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# COMPOSITIONS OF GRAPHS AND POLYHEDRA III: GRAPHS WITH NO $W_4$ MINOR*

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**Abstract.** The authors characterize the stable set polytope for graphs that do not have a 4-wheel as a minor. The authors prove that the nontrivial facets are either "edge" inequalities or can be obtained by composing "odd cycles" and "subdivisions of $K_4$." By adding some extra variables, it is shown that the stable set problem for these graphs can be formulated as a linear program of polynomial size.

**Key words.** polyhedral combinatorics, composition of polyhedra, stable set polytope, compact systems

**AMS subject classifications.** 05C85, 90C27

**1. Introduction.** Given a graph $G = (V, E)$, a set $S \subseteq V$ is called a *stable set* if no two nodes in $S$ are adjacent. Given a stable set $S$, the incidence vector of $S$, $x^S \in \Re^V$ is defined by

$$x^S(u) = \begin{cases} 1 & \text{if } u \in S, \\ 0 & \text{if } u \notin S. \end{cases}$$

The *stable set polytope* of $G$, denoted by $P(G)$, is the convex hull of incidence vectors of stable sets of $G$. The maximum stable set problem is NP-hard, so it seems difficult to find a complete characterization of $P(G)$ for general graphs. To our knowledge, the only classes of graphs, besides perfect graphs, for which this polytope has been characterized are line graphs [4], series parallel graphs [2], [13], almost bipartite graphs [5], and graphs with no odd $K_4$ [7]. The class studied by Gerards and Schrijver [7] contains the classes studied by Boulala, Fonlupt, and Uhry. In this case, the only nontrivial facets correspond to edges and odd holes. All the linear systems mentioned above consist of inequalities with 0-1 coefficients.

In this paper, we characterize the stable set polytope for graphs that do not have a 4-wheel as a minor. The inequalities are more difficult to describe than in the preceding cases, and they may have arbitrarily large coefficients.

Graphs in this class can be decomposed by two-vertex cuts [10]. We use this property to prove that the nontrivial facets are either edges or can be obtained by composing odd cycles and subdivisions of $K_4$. A list of the facets for subdivisions of $K_4$ is also given. We also show that, by adding some extra variables, the stable set problem in graphs with no 4-wheel minor can be formulated as a linear program of polynomial size. A polynomial combinatorial algorithm for the stable set problem in this class can be easily derived [1].

If $G$ has a one-node or a two-node cutset, then $G$ decomposes into $G_1$ and $G_2$. In a companion paper [1], we gave a technique to characterize $P(G)$ starting from systems related to $G_1$ and $G_2$. In this paper, we apply that technique.

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A connected graph $G$ is said to have the graph $H$ as a *minor* if $H$ can be obtained from $G$ by deleting some edges and by a sequence of elementary contractions in which a pair of adjacent vertices is identified and all other adjacencies between vertices are preserved (multiple edges arising from the identification being replaced by single edges).

We denote a 4-wheel by $W_4$; see Fig. 1.1.

This paper is organized as follows. In §2 we summarize our composition techniques. In §3 we describe the decomposition of graphs with no $W_4$ minor. In §4 we state our main result. Sections 5 and 6 are devoted to the study of the subdivisions of $K_4$.

We conclude this introduction with a few definitions.

The polytope $P(G)$ is full-dimensional. This implies that (up to multiplication by a positive constant) there is a unique nonredundant inequality system $Ax \leq b$ such that $P(G) = \{x : Ax \leq b\}$. These inequalities define the *facets* of $P(G)$. In many cases, we say that the inequality $ax \leq \alpha$ *is a facet* instead of saying that it defines a facet. If the inequality has at least two nonzero coefficients, we say that it is a *nontrivial* facet.

If $ax \leq \alpha$ defines a facet of $P(G)$, we denote by $V_a$ the set

$$V_a = \{v : a_v > 0\}.$$

The subgraph induced by $V_a$ is denoted by $G_a$, and it is called the *support* of the facet.

We denote by $uv$ the edge whose endnodes are $u$ and $v$. If $U \subseteq V$, then $E(U)$ denotes the set of edges with both endnodes in $U$, and $(U, E(U))$ is the *subgraph induced* by $U$. An odd cycle with no chord is called an *odd hole*. A maximal complete graph is called a *clique*.

If $K$ is a clique, then the inequality $\sum_{u \in K} x(u) \leq 1$ defines a facet of $P(G)$ [14]. This is called a *clique inequality*. If the clique is an edge, it is called an *edge inequality*.

If $H$ is an odd hole, then the inequality

$$\sum_{u \in H} x(u) \leq \frac{|H| - 1}{2}$$

is valid for $P(G)$; this is called an *odd hole inequality*. Under some conditions, these inequalities define facets of $P(G)$ [14].

The trivial facets of $P(G)$ are $x(v) \geq 0$ for $v \in V$.

A graph $G$ is called *t-perfect* if the only nontrivial facets of $P(G)$ are the odd hole and the edge inequalities. Chvátal [3] introduced this class of graphs and conjectured that series-parallel graphs are $t$-perfect.

