, a contradiction.

*Case 2:* $m_2 < k$.

We then have

$$\begin{aligned} k = \bar{x}(\delta(W)) &= \bar{x}([W \cap V_1, \overline{W} \cap V_2]) + \bar{x}([W \cap V_2, \overline{W} \cap V_1]) \\ &\geqslant m_1(k - m_2) + m_2(k - m_1) \\ &\geqslant m_1 + m_2 \\ &\geqslant k. \end{aligned}$$

The two last inequalities come from the fact that $k - m_i \geqslant 1$, $i = 1, 2$, and $|W| = m_1 + m_2 \geqslant k$. Thus all the above inequalities are satisfied with equality. Therefore we obtain that

$$\begin{aligned} &\bar{x}([W \cap V_1, \overline{W} \cap V_2]) = m_1(k - m_2), \\ &k = m_1 + m_2, \\ &\bar{x}([W \cap V_2, \overline{W} \cap V_1]) = m_2(k - m_1). \end{aligned}$$

This implies that

$$\begin{aligned} &\bar{x}(\delta(v)) = k \qquad \forall\, v \in W, \\ &\bar{x}(e) = 1 \qquad \forall\, e \in [W \cap V_1, W \cap V_2]. \end{aligned}$$

We again obtain that $x(\delta(W)) = k$ is redundant in system $S(\bar{x})$, which is impossible. This ends the proof. $\square$

Along the same lines as in the proof of Theorem 4.4, we can also show the following.

**Theorem 4.5.** *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph without multiple edges. If* $\min\{|V_1|, |V_2|\} \leqslant k + 1$, *then G is perfectly-kEC.*

## 5. Concluding remarks

We have introduced the concept of critical extreme points of the polytope $P(G, k)$ and described some structural properties of these extreme points. Using this we characterized two classes of perfectly $k$-edge connected graphs. These results can be seen as a first step toward a complete characterization of this class of graphs.

In a forthcoming paper we will discuss some polyhedral and algorithmic consequences of these results. In particular we will describe a large class of facets for the $k$-edge connected subgraph polytope and show that critical extreme points may be separated from that polytope in polynomial time using those facets. We will also describe some separation techniques. Using this we will devise a Branch&Cut algorithm for the $k$-edge connected subgraph problem. The reduction operations $\theta_1', \ldots, \theta_4'$ may be effective in solving the $k$-edge connected subgraph problem. In fact they may be used in a preprocessing phase of the Branch&Cut and then permit to considerably accelerate the separation process. A Branch&Cut algorithm based on the critical extreme points of the 2-edge connected subgraph polytope is discussed in [22].

## References


[1] M. Baïou, A.R. Mahjoub, Steiner 2-edge connected subgraph polytopes on series–parallel graphs, SIAM J. Discrete Math. 10 (1997) 505–514.

[2] F. Barahona, A.R. Mahjoub, On two-connected subgraph polytopes, Discrete Math. 147 (1995) 19–34.

[3] D. Bienstock, E.F. Brickell, C.L. Monma, On the structure of minimum weight $k$-connected spanning networks, SIAM J. Discrete Math. 3 (1990) 320–329.

[4] S.C. Boyd, T. Hao, An integer polytope related to the design of survivable communication networks, SIAM J. Discrete Math. 5 (1992) 321–337.

[5] S. Chopra, The $k$-edge connected spanning subgraph polyhedron, SIAM J. Discrete Math. 7 (1994) 245–259.

[6] G. Cornuéjols, J. Fonlupt, D. Naddef, The traveling salesman problem on a graph and some related integer polyhedra, Math. Programming 33 (1985) 1–27.

[7] R. Coullard, A. Rais, R.L. Rardin, D.K. Wagner, The 2-connected-Steiner subgraph polytope for series–parallel graphs, Technical Report, School of Industrial Engineering, Purdue University, 1991.

[8] R. Coullard, A. Rais, R.L. Rardin, D.K. Wagner, Linear-time algorithm for the 2-connected Steiner subgraph problem on special classes of graphs, Networks 23 (1993) 195–206.

[9] R. Coullard, A. Rais, R.L. Rardin, D.K. Wagner, The dominant of the 2-connected-Steiner subgraph polytope for $W_4$-free graphs, Discrete Appl. Math. 66 (1996) 33–43.

[10] M. Didi Biha, A.R. Mahjoub, $k$-edge connected polyhedra on series–parallel graphs, Oper. Res. Lett. 19 (1996) 71–78.

[11] M. Didi Biha, A.R. Mahjoub, Steiner $k$-edge connected subgraph polyhedra, J. Combin. Optim. 4 (2000) 131–144.

[12] E. Dinits, Algorithm for solution of a problem of maximum flow in a network with power estimation, Soviet Math. Dokl. 11 (1970) 1277–1280.

[13] J. Edmonds, R. Karp, Send-and-split method for minimum-concave-cost network flows, J. Assoc. Comput. Mach. 19 (1972) 248–264.

[14] J. Fonlupt, A.R. Mahjoub, Critical extreme points of the 2-edge connected spanning subgraph polytope, Proceedings IPCO'99, LNCS 1610, (1999), pp. 166–183.

[15] J. Fonlupt, D. Naddef, The traveling salesman problem in graphs with some excluded minors, Math. Programming 53 (1992) 147–172.

[16] M. Grötschel, L. Lovász, A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (1981) 70–89.

[17] M. Grötschel, C.L. Monma, Integer polyhedra arising from certain net work design problems with connectivity constraints, SIAM J. Discrete Math. 3 (1990) 502–523.

[18] M. Grötschel, C.L. Monma, M. Stoer, Polyhedral approaches to network survivability, in: Reliability of computer and Communication Networks, 1991, pp. 121–141.

[19] M. Grötschel, C.L. Monma, M. Stoer, Computational results with a cutting plane algorithm for designing communication networks with low-connectivity constraints, Oper. Res. 40 (1992) 309–330.

[20] M. Grötschel, C.L. Monma, M. Stoer, Facets for polyhedra arising in the design of communication networks with low-connectivity constraints, SIAM J. Optim. 2 (1992) 474–504.

[21] M. Grotschel, C.L. Monma, M. Stoer, Polyhedral and computational investigations for designing communication networks with high survivability requirements, Oper. Res. 43 (1995) 1012–1024.

[22] H. Kerivin, A.R. Mahjoub, C. Nocq, (1,2)-Survivable networks: facets and branch-and-cut, in: M. Grötschel (Ed.), The Sharpest Cut, MPS–SIAM Series in Optimization, 2004, pp. 121–152.

[23] C.W. Ko, C.L. Monma, Heuristic methods for designing highly survivable communication networks, Technical Report, Bell Communication Research, 1989.

[24] A.R. Mahjoub, Two-edge connected spanning subgraphs and polyhedra, Math. Programming 64 (1994) 199–208.

[25] A.R. Mahjoub, On perfectly 2-edge connected graphs, Discrete Math. 95 (1997) 153–172.

[26] C.L. Monma, D.F. Shallcross, Methods for designing communication networks with certain two-connected survivability constraints, Oper. Res. 37 (1989) 531–541.

[27] M. Stoer, Design of Survivable Networks, Lecture Notes in Mathematics, Springer-Verlag, 1992.