# FACETS OF THE BALANCED (ACYCLIC) INDUCED SUBGRAPH POLYTOPE 

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#### Abstract

A signed graph is a graph whose edges are labelled positive or negative. A signed graph is said to be balanced if the set of negative edges form a cut. The balanced induced subgraph polytope $P(G)$ of a graph $G$ is the convex hull of the incidence vectors of all node sets that induce balanced subgraphs of $G$. In this paper we exhibit various (rank) facet defining inequalities. We describe several methods with which new facet defining inequalities of $P(G)$ can be constructed from known ones. Finding a maximum weighted balanced induced subgraph of a series parallel graph is a polynomial problem. We show that for this class of graphs $P(G)$ may have complicated facet defining inequalities. We derive analogous results for the polytope of acyclic induced subgraphs.


Key words: Balanced subgraphs, acyclic subgraphs, facets of polyhedra.

## 1. Introduction

The graphs we consider are finite, undirected, without loops and may have multiple edges. A graph is denoted by $G=(V, E)$ where $V$ is the node set and $E$ is the edge set of $G$. An edge $e \in E$ whose vertices are $u$ and $v$ can also be denoted by $u v$. If $W \subseteq V$, then $E(W)$ denotes the set of all edges of $G$ with both endnodes in $W$. The graph $H=(W, E(W))$ is the subgraph of $G$ induced by $W$.

A signed graph $G=(V, E)$ is a graph whose edges are labelled positive or negative. Signed graphs have been introduced by Harary [6]. A signed graph is said to be balanced if its set of negative edges form a cut. In other words, a signed graph $G=(V, E)$ is balanced if the node set $V$ can be partitioned into $U$ and $\bar{U}$ in such a way that $E(U) \cup E(\bar{U})$ is the set of positive edges, $U$ (or $\bar{U}$ ) may be empty. If $W \subseteq V$ and $H=(W, E(W))$ is balanced, then $H$ is called a balanced induced subgraph of $G$ (BIS for short). When $G=(V, E)$ is a signed graph all of whose edges are labelled negative, a BIS of $G$ is a bipartite induced subgraph of $G$. Given a signed graph $G=(V, E)$ with node weights $c(v)$ for all $v \in V$, the maximum BIS

[^0]problem consists of finding a BIS $(W, E(W))$ of $G$ such that $c(W)=\sum_{v \in W} c(v)$ is as large as possible.

The maximum BIS problem is a generalization of the maximum stable set problem. In fact, if $H=(V, F)$ is a graph then the maximum stable set problem in $H$ can be reduced to a maximum BIS problem in the signed graph $G=(V, E)$ that is obtained from $H$ by replacing each edge in $F$ by a positive and negative edge. This implies that the maximum BIS problem is NP-hard.

If $W \subseteq V$ let $x^{W} \in \mathbb{R}^{V}$ where $x^{W}(u)=1$ if $u \in W$ and $x^{W}(u)=0$ otherwise; $x^{W}$ is called the incidence vector of $W$.

The convex hull of the incidence vectors of all BIS of $G$, denoted by $P(G)$, is called the BIS polytope, i.e.

$$
P(G)=\operatorname{Conv}\left\{x^{W} \in \mathbb{R}^{V} \mid W \subseteq V,(W, E(W)) \text { is balanced }\right\}
$$

Thus the maximum BIS problem in $G$ may be stated as the following linear program

$$
\max \{c x, x \in P(G)\}
$$

In [2] we gave a polynomial algorithm to solve the maximum BIS problem in series parallel graphs. The development of a polynomial algorithm for combinatorial optimization problems has often been closely related to the characterization of a system of inequalities that defines the corresponding polytope. This is the case for the maximum stable set problem in series parallel graphs, where the stable set polytope is defined by the clique and the odd cycle inequalities, of. [3]. If such a system of inequalities is known then linear programming duality can be used to prove optimality and to derive a max-min relation. Our algorithm for the BIS problem does not provide an explicit description of the polytope. In this paper we shall derive a partial description of those inequalities. We shall show that such a system is "less simple" than for the stable set polytope.

A partial knowledge of the facets of combinatorial polyhedra may have algorithmic use. Some instances of NP-hard problems can be solved using linear programming techniques. The Travelling Salesman Problem [4], the Linear Ordering Problem [5] and the Max-Cut Problem [1] are examples of this.

In this paper we shall exhibit various classes of (rank) facet defining inequalities of $P(G)$. We describe several methods to derive "facets from facets", examples of these methods are subdivision of edges and addition of nodes.

We shall give a method to derive facets of $P(G)$ from facets of the Stable Set Polytope. Using these constructions we exhibit complicated facet defining inequalities of $P(G)$. Analogous results for the acyclic induced subgraph polytope will also be presented.