A graph is called *series-parallel* if it does not contain $K_4$ as a minor. Boulala and Uhry [2] proved that series-parallel graphs are $t$-perfect; i.e., they characterized $P(G)$ for graphs that do not have a 3-wheel as a minor. A short proof of this appears in [13].

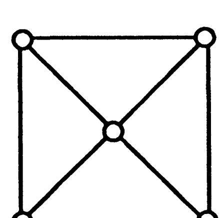

FIG. 1.1

**2. Compositions of polyhedra.** This section is devoted to survey the composition/decomposition techniques that we need.

Let $G = (V, E)$ be a graph such that $V = V_1 \cup V_2$, $W = V_1 \cap V_2 \neq \emptyset$, $(W, E(W))$ is a clique, and $(V \backslash W, E(V \backslash W))$ is disconnected. Chvátal [3] proved the following result.

THEOREM 2.1. *If $G_1 = (V_1, E(V_1))$ and $G_2 = (V_2, E(V_2))$, then a system that defines $P(G)$ is obtained by taking the union of the systems that define $P(G_1)$ and $P(G_2)$ and identifying the variables associated with the nodes in $W$.*

This theorem applies to the case where $G$ has a one-node or a two-node cutset $\{u, v\}$ with $uv \in E$. This is called Case 1.

Now we must treat Case 2, i.e., when $G$ has a two-node cutset $\{u, v\}$ and $uv \notin E$.

In the remainder of this section, we assume that

(i) $V = V_1 \cup V_2$,

(ii) $V_1 \cap V_2 = \{u, v\}$,

(iii) $uv \notin E$, and

(iv) $G \backslash \{u, v\}$ is disconnected.

We decompose into two pieces and add a 5-cycle to both of them; see Fig. 2.1. For $k = 1, 2$, we define $\bar{G}_k = (\bar{V}_k, \bar{E}_k)$ as

(i) $\bar{V}_k = V_k \cup \{w_1, w_2, w_3\}$,

(ii) $\bar{E}_k = E(V_k) \cup \{uw_1, vw_1, uw_2, w_2w_3, w_3v\}$.

To study the facets of $P(\bar{G}_k)$, we present two lemmas. Their proofs appear in [13].

LEMMA 2.2. *Let $ax \leq \alpha$ be a facet of $P(G)$. If $G_a$ has a path with vertices $p$, $u$, $v$, $q$, where $u$ and $v$ have degree 2 in $G_a$, then $a_u = a_v$.*

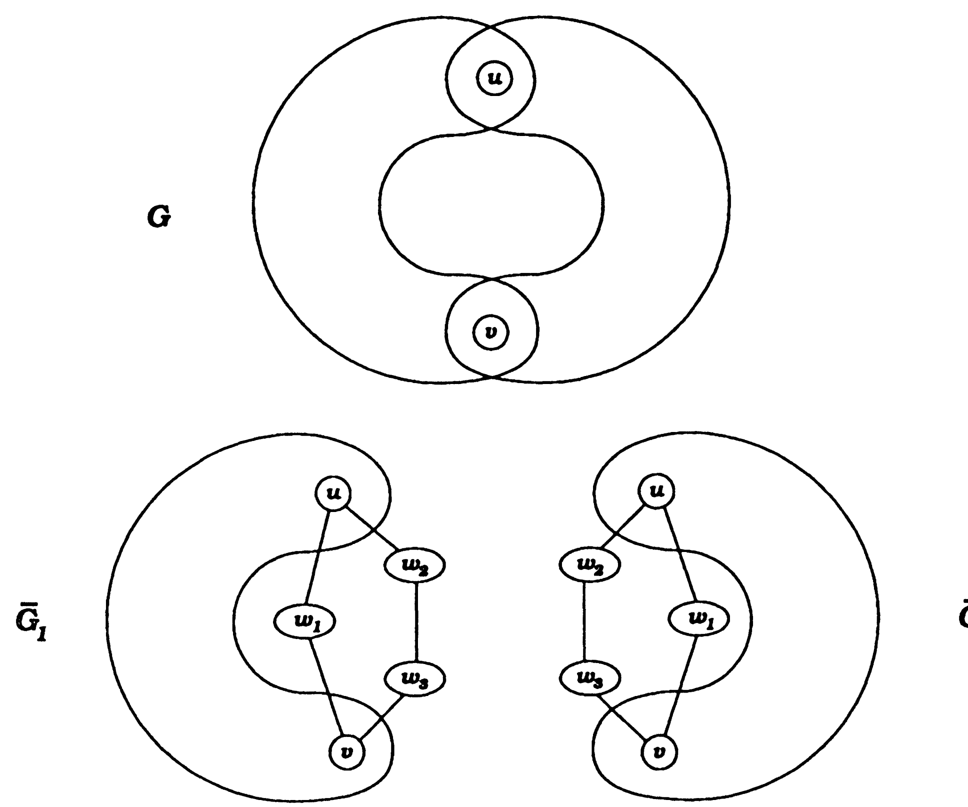


FIG. 2.1

LEMMA 2.3. *Let $ax \leq \alpha$ be a facet of $P(G)$. If $G_a$ is different from an odd hole, then it does not contain between two given nodes $p$ and $q$ two paths such that each node of them, different from $p$, $q$, has degree 2 in $G_a$.*

Lemmas 2.2 and 2.3 imply that the facets of $P(\bar{G}_k)$ for $k = 1, 2$ can be classified in the following ten types:

(a) $\sum_{j \in V_k} a_{ij}^k x(j) \leq \alpha_i^k$, $i \in I_1^k$,
(b) $\sum_{j \in V_k} a_{ij}^k x(j) + x(w_1) \leq \alpha_i^k$, $i \in I_2^k$,
(c) $\sum_{j \in V_k} a_{ij}^k x(j) + x(w_2) + x(w_3) \leq \alpha_i^k$, $i \in I_3^k$,
(d) $x(u) + x(w_1) \leq 1$,
(e) $x(u) + x(w_2) \leq 1$,
(f) $x(v) + x(w_1) \leq 1$,
(g) $x(v) + x(w_3) \leq 1$,
(h) $x(w_2) + x(w_3) \leq 1$,
(i) $x(u) + x(v) + x(w_1) + x(w_2) + x(w_3) \leq 2$,
(j) $x(j) \geq 0, j \in \bar{V}_k$,

where $I_1^k$ is the set of inequalities whose support has empty intersection with $\{w_1, w_2, w_3\}$, $I_2^k$ is the set of inequalities whose support contains $w_1$ and has empty intersection with $\{w_2, w_3\}$, and $I_3^k$ is the set of inequalities whose support contains $\{w_2, w_3\}$ and not $w_1$.

Now we can present the necessary polyhedral composition theorems.

Let $\bar{G} = (\bar{V}, \bar{E})$ be the union of $\bar{G}_1$ and $\bar{G}_2$, i.e.,

$$\bar{V} = \bar{V}_1 \cup \bar{V}_2, \qquad \bar{E} = \bar{E}_1 \cup \bar{E}_2.$$

The equation

$$x(u) + x(v) + x(w_1) + x(w_2) + x(w_3) = 2 \tag{2.1}$$

defines a facet $F(\bar{G})$ of $P(\bar{G})$; it also defines a facet $F(\bar{G}_k)$ of $P(\bar{G}_k)$ for $k = 1, 2$. The polytope $P(G)$ is a projection of $F(\bar{G})$ along the variables $\{x(w_i)\}$.

Now we state two theorems that appear in [1].

THEOREM 2.4. *The facet $F(\bar{G})$ is defined by the union of the systems that define $F(\bar{G}_1)$ and $F(\bar{G}_2)$.*

THEOREM 2.5. *The polytope $P(G)$ is defined by* (a), *together with $x(j) \geq 0$ and the mixed inequalities*

$$\sum_{j \in V_k} a_{ij}^k x(j) + \sum_{j \in V_l} a_{sj}^l x(j) - x(u) - x(v) \leq \alpha_i^k + \alpha_s^l - 2 \tag{2.2}$$

$$\text{for } k = 1, 2;\ l = 1, 2;\ k \neq l;\ i \in I_2^k,\ s \in I_3^l.$$

*Moreover, all these inequalities define facets of $P(G)$.*

**3. Graphs with no $W_4$ minor.** Graphs with no $W_4$ minor can be easily decomposed [10]. Gan and Johnson [6] used this property to study the Chinese postman problem in these graphs. More precisely, if $G$ has no $W_4$ as a minor and has at least five nodes, then $G$ has a one-node or a two-node cutset where one of the pieces is a path or the *Wheatstone bridge*; see Fig. 3.1.

Now it is clear how to apply the decomposition techniques of §2. If the cutset is $\{u, v\}$ and $uv \in E$, then we just separate the two pieces. If $uv \notin E$, then we separate the two pieces and add a 5-cycle to both of them. In what follows, we formalize this procedure. Let us denote by $n$ the number of nodes of $G$; we prove that the total number of nodes after decomposing is $O(n)$.

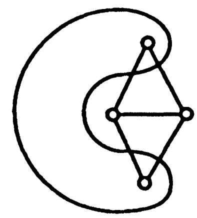

FIG. 3.1

We recursively apply the procedure below.

(a) If $G$ has at most four nodes, stop.

(b) If $G$ has a one-node cutset, we decompose it into the two blocks.

(c) Suppose that $G$ has a two-node cutset $\{u, v\}$, where the second block is a path with two edges or the Wheatstone bridge.

(i) If $uv \notin E$, we decompose $G$ into the two blocks, we add the edge $uv$ to both blocks, and we label these two new edges as "artificial." This corresponds to Case 2 of §2. Artificial edges represent the 5-cycles that are added to both pieces.

(ii) If $uv \in E$ (where $uv$ is not artificial), we decompose into the two blocks. This corresponds to Case 1 of §2.

(iii) If $uv \in E$ and $uv$ is artificial, we decompose into the two blocks, we leave the artificial edge $uv$ only in the first block (if there are parallel artificial edges between $u$ and $v$, we leave them all in the first block), and we add a new artificial edge $uv$ to each block. This corresponds to Case 2 of §2.