Now we shall introduce some notation.
A path $P$ in $G=(V, E)$ is a sequence of edges $e_{1}, e_{2}, \ldots, e_{k}$ such that $e_{1}=v_{0} v_{1}$, $e_{2}=v_{1} v_{2}, \ldots, e_{k}=v_{k-1} v_{k}$ and such that $v_{i} \neq v_{j}$ for $i \neq j$. The nodes $v_{0}$ and $v_{k}$ are the endnodes of $P$ and we say that $P$ links $v_{0}$ and $v_{k}$ and that $v_{1}, \ldots, v_{k-1}$ are the internal nodes of $P$. The number $k$ of edges of $P$ is called the length of $P$.

If $P=e_{1}, e_{2}, \ldots, e_{k}$ is a path linking $v_{0}$ and $v_{k}$, and $e_{k+1}=v_{0} v_{k} \in E$, then the sequence $e_{1}, e_{2}, \ldots, e_{k}, e_{k+1}$ is called a cycle of length $k+1$. We say that a path is odd (even) if it contains an odd (even) number of negative edges. If $u v$ is an edge then $G \backslash u v$ denotes the graph obtained by deleting $u v$. A clique is a maximal complete subgraph. A negative graph is a signed graph all of whose edges are negative.

A polyhedron $P \subseteq \mathbb{R}^{n}$ is the intersection of finitely many halfspaces in $\mathbb{R}^{m}$. A polytope is a bounded polyhedron or equivalently the convex hull of finitely many points. The dimension of a polyhedron $P$, denoted by $\operatorname{dim} P$, is the maximum number of affinely independent points in $P$ minus one.

If $a \in \mathbb{R}^{m} \backslash\{0\}, a_{0} \in \mathbb{R}$, then the inequality $a x \leqslant a_{0}$ is said to be valid with respect to a polyhedron $P \subseteq \mathbb{R}^{m}$ if $P \subseteq\left\{x \in \mathbb{R}^{m} \mid a x \leqslant a_{0}\right\}$. We say that a valid inequality $a x \leqslant a_{0}$ supports $P$ or define a face of $P$ if $\emptyset \neq P \cap\left\{x \mid a x=a_{0}\right\} \neq P$. A valid inequality $a x \leqslant a_{0}$ defines a facet of $P$ if it defines a face of $P$ and if there exist $\operatorname{dim} P$ affinely independent points in $P \cap\left\{x \mid a x=a_{0}\right\}$.

For every facet $F$ of a full-dimensional polyhedron $P \subseteq \mathbb{R}^{m}$, i.e. $\operatorname{dim} P=m$, there exists a unique (up to multiplication by a positive constant) valid inequality $a x \leq a_{0}$ such that $F=\left\{x \in P \mid a x=a_{0}\right\}$. This implies that (up to multiplication by a positive constant) there is a unique minimal system $A x \leqslant b$ such that $P=\left\{x \in \mathbb{R}^{m} \mid A x \leqslant b\right\}$.

## 2. Facets of $\boldsymbol{P ( G )}$

Given a signed graph $G=(V, E)$ let

$$
\beta(G)=\{W \subseteq V \mid(W, E(W)) \text { is balanced }\}
$$

denote the family of node sets of balanced induced subgraphs. Clearly, $\beta(G)$ is an independence system on $V$, i.e., $\emptyset \in \beta(G)$ and $W^{\prime} \subseteq W \in \beta(G) \Rightarrow W^{\prime} \in \beta(G)$. So if $a x \leqslant \alpha$ defines a facet of $P(G)$ with $\alpha>0$, then $a \geqslant 0$.

For every set $Y \subseteq V$ the number

$$
r(Y)=\max \{|W|: W \subseteq Y, W \in \beta(G)\}
$$

is called the rank of $Y$.
If $Y \subseteq V$ then the $0 / 1$-inequality

$$
\begin{equation*}
x(Y)=\sum_{u \in Y} x(u) \leqslant r(Y) \tag{2.1}
\end{equation*}
$$

is a supporting inequality of $P(G)$. The inequalities of type (2.1) will be called rank inequalities.

In this section we shall introduce some classes of facet defining rank inequalities, further such inequalities will be derived in the next section.

Given a signed graph $G=(V, E)$, a cycle of $G$ will be called frustrated if it contains an odd number of negative edges. If $W$ is the node set of a frustrated cycle a BIS contains at most $|W|-1$ elements of $W$.

An integer vector $x \in \mathbb{R}^{V}$ is the incident vector of a BIS if and only if it satisfies

$$
\begin{align*}
& -x(u) \leqslant 0 \text { for all } u \in V  \tag{2.2}\\
& x(u) \leqslant 1 \text { for all } u \in V  \tag{2.3}\\
& x(W) \leqslant|W|-1 \quad \text { for all } W \subseteq V \text { such that }(W, E(W)) \text { is a frustrated cycle. } \tag{2.4}
\end{align*}
$$

Constraints (2.2) and (2.3) will be called trivial. It is easy to see that (2.2) defines facets for $P(G)$. This is not necessarily the case for (2.3) and (2.4). In what follows we shall study when those inequalities define facets.