Note that a two vertex cutset could be used several times in this decomposition and that that would create parallel artificial edges. The number of nodes of the larger block decreases each time we decompose, so the number of artificial edges is bounded by $2n$. The resulting pieces are single edges, sets of parallel edges, triangles, or copies of $K_4$. Therefore, after applying this procedure, the total number of edges is $O(n)$. These pieces may have parallel artificial edges.

Figure 3.2 shows an example of this decomposition. Dashed lines represent artificial edges. The set $\{u, v\}$ has been used twice in the decomposition.

Now we must treat the blocks that have parallel artificial edges. Given a block with more than two nodes and parallel artificial edges between $u$ and $v$, we decompose into two blocks. One of them consists of all those parallel edges. We add a new artificial edge to each block. Figure 3.3 shows the result of this for the example in Fig. 3.2.

Now let us assume that we have a block that consists of two nodes and $p$ parallel edges, $p \geq 4$. The following procedure is applied recursively. We separate into two blocks, the first with $\lfloor p/2 \rfloor$ edges and the second with the remainder. We add one artificial edge to each block. We can prove by induction that this procedure creates less than $2p$ new

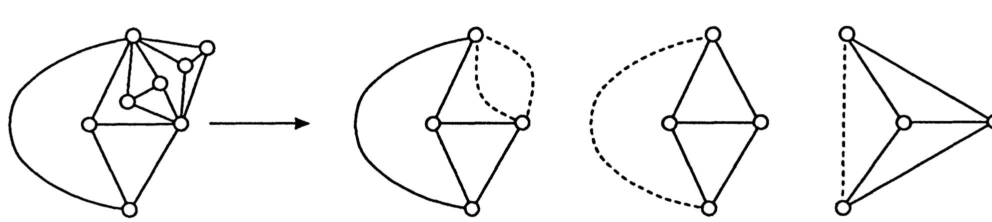

FIG. 3.2

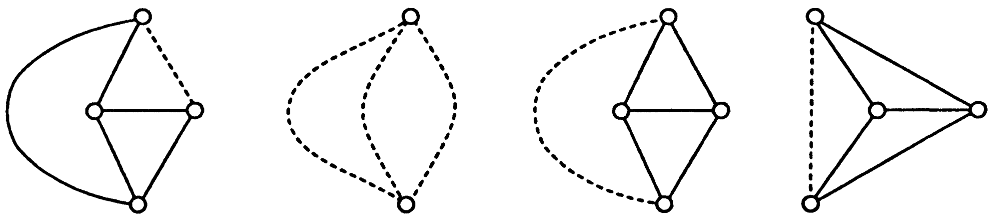

FIG. 3.3

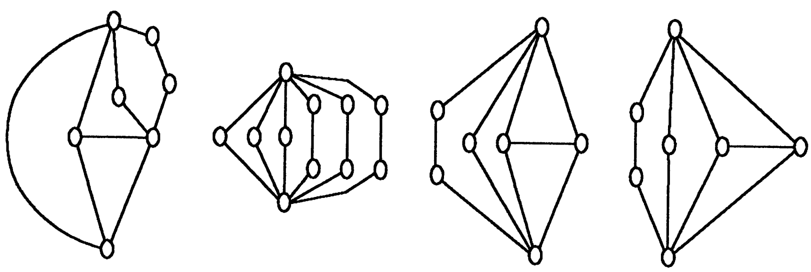

FIG. 3.4

artificial edges. Therefore the total number of edges is $O(n)$. Now the pieces are single edges, triangles, copies of $K_4$, and sets of at most three parallel edges. The first three types do not have parallel edges. Finally, Operation $\mathcal{O}$, given below, is applied to every artificial edge.

*Operation* $\mathcal{O}(uv)$. Remove the edge $uv$. Add the nodes $w_1$, $w_2$, and $w_3$; add the edges $uw_1$, $vw_1$, $uw_2$, $w_2w_3$, and $w_3v$.

Figure 3.4 shows the result of applying this to the pieces in Fig. 3.3.

Let us remark that the final pieces are series-parallel graphs with at most eleven nodes (like the second block in Fig. 3.4) and graphs obtained by applying $\mathcal{O}$ to $K_4$.

**4. On the stable set polytope of graphs with no $W_4$ minor.** In this section, we state our main result. The facets of $P(G)$ are not described in a simple way as "odd holes" or "cliques." We present a combinatorial procedure that produces all of them. We first present three theorems to derive "facets from facets." To make the notation less cumbersome, we use $a(u)$ instead of $a_u$ to denote the coefficients of the inequalities.