A signed graph is called bicomplete if every two different nodes are linked by a positive and a negative edge. A biclique of a graph $G$ is a maximal bicomplete subgraph of $G$.

Theorem 2.5. Let $G=(V, E)$ be a signed graph and $H=(W, E(W))$ be a bicomplete subgraph of $G$. Then

$$
\begin{equation*}
x(W) \leqslant 1 \tag{2.6}
\end{equation*}
$$

defines a facet of $P(G)$ if and only if $H$ is a biclique of $G$.

Proof. It is clear that (2.6) is valid for $P(G)$. Suppose that $W=\left\{v_{1}, \ldots, v_{k}\right\}, k \geqslant 1$, and that $H=(W, E(W))$ is a biclique of $G$, then for every node $v \in V \backslash W$ there exists at least one node in $W$, say $v_{i}$, such that $\bar{W}_{v}=\left\{v, v_{i}\right\} \in \beta(G)$. Now consider the sets

$$
\begin{array}{ll}
W_{v}=\{v\} & \text { if } v \in W \\
W_{v}=\bar{W}_{v} & \text { if } v \in V \backslash W
\end{array}
$$

Then $W_{v} \in \beta(G)$ for all $v \in V$. Moreover, the incidence vectors $x^{W_{v}}, v \in V$, are linearly independent and satisfy (2.6) with equation. This implies that (2.6) defines a facet. Conversely, if (2.6) defines a facet of $P(G)$ then $W$ must be the node set of a bicomplete subgraph of $G$. Assume that this subgraph is not a biclique then there exists a node $v_{0} \in V \backslash W$ such that $W \cup\left\{v_{0}\right\}$ induces a bicomplete subgraph of $G$. Thus every node set $Y \subseteq V$, such that $Y \in \beta(G)$ and its incidence vector $x^{Y}$ satisfies (2.6) with equality, cannot contain $v_{0}$, a contradiction.

Theorem 2.5 implies that a constraint $x_{u} \leqslant 1$ for $u \in V$ defines a facet for $P(G)$ if and only if $u$ is not linked to another node in $V$ by a positive edge and a negative edge.

Theorem 2.7. A constraint of type (2.4) defines a facet if and only if
(i) $(W, E(W))$ is a chordless cycle,
(ii) for each node $v \in V \backslash W$ the subgraph induced by $W \cup\{v\}$ contains at most two frustrated cycles.

Proof. Let $W=\left\{w_{1}, \ldots, w_{p}\right\}$.
Suppose that an inequality (2.4) defines a facet.
If there exists a chord $w_{1} w_{k}$ then the subgraph induced by $\boldsymbol{W}^{\prime}=\left\{w_{1}, \ldots, w_{k}\right\}$ or $W^{\prime}=\left\{w_{k}, w_{k+1}, \ldots, w_{1}\right\}$ has a frustrated cycle. The inequality

$$
\begin{equation*}
x\left(W^{\prime}\right) \leqslant\left|W^{\prime}\right|-1 \tag{2.8}
\end{equation*}
$$

is valid for $P(G)$.
Inequality (2.4) can be obtained by summing (2.8) and $x(v) \leqslant 1$, for $v \in W \backslash W^{\prime}$, a contradiction.

If there is a nodes $v_{0}$ such that the subgraph induced by $W \cup\left\{v_{0}\right\}$ contains more than two frustrated cycles then it contains at least four. This implies that the subgraph induced by $\left(W \cup\left\{v_{0}\right\}\right) \backslash\{v\}$ has a frustrated cycle, for any $v \in W$. In this case the inequality

$$
\begin{equation*}
x(W)+x\left(v_{0}\right) \leqslant|W|-1 \tag{2.9}
\end{equation*}
$$

is valid for $P(G)$. Inequalities (2.9) and $-x\left(v_{0}\right) \leqslant 0$ imply (2.4).
Conversely, suppose that (i) and (ii) hold. We shall exhibit a set of $|V|$ linearly independent vectors that satisfy (2.4) as equation.

For $j=1, \ldots, k$ set $W_{j}=W \backslash\left\{w_{j}\right\}$.
Let $V \backslash W=\left\{w_{k+1}, \ldots, w_{n}\right\}$.
For $j=k+1, \ldots, n$ there are two cases.
Case 1. $\left(W \cup\left\{w_{j}\right\}, E\left(W \cup\left\{w_{j}\right\}\right)\right)$ contains only one frustrated cycle. We set $W_{j}=$ $\left(W \backslash\left\{w_{1}\right\}\right) \cup\left\{w_{j}\right\}$.
Case 2. $\left(W \cup\left\{w_{j}\right\}, E\left(W \cup\left\{w_{j}\right\}\right)\right)$ contains two frustrated cycles. Let $w_{j} w_{l}$, $w_{l} w_{l+1}, \ldots, w_{r} w_{j}$ be the frustrated cycle that contains $w_{j}$. We set $W_{j}=\left(W \backslash\left\{w_{l}\right\}\right) \cup\left\{w_{j}\right\}$.

The vectors $x^{W_{1}}, \ldots, x^{W_{n}}$ form a matrix $M$ with the following structure:

$$
M=\left[\begin{array}{ll}
A & R \\
O & I
\end{array}\right]
$$

where $I$ denotes an identity matrix and $A$ denotes a matrix like

$$
\left[\begin{array}{lllll}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 \\
& & & & \\
1 & 1 & 1 & \cdots & 0
\end{array}\right] .
$$

Corollary 2.10. If $G$ is series parallel then chordless frustrated cycles induce facets of $P(G)$.

Theorem 2.11. Let $G=(V, E)$ be a negative graph and $W \subseteq V$ a set of nodes. The inequality

$$
\begin{equation*}
x(W) \leqslant 2 \tag{2.12}
\end{equation*}
$$

defines a facet of $P(G)$ if and only if $|W| \geqslant 3$ and $W$ induces a clique of $G$.


Fig. 1.
The proof of this is analogous to the proof of Theorem 2.5.
Corollary 2.13. If $G=(V, E)$ is a negative graph, then every facet defining inequality $a x \leqslant \alpha$ of $P(G)$ not of type (2.3), with integral coefficients and $\alpha=2$ is of type (2.12).

Proof. First of all, it is clear that $a_{v} \geqslant 0$ for $v \in V$. If $a_{v_{0}}=2$ then this is the inequality

$$
2 x\left(v_{0}\right) \leqslant 2
$$

hence $a_{v} \leqslant 1$ for all $v \in V$.
This is not true for all signed graphs. In fact, consider the graph $G=(V, E)$ in Fig. 1, where the dashed lines correspond to positive edges.

The inequality

$$
x(1)+x(2)+x(3)+2 x(4) \leqslant 2
$$

defines a facet of $P(G)$.

## 3. Construction of facets

In this section we shall derive facet defining inequalities using transformations of the graph. We first state a standard lifting theorem.

Theorem 3.1. Let $G=(V, E)$ be a signed graph and $a x \leqslant \alpha$ be a non trivial facet defining inequality for $P(G)$. Let uv be a negative (positive) edge of $G$ such that $a_{u}, a_{v}>0$, set $\gamma=\max \left\{a x^{W}: W \in \beta(G \backslash u v)\right\}$. Let $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ be the graph obtained from $G$ as follows: The edge $u v$ is deleted; a new node $v_{0}$ is added; the path $u v_{0}, v_{0} v$ is added; the edges $u v_{0}, v_{0} v$ are labelled in such a way that this path is odd (even). Set

$$
\bar{a}_{u}=a_{u} \text { for all } u \in V \cap V^{\prime}, \quad \bar{a}_{v_{0}}=\gamma-\alpha, \quad \bar{\alpha}=\gamma,
$$

then

$$
\begin{equation*}
\bar{a} x \leqslant \bar{a} \tag{3.2}
\end{equation*}
$$

defines a facet of $P\left(G^{\prime}\right)$.

Corollary 3.3. (Replacing a negative (positive) edge by an even (odd) path.) Let $G=(V, E)$ be a signed graph and ax $\leqslant \alpha$ be a nontrivial facet defining inequality of $P(G)$. Let uv $\in E$ be a negative (positive) edge such that $a_{u}, a_{v}>0$ and $u$ is of degree two. Let $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ be the graph obtained from $G$ in the following way. A set of new nodes $\left\{v_{1}, v_{2}, \ldots, v_{k}\right\}$ is added. The edges set $P=\left\{u v_{1}, v_{1} v_{2}, \ldots, v_{k} v\right\}$ is added (i.e. a path is added). The new edges are signed in such a way that $P$ is odd (even). The edge $u v$ is removed. Let $\bar{a} \in \mathbb{R}^{V^{\prime \prime}}$ be defined as follows:

$$
\bar{a}_{w}=a_{w} \text { for all } w \in V \cap V^{\prime}, \quad \bar{a}_{v_{i}}=a_{u}, \quad \bar{\alpha}=k a_{u}-\alpha,
$$

then $\bar{a} x \leqslant \bar{\alpha}$ defines a facet of $P\left(G^{\prime}\right)$.
Proof. Without loss of generality we can suppose that path $P$ is of size two (i.e. $k=1$ ). Since $a x \leqslant \alpha$ defines a nontrivial facet, there must exist a node set $W_{0} \subseteq V \backslash\{u\}$ such that $W_{0} \in \beta(G)$ and $a x^{W_{0}}=\alpha$. Since $u$ is of degree two it is clear that $W_{0} \cup\{u\} \in$ $\beta(G \backslash u v)$ and thus $\alpha+a_{u}=\max \left\{a^{\mathrm{T}} x^{W}: W \in \beta(G \backslash u v)\right\}$. So from Theorem 3.1 it follows that $\bar{a} x \leqslant \bar{\alpha}$ defines a facet of $P\left(G^{\prime}\right)$.