THEOREM 4.1 (subdivision of an edge [15]). *Let $G = (V, E)$ be a graph and let $uv$ be an edge of $G$ and $\bar{G} = G\backslash uv$. Let $ax \leq \alpha$ be a facet-defining inequality of $P(G)$ different from $x(u) + x(v) \leq 1$. If $z = \max\ \{ax : x \in P(\bar{G})\}$ has a solution with $x(u) = x(v) = 1$, then $ax + \beta x(w) + \beta x(y) \leq z$ defines a facet of $P(G')$, where $G'$ is the graph obtained from $G$ by replacing the edge $uv$ by the path $(u, w, y, v)$, and $\beta = z - \alpha$.*

THEOREM 4.2 (contraction of an odd path [1]). *Let $G = (V, E)$ be a graph and let $ax \leq \alpha$ be a facet-defining inequality of $P(G)$. Suppose that $G$ contains a path $(pu, uv, vq)$ such that $u$ and $v$ are of degree 2. Assume also that $a(p) = a(u) = a(v) = \beta$. Let $G' = (V', E')$ be the graph obtained from $G$ by replacing that path by the edge $pq$. Let*

$$\bar{a}(u) = a(u) \quad \textit{for } u \in V',$$

$$\bar{\alpha} = \alpha - \beta;$$

*then $\bar{a}x \leq \bar{\alpha}$ defines a facet of $P(G')$.*

THEOREM 4.3 (subdivision of a star [1]). *Let $G$ be a graph and let $ax \leq \alpha$ be a nontrivial facet that is not an edge inequality. Let $v$ be a node of $G$ and let $N = \{v_0, \ldots, v_{k-1}\}$ be its neighbor set. Suppose that, for each $v_i$, there is a stable set $S_i$ such that $ax^{S_i} = \alpha$ and $S_i \cap N = \{v_i, v_{i+1}, \ldots, v_{i+p-1}\}$, where $p \geq 1$ is a fixed integer and the indices are numbers modulo $k$. Suppose also that $p$ and $k$ are relatively prime and $a(v_0) = \cdots = a(v_k) = a(v)/p$. Let $G' = (V', E')$ be the graph obtained from $G$ by adding on each edge $vv_i$ a new node $v'_i$ for $0 \leq i \leq k-1$. Set*

$$\bar{a}(u) = a(u) \quad \text{for } u \in V\backslash\{v\},$$

$$\bar{a}(v) = a(v)(k-p)/p,$$

$$\bar{a}(v'_i) = a(v)/p \quad \text{for } 0 \leq i \leq k-1,$$

$$\bar{\alpha} = \alpha + a(v)(k-p)/p.$$

*Then $\bar{a}x \leq \bar{\alpha}$ defines a facet of $P(G')$.*

It follows from Lemma 2.3 that, if we apply Operation $\mathcal{O}$ of §3 to $K_4$, the only nontrivial facets with support different from odd holes and edges, have as support a subdivision of $K_4$. Now we characterize those facets.

THEOREM 4.4. *If $G$ is a subdivision of $K_4$, then the nontrivial facets of $P(G)$ are either odd holes or edges or have been obtained by applying Theorems* 4.1 *and* 4.3, *starting from the clique inequality of $K_4$.*

The proof of this is the subject of the next two sections. We prove that there are 16 cases to study and we give an explicit list of the facets for each case. We call them $K_4$ inequalities.

Now let $G$ be a graph with $n$ nodes that has no $W_4$ minor. Suppose that it is decomposed, as described in §3. For each piece, we have an explicit list of the facets; the total number of them is $O(n)$. The facets of $P(G)$ are obtained by composing those inequalities according to Theorem 2.5.

Our main result can be stated as follows.

THEOREM 4.5. *If $G$ is a graph with no $W_4$ minor, then the nontrivial facets defining inequalities of $P(G)$ are either edge inequalities or can be constructed by composing inequalities from the following two families:* (i) *odd hole inequalities and* (ii) *a set of* 19 *$K_4$ inequalities.*

This last theorem gives a system that may have exponentially many inequalities. Suppose now that we use Theorem 2.4 instead of Theorem 2.5; i.e., we do not project the extra variables associated with the extra nodes. Then we can describe a polytope $Q = \{(x, y) : Ax + By \leq b\}$ such that $P(G) = \{x : \text{there is a vector } y, \text{ with } (x, y) \in Q\}$; i.e., $P(G)$ is a projection of $Q$. The decomposition of §3 gives a set of pieces that are series-parallel graphs with at most 11 nodes and copies of $K_4$ with some edges replaced by a 5-cycle. The total number of nodes is $O(n)$. Thus the number of variables and the number of inequalities in the system that defines $Q$ is $O(n)$. Moreover, the coefficients in those inequalities are integer numbers of absolute value at most 2.

Therefore, for this class of graphs, the stable set problem can be formulated as a linear program of polynomial size.

For a polytope $P$ the so-called *separation* problem is: Given a vector $\bar{x}$, decide whether $\bar{x} \in P$ and, if not, find a hyperplane that separates $\bar{x}$ from $P$.

Grötschel, Lovász, and Schrijver [8], [9] have shown that, if the optimization problem can be solved in polynomial time, then the separation problem can also be solved in polynomial time by means of the ellipsoid method.