Now we present a different lifting procedure that can be visualized as adding nodes to the graph.

Theorem 3.4. Let $G=(V, E)$ be a signed graph and $a x \leqslant \alpha$ a nontrivial facet defining inequality of $P(G)$. Let $p$ and $q$ be two nodes such that $0 \leqslant a_{p} \leqslant a_{q}$. Suppose that every maximal BIS of $G$ intersects $\{p, q\}$. Consider the graph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ obtained from $G$ in the following way: add three new nodes $v_{1}, v_{2}, v_{3}$ and the edges $\left\{p v_{1}, p v_{2}, q v_{2}, q v_{3}, v_{1} v_{2}, v_{2} v_{3}\right\}$, labelled negative. See Fig. 2.

Set

$$
\bar{a}_{u}=a_{u} \text { for all } u \in V, \quad \bar{a}_{v_{1}}=\bar{a}_{v_{2}}=\bar{a}_{v_{s}}=a_{p}, \quad \bar{\alpha}=\alpha+2 a_{p}
$$

Then $\bar{a} x \leqslant \bar{\alpha}$ defines a facet for $P\left(G^{\prime}\right)$.

Proof. First, we show that $a x \leqslant \alpha$ is valid for $P\left(G^{\prime}\right)$.


Fig. 2.

If $W^{\prime} \subseteq V^{\prime}$ induces a BIS then the node set

$$
W=\left(W^{\prime} \backslash\left\{v_{1}, v_{2}, v_{3}\right\}\right) \subseteq V
$$

induces a BIS of $G$. So if $\{p, q\} \cap W \neq \emptyset$ then $\left|\left\{v_{1}, v_{2}, v_{3}\right\} \cap W^{\prime}\right| \leqslant 2$ and thus $\bar{a} x^{W^{\prime}} \leqslant \bar{\alpha}$. If $\{p, q\} \cap W=\emptyset$ we can extend $W$ to a node set $W_{1} \subseteq V$ such that $W_{1} \in \beta(G)$ and $\{p, q\} \cap W_{1} \neq \emptyset$. Hence $a x^{W} \leqslant \alpha-a_{p}$ which implies that $\bar{a} x^{W^{\prime}} \leqslant \bar{\alpha}$.

Now let us assume that

$$
\{x \mid \bar{a} x=\bar{\alpha}\} \cap R\left(G^{\prime}\right) \subseteq\{x \mid b x=\gamma\}
$$

where $b x \leqslant \gamma$ is a facet defining inequality of $P\left(G^{\prime}\right)$.
Since $a x \leqslant \alpha$ defines a facet of $P(G)$ there are $n=|V|$ nodes sets $W_{1}, \ldots, W_{n}$ in $\beta(G)$ whose incidence vectors are linearly independent and satisfy $a x^{W_{i}}=\alpha$ for $i=1, \ldots, n$.

Set $W_{i}^{\prime}=W_{i} \cup\left\{v_{1}, v_{3}\right\}$, for $i=1, \ldots, n$. It is clear that $W_{i}^{\prime}, \ldots, W_{n}^{\prime} \in \beta\left(G^{\prime}\right)$ and $\bar{a} x^{W_{i}^{\prime}}=\bar{\alpha}, i=1, \ldots, n$. Then $b x^{W_{i}^{\prime}}=\gamma$, for $i=1, \ldots, n$. We can conclude that $b_{u}=\rho \bar{a}_{u}$ for $u \in V^{\prime} \backslash\left\{v_{1}, v_{2}, v_{3}\right\}$.

Since $a x \leqslant \alpha$ is a nontrivial inequality and from our hypothesis we can conclude that among sets $W_{1}, \ldots, W_{n}$ there are two sets, say $W_{1}$ and $W_{2}$, such that $\{p, q\} \cap$ $W_{1}=\{p\}$ and $\{p, q\} \cap W_{2}=\{q\}$.

Let $W_{1}^{\prime \prime}=W_{1} \cup\left\{v_{2}, v_{3}\right\}$ and $W_{2}^{\prime \prime}=W_{2} \cup\left\{v_{1}, v_{2}\right\}$, then $\bar{a} x^{W_{1}^{\prime \prime}}=\bar{a} x^{W_{2}^{\prime \prime}}=\bar{\alpha}$, we have that

$$
0=b x^{W_{1}^{\prime}}-b x^{W_{1}^{\prime}}=b_{v_{1}}-b_{v_{2}}, \quad 0=b x^{W_{2}^{\prime}}-b x^{W_{2}^{\prime \prime}}=b_{v_{2}}-b_{v_{3}},
$$

thus $b_{v_{1}}=b_{v_{2}}=b_{v_{3}}$.
Now consider the set $W_{3}^{\prime \prime}=\left(W_{1} \backslash\{p\}\right) \cup\left\{v_{1}, v_{2}, v_{3}\right\}$, it is clear that $W_{3}^{\prime \prime} \in \beta\left(G^{\prime}\right)$ and $\bar{a} x^{w_{3}^{\prime \prime}}=\bar{\alpha}$. Then