In our case, by the Farkas lemma, if $\bar{x} \notin P(G)$, there is a vector $\pi$ such that

$$\pi B = 0, \quad \pi \geq 0, \quad \text{and} \quad \pi(b - A\bar{x}) < 0.$$

So $\pi Ax \leq \pi b$ is the required inequality. Thus, the separation problem can be solved in polynomial time by means of any polynomial algorithm for linear programming; cf. Khachiyan [12] and Karmarkar [11], for instance.

Solving the stable set problem with a cutting plane approach based on this separation algorithm is equivalent to applying Benders decomposition to the linear program

$$\text{maximize } wx \quad \text{s.t. } Ax + By \leq b.$$

**5. Technical lemmas.** In this section, we present a series of lemmas that lead to the characterization of $P(G)$, when $G$ is a subdivision of $K_4$. First, note the following remarks.

*Remark* 5.1. It follows from the results of §2 that it is enough to characterize the polytope for graphs that are obtained by replacing some edges of $K_4$ by paths with two or three edges.

*Remark* 5.2. It is enough to consider the case where the four faces of the graph are odd (a planar graph has an even number of odd faces). If only two of them are odd, then there is a node that covers them; i.e., the removal of this node will leave a bipartite graph. In this case, the graph is $t$-perfect, as shown by Fonlupt and Uhry [5].

*Remark* 5.3. Suppose that we have a graph that has been obtained from $K_4$ by replacing some edges by paths of two edges, with the additional condition that every original node has at least one incident edge that has not been replaced. Since we should have four odd faces, the only graph of this kind to be studied is the graph of Fig. 5.1. This has been shown to be $t$-perfect by Gerards and Schrijver [7].

*Remark* 5.4. Consider a graph $G = (V, E)$ that is a subdivision of $K_4$ and let $ax \leq \alpha$ with $\alpha > 0$ be a facet of $P(G)$ whose support is not a clique or an odd hole. Since $G\backslash v$ is a series-parallel graph, we have that $a(v) > 0$ for all $v \in V$.

From the first three remarks, it follows that we must study the 16 cases shown in Fig. 5.2. We first state a lemma that will be used in this section.

LEMMA 5.5. *Let $ax \leq \alpha$ be a facet of $P(G)$. Let $u$ be a node of degree 2 in $G_a$ and let $v$, $w$ be the neighbors of $u$ in $G_a$. Then $a(v) \geq a(u) \leq a(w)$.*

*Proof.* The equation

$$ax = \alpha \tag{5.1}$$

defines a hyperplane different from those of $x(v) + x(w) = 2$ and $x(v) - x(w) = 0$. So there is a stable set $S$ such that $ax^S = \alpha$ and $|S \cap \{v, w\}| = 1$. Assume that $v \in S$. Since $(S\backslash\{v\}) \cup \{u\}$ is a stable set, we can conclude that $a(u) \leq a(v)$.

Since (5.1) also defines a hyperplane different from that of $x(u) + x(v) = 1$, there is a stable set $T$ such that $ax^T = \alpha$, and $T \cap \{u, v, w\} = \{w\}$. This implies that $a(u) \leq a(w)$. $\square$

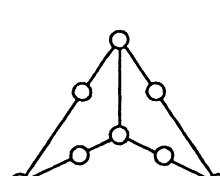

FIG. 5.1

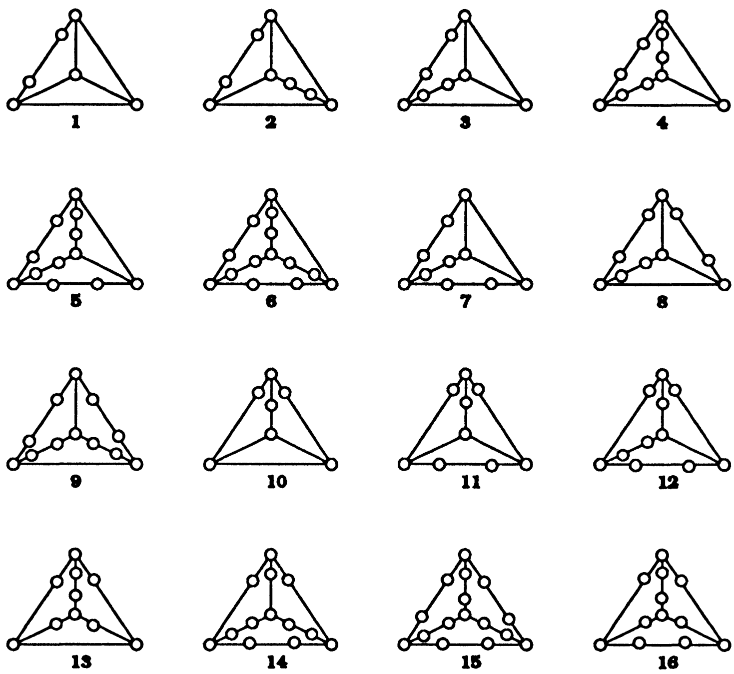


FIG. 5.2

The following lemma will be used to solve nine cases.