$$
0=b x^{W_{1}^{\prime \prime}}-b x^{W_{3}^{\prime \prime}}=b_{p}-b_{v_{2}}
$$

which implies that $\rho \bar{a}_{p}=b_{p}=b_{v_{2}}$, then $b=\rho \bar{a}$.
Since $b x \leqslant \gamma$ is a nontrivial inequality, we have that $\rho>0$.
We say that a pair of nodes $(u, v)$ satisfy the property $\pi$ if $u$ has degree two and they are linked by a path all of whose internal nodes are of degree two.

Note that if ( $u, v$ ) satisfy the property $\pi$ then

$$
W \in \beta(G) \text { and } u, v \notin W \Rightarrow W \cup\{u\} \in \beta(G)
$$

Corollary 3.5. Let $G=(V, E)$ be a signed graph and $a x \leqslant \alpha$ a nontrivial facet defining inequality of $P(G)$. Let $p$ and $q$ be two nodes of $G$ such that $a_{p}, a_{q}>0$ and $(p, q)$ satisfies the property $\pi$. Let $G^{\prime}$ and $\bar{a} x \leqslant \bar{\alpha}$ be defined as in Theorem 3.4. The inequality $\bar{a} x \leqslant \bar{a}$ defines a facet of $P\left(G^{\prime}\right)$.

Proof. Since $(p, q)$ satisfies the property $\pi$, every maximal BIS of $G$ intersects $\{p, q\}$. Since $a x^{W} \leqslant \alpha$ is nontrivial, there is a set, $W \subseteq V \backslash\{p\}$ such that $W \in \beta(G)$ and $a x^{W}=\alpha$. Since $a_{p}, a_{q}>0$, we have that $q \in W$. Let $W^{\prime}(W \backslash\{q\}) \cup\{p\}$. It is clear that $W^{\prime} \in \beta(G)$, then $a_{p} \leqslant a_{q}$. The result follows from Theorem 3.4.

## 4. $P(G)$ and the stable set polytope

Given a graph $G=(V, E)$, a stable set $S$ is a set of nodes such that no edge has both endnodes in it. The stable set polytope $S(G)$ is the convex hull of incidence vectors of the stable sets of $G$. Several classes of facet defining inequalities have been characterized for this polytope. We shall present a procedure to derive facets of $P(G)$ from facets of the stable set polytope.

If $a x \leqslant \alpha$ defines a facet of $S(G)$, we denote by $V_{a}$ the set

$$
V_{a}=\left\{u \in V \mid a_{u} \neq 0\right\} .
$$

Theorem 4.1. Let $G=(V, E)$ be a graph and $a x \leqslant \alpha$ a facet defining inequality different from $x(u) \geqslant 0, u \in V$. Let $E=\left\{u_{1} v_{1}, \ldots, u_{m} v_{m}\right\}$, suppose that for every edge $u_{i} v_{i} \in$ $E\left(V_{a}\right)$ there exists a stable set $S_{i}^{*}$ such that $S_{i}^{*} \cap\left\{u_{i}, v_{i}\right\} \neq \emptyset$, say $u_{i} \in S_{i}^{*}, S_{i}^{*} \backslash\left\{u_{i}\right\} \subseteq$ $V \backslash\left\{w \in V \mid v_{i} w \in E\right\}$ and $a x^{S_{i}^{*}}=\alpha$. Let $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ be the signed graph obtained from $G$ by adding $m$ new nodes $w_{1}, \ldots, w_{m}$ and the edges $\left\{w_{i} u_{i}, w_{i} v_{i} ; i=1,2, \ldots, m\right\}$. All the edges of $G^{\prime}$ are labelled negative. Set

$$
\begin{aligned}
& \bar{a}_{u}=a_{u} \quad \text { for } u \in V \\
& \bar{a}_{w_{i}}=\min \left\{a_{u_{i}}, a_{v_{i}}\right\} \quad \text { for } i=1, \ldots, m \\
& \bar{\alpha}=\alpha+\sum_{i=1}^{m} a_{w_{i}}
\end{aligned}
$$

The inequality $\bar{a} x \leqslant \bar{\alpha}$ defines a facet of $P\left(G^{\prime}\right)$.
Proof. First we show that this inequality is valid for $P\left(G^{\prime}\right)$. Let $W^{\prime} \subseteq V^{\prime}$ be a node set that induces a BIS. Set

$$
W=\left(W^{\prime} \cap V_{a}\right) \backslash\left\{n_{i} \mid w_{i} \notin W^{\prime}, 1 \leqslant i \leqslant m\right\},
$$

where

$$
n_{i}=\left\{\begin{array}{ll}
u_{i} & \text { if } a_{u_{i}} \leqslant a_{v_{i}}, \\
v_{i} & \text { otherwise },
\end{array} \quad 1 \leqslant i \leqslant m .\right.
$$

It is easy to see that $W$ defines a stable set of $G$. So $a x^{W} \leqslant \alpha$ and then $\bar{a} x^{W^{\prime}} \leqslant \bar{\alpha}$. Now let us assume that

$$
\left\{x \in P\left(G^{\prime}\right) \mid \bar{a} x=\bar{\alpha}\right\} \subseteq\left\{x \in P\left(G^{\prime}\right) \mid b x=\gamma\right\}
$$

where $b x \leqslant \gamma$ defines a facet of $P\left(G^{\prime}\right)$.
Since $a x \leqslant \alpha$ defines a facet of $S(G)$, there are $n=|V|$ stable sets $S_{1}, \ldots, S_{n}$ whose incidence vectors are linearly independent and satisfy $a x^{S_{i}}=\alpha, i=1, \ldots, n$.

Set $W_{i}^{\prime}=S_{i} \cup\left\{w_{1}, \ldots, w_{m}\right\}$, for $i=1, \ldots, n$.
It is clear that $W_{i} \in \beta\left(G^{\prime}\right)$ and $\bar{a} x^{W_{i}}=\bar{\alpha}$ for $i=1, \ldots, n$. Then $b x^{W_{i}^{\prime}}=\gamma$ for $i=1, \ldots, n$. Since these vectors are linearly independent, we can conclude that

$$
\bar{a}_{u}=\rho b_{u} \quad \text { for } u \in V
$$

Now for $i=1, \ldots, m$ set

$$
\begin{aligned}
& W_{i}^{*}=S_{i}^{*} \cup\left\{w_{1}, \ldots, w_{m}\right\}, \\
& \bar{W}_{i}=S_{i}^{*} \cup\left\{w_{j}, 1 \leqslant j \leqslant m, j \neq i\right\} \cup\left\{v_{i}\right\} .
\end{aligned}
$$

Thus $W_{i}^{*}, \bar{W}_{i} \in \beta\left(G^{\prime}\right)$ and $\bar{a} x^{W_{i}^{*}}=\bar{a} x \bar{W}_{i}=\alpha$ for $i=1, \ldots, m$.
Hence

$$
0=b x^{\bar{w}_{i}}-b x^{W_{i}^{*}}=b_{v_{i}}-b_{w_{i}} ;
$$

this implies that

$$
b_{v_{i}}=b_{w_{i}}=\frac{1}{\rho} \bar{a}_{v_{i}} .
$$

Since $\bar{a}$ and $\bar{b}$ are non negative we have that $\rho>0$.

## 5. Some examples

Consider an odd cycle $C=(U, T)$ where $U=\left\{u_{1}, \ldots, u_{2 k+1}\right\}, k \geqslant 1$, and $T=$ $\left\{u_{1}, u_{2}, \ldots u_{2 k} u_{2 k+1}, u_{2 k+1} u_{1}\right\}$. The constraint

$$
x(U) \leqslant k
$$

defines a facet of the stable set polytope of $C$. Consider the signed graph $G=(V, E)$ obtained from $C$ by replacing each edges $u v$ by a frustrated cycle whose nodes different from $u$ and $v$ are of degree two. Then the inequality

$$
x(V) \leqslant|V|-(k+1)
$$

defines a facet of $P(G)$.
This can be proved by replacing each edge of $C$ by a positive and a negative edge and then applying Theorem 3.1 repeatedly.

Let us denote by $\boldsymbol{G}$ the class of signed graphs defined in this way. The graph of Fig. 3 belongs to $\boldsymbol{G}$. Dashed lines correspond to positive edges.


Fig. 3.

Let $\boldsymbol{H}$ be the class of graphs that consist of two frustrated cycles with one node in common. If $F$ is a signed graph, adding $H \in \boldsymbol{H}$ to $F$ means that we identify two nodes $p, q$ in different frustrated cycles of $H$ with two nodes $p^{\prime}, q^{\prime}$ in $F$ such that ( $p^{\prime}, q^{\prime}$ ) satisfies property $\pi$.

Let $G=(V, E)$ a graph obtained by adding graphs $H \in \boldsymbol{H}$ to a graph $F \in \boldsymbol{G}$. Then the inequality

$$
x(V) \leqslant r(V)
$$

defines a facet of $P(G)$.
This can be proved by applying Theorems 3.4 and 3.1 repeatedly. The graph of Fig. 4 has been obtained by adding graphs $\boldsymbol{H} \in \boldsymbol{H}$ to the graph of Fig. 3.

Let us remark that all the graphs obtained in this way are series parallel.


Fig. 4.

## 6. Acyclic induced subgraphs

Let $D=(V, A)$ be a directed graph, the induced subgraph $(W, A(W))$ is called acyclic if it does not contain a directed cycle. We say that $(W, A(W))$ is an AIS.