LEMMA 5.6. *Let $G = (V, E)$ be a graph and let $(u, w, y, v)$ be a path in $G$, where the nodes $w$ and $y$ have degree 2. Let $G'$ be the graph obtained from $G$ by replacing this path by one edge; see Fig. 5.3. Let $N(u)$ and $N(v)$ be the neighbor sets of $u$ and $v$ in $G$, respectively. Suppose that $ax \leq \alpha$ is the only facet-defining inequality of $P(G')$ whose support is $G'$ and let $bx \leq \beta$ be the facet of $P(G)$ obtained from $ax \leq \alpha$ by the procedure described in Theorem 4.1. Suppose that, for every facet $\bar{a}x \leq \bar{\alpha}$ of $P(G)$ whose support is $G$, there exists a stable set $S$ of $G$ such that $\bar{a}x^S = \bar{\alpha}$ and either $S \cap (N(u)\backslash\{w\}) = \emptyset$ or $S \cap (N(v)\backslash\{y\}) = \emptyset$. Then $bx \leq \beta$ is the only facet of $P(G)$ whose support is $G$.*

*Proof.* Let $\bar{a}x \leq \alpha$ be a facet of $P(G)$, whose support is $G$. Lemma 5.5 implies that $\bar{a}(w) = \bar{a}(y) \leq \min\{\bar{a}(u), \bar{a}(v)\}$. If there exists a stable set $S$ of $G$ such that $\bar{a}x^S = \bar{\alpha}$ and $S \cap (N(u)\backslash\{w\}) = \emptyset$, say, then $w \in S$ and $S' = (S\backslash\{w\}) \cup \{u\}$ is also a stable set in $G$. Hence $\bar{a}(u) \leq \bar{a}(w) = \bar{a}(y) \leq \bar{a}(u)$. From Theorem 4.2, we have that the inequality

$$a'x \leq \alpha' \tag{5.2}$$

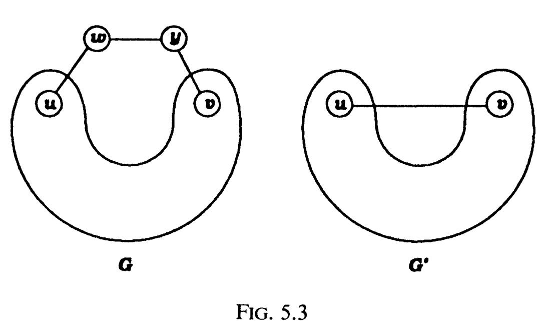


FIG. 5.3

defines a facet of $P(G')$, where

$$a'(i) = \bar{a}(i) \quad \text{for } i \in V\backslash\{w, y\}$$

and

$$\alpha' = \bar{\alpha} - \bar{a}(u).$$

Since the support of (5.2) is $G'$, we have that $a' = a$ and $\alpha' = \alpha$. Thus $\bar{a} = b$ and $\bar{\alpha} = \beta$. The proof is complete. □

The next lemma will allow us to solve two cases.

LEMMA 5.7. *Let $G$ be a graph obtained from $K_4$ in such a way that at least two edges have not been subdivided. Let $G'$ be the graph obtained from $G$ by replacing one of those two edges by a path of three edges; see Fig. 5.4. If $ax \leq \alpha$ is the only facet of $P(G)$ having $G$ as support, then the only facet of $P(G')$ having $G'$ as support is the inequality $a'x \leq \alpha'$ obtained by applying the procedure of Theorem 4.1 to $ax \leq \alpha$.*

*Proof.* Let $bx \leq \beta$ be a facet of $P(G')$ whose support is $G'$. Let $C$ be the odd hole of $G'$ defined by the edge $\{1, 3\}$ and the paths 1-2 and 2-3. Since $bx \leq \beta$ is different from the facet associated with $C$, there is a stable set $S$ such that $bx^S = \beta$ and $|C \cap S| < (|C| - 1)/2$.

If $|C| = 3$, then $S \cap C = \emptyset$. If $|C| = 5$ or $|C| = 7$, then $S$ can be chosen so that $S \cap C = Z$, where $Z$ is the set of nodes of $C$ adjacent to the node 2 and different from nodes 1 and 3. In these three cases, we have that $S \cap (N(1)\backslash\{w\}) = \emptyset$. From the previous lemma we have that $b = \rho a'$ and $\beta = \rho\alpha'$, for some $\rho > 0$. □

The next three lemmas will enable us to solve four cases.

LEMMA 5.8. *Let $v$ be a node of $G$. Let $N(v)$ be the neighbor set of $v$. Let $ax \leq \alpha$ be a facet-defining inequality of $P(G)$ whose support is not an edge; then*

$$a(v) \leq a(N(v)\backslash\{u\})$$

*for all $u \in N(v)$.*

*Proof.* Let $u \in N(v)$. Since $ax \leq \alpha$ is a facet not associated with an edge, then there is a stable set $S$ such that

$$ax^S = \alpha$$

and $\{u, v\} \cap S = \emptyset$.