Let $Q(D)$ be the acyclic induced subgraph (AIS) polytope of $D$, i.e.,

$$
Q(D)=\operatorname{Conv}\left\{x^{W} \mid(W, A(W)) \text { is acyclic }\right\}
$$

All our results about facets of the BIS polytope can be translated into characterizations of facets of the AIS polytope.

Inequalities of type (2.2) also define facets of $Q(D)$. The analogue of Theorem 2.5 holds for $Q(D)$ where bicomplete subgraph should be replaced by bidirected subgraph. A bidirected graph $D=(V, A)$ is a graph such that for each pair of nodes $i$ and $j$ the arcs $(i, j)$ and $(j, i)$ belong to $A$.

The analogue of Theorem 2.7 holds for $Q(D)$ where frustrated cycle should be replaced by directed cycle.

Let $\boldsymbol{A}(D)=\{W \subseteq V \mid(W, A(W))$ is acyclic $\}$, the analogue of Theorem 3.1 is the following.

Theorem 6.1. Let $D=(V, A)$ be a directed graph and $a x \leqslant \alpha$ a nontrivial facet defining inequality for $Q(D)$. Let $(u, v)$ be an arc such that $a_{u}, a_{v}>0$, set $\gamma=$ $\max \left\{a x^{W}: W \in \boldsymbol{A}(D \backslash(u, v))\right\}$. Let $D^{\prime}=\left(V^{\prime}, A^{\prime}\right)$ be the graph obtained from $D$ as follows: The arc $(u, v)$ is deleted; a new node $v_{0}$ is added; the path $\left(u, v_{0}\right),\left(v_{0}, v\right)$ is added. Set

$$
\bar{a}_{u}=a_{u} \text { for } u \in V \cap V^{\prime}, \quad \bar{a}_{v_{0}}=\gamma-\alpha, \quad \bar{\alpha}=\gamma,
$$

then

$$
\bar{a} x \leqslant \bar{\alpha}
$$

defines a facet of $Q\left(D^{\prime}\right)$.

The analogue of Corollary 3.3 also holds.
The analogue of Theorem 3.4 is the following.

Theorem 6.2. Let $D=(V, A)$ be a directed graph and $a x \leqslant \alpha$ a non trivial facet defining inequality of $Q(D)$. Let $p$ and $q$ be two nodes such that $0 \leqslant a_{p} \leqslant a_{q}$. Suppose that every maximal AIS of D intersects $\{p, q\}$. Consider the directed graph $D^{\prime}=\left(V^{\prime}, A^{\prime}\right)$ obtained from $D$ as follows: add three nodes $v_{1}, v_{2}, v_{3}$ and arcs in such a way that $\left\{p, v_{1}, v_{2}\right\}$ and $\left\{q, v_{2}, v_{3}\right\}$ induce two directed cycles. Set

$$
\begin{aligned}
& \bar{a}_{u}=a_{u} \quad \text { for } u \in V \cap V^{\prime}, \\
& \bar{a}_{v_{1}}=\bar{a}_{v_{2}}=\bar{a}_{v_{s}}=a_{p}, \\
& \bar{\alpha}=\alpha+2 a_{p} .
\end{aligned}
$$

Then $\bar{a} x \leqslant \bar{a}$ defines a facet of $Q\left(D^{\prime}\right)$.

The analogue of Corollary 3.5 also holds.
The analogue of Theorem 4.1 also holds. Let $G^{\prime}$ and $\left\{u_{i}, v_{i}, w_{i}\right\}, 1 \leqslant i \leqslant m$, be defined as in Theorem 4.1. $D^{\prime}$ is obtained by giving an orientation to $G^{\prime}$ in such a way that $\left\{u_{i}, v_{i}, w_{i}\right\}$ induces a directed cycle for $i=1, \ldots, m$.

Examples analogous to those of Section 5 can also be derived.

## References

[1] F. Barahona, M. Grötschel, M. Jünger and G. Reinelt, "An application of combinatorial optimization to statistical physics and circuit layout design," Operations Research 36 (1988) 493-513.
[2] F. Barahona and A.R. Mahjoub, "Compositions of graphs and polyhedra I," Research Report CORR 86-16, University of Waterloo (Waterloo, Ontario, 1986).
[3] M. Boulala and J.P. Uhry, "Polytope des indépendants d'un graphe série-paralèle," Discrete Mathematics 27 (1979) 225-243.
[4] H. Crowder and M.W. Padberg, "Solving large scale symmetric travelling salesman problems," Management Science 26 (1980) 495-509.
[5] M. Grötschel, M. Jünger and G. Reinelt, "A cutting plane algorithm for the linear ordering problem," Operations Research 32 (1984) 1195-1220.
[6] F. Harary, "On the notion of balance of a signed graph," Michigan Mathematical Journal 2 (1952) 143-146.
[7] M.W. Padberg, "On the facial structure of set packing polyhedra," Mathematical Programming 5 (1973) 199-215.


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