Then $(S\backslash(N(v)\backslash\{u\})) \cup \{v\}$ is a stable set in $G$. This implies that

$$a(v) \leq a(N(v)\backslash\{u\}). \qquad \square$$

LEMMA 5.9. *Let $G$ be a graph as in Fig. 5.5, where the dashed lines represent paths with one or more edges, and $v$ (respectively, $w$) is a node adjacent to $v_3$ (respectively, $v_1$) in the path that replaces the edge $v_3v_4$ (respectively, $v_1v_4$) of $K_4$. If $ax \leq \alpha$ defines a facet*

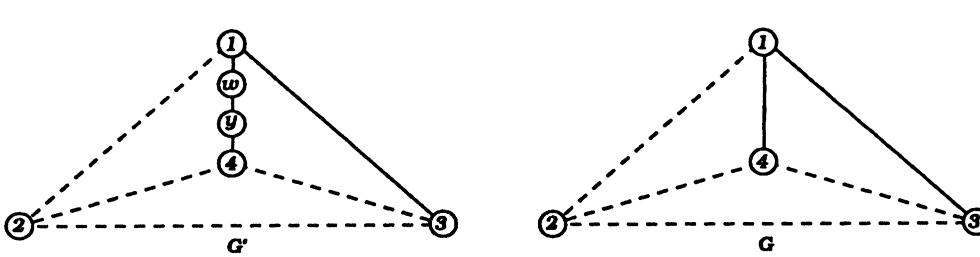


FIG. 5.4

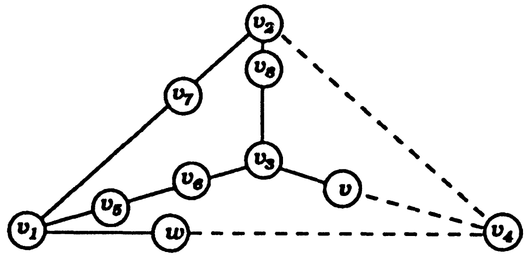


FIG. 5.5

*of $P(G)$ whose support is $G$, then we have the following*:

(a) *Either* (i) $a(v_1) = a(v_5) + a(v_7)$ *and* $a(v_3) = a(v_6) + a(v_8)$, *or* (ii) $a(v_2) = a(v_7) + a(v_8)$, $a(v_1) = a(w)$, *and* $a(v_3) = a(v)$;

(b) *If the edge $v_2v_4$ is subdivided and $u$ is the node adjacent to $v_2$ in this path, then*

(b1) *Either* (i) *and* $a(v_2) = a(u)$ *hold, or* (ii) *holds,*

(b2) *If the path between $u$ and $v_4$ is $(uy, yv_4)$ and $a(v_2) = a(u)$, then $ax \leq \alpha$ is obtained from a facet of $P(G')$ using the procedure of Theorem* 4.1, *where $G'$ is the graph obtained from $G$ by contracting the edges $uy$ and $yv_4$.*

*Proof.* (a) Let $C$ denote the cycle $(v_1, v_5, v_6, v_3, v_8, v_2, v_7, v_1)$. Since $ax \leq \alpha$ has as support the graph $G$ and $C$ is an odd hole, there is a stable set $S$ in $G$ such that $|S \cap C| < 3$ and $ax^S = \alpha$.

*Case* 1. $\{v_1, v_3\} \subseteq S$.

Thus $S \cap C = \{v_1, v_3\}$. Since $(S\backslash\{v_1\}) \cup \{v_5, v_7\}$ is a stable set, it follows that $a(v_1) \geq a(v_5) + a(v_7)$. From Lemma 5.8, we have that $a(v_1) = a(v_5) + a(v_7)$.

Since $(S\backslash\{v_3\}) \cup \{v_6, v_8\}$ is also a stable set in $G$, we obtain $a(v_6) + a(v_8) = a(v_3)$ in a similar way.

*Case* 2. $\{v_1, v_3\} \not\subseteq S$.

We should have that $\{v_1, v_3\} \cap S = \emptyset$. If, for instance, $v_1 \in S$ and $v_3 \notin S$, then $\{v_2, v_6\} \subseteq S$ or $\{v_8, v_6\} \subseteq S$, and $|S \cap C| = 3$, which is a contradiction.

Therefore $S$ must contain $v_2$ and a node from $\{v_5, v_6\}$, say $v_6$. Then $(S\backslash\{v_2\}) \cup \{v_7, v_8\}$ is a stable set, which implies that $a(v_2) \geq a(v_7) + a(v_8)$, and, from Lemma 5.8, we have that $a(v_2) = a(v_7) + a(v_8)$.

Furthermore, $w \in S$; otherwise, $v_1 \in S$, which is a contradiction. So $(S\backslash\{w\}) \cup \{v_1\}$ is a stable set. Then $a(v_1) \leq a(w)$, and, from Lemma 5.5, we have that $a(v_1) = a(w)$. Also, we have $v \in S$. If not, $(S\backslash\{v_6\}) \cup \{v_3, v_5\}$ is a stable set. Since $a(v_5) = a(v_6)$ and $a(v_3) > 0$, we have a contradiction.

Since $(S\backslash\{v, v_6\}) \cup \{v_5, v_3\}$ is a stable set, we have that $a(v_3) \leq a(v)$, and hence $a(v_3) = a(v)$.

We now prove (b).

(b1) If $\{v_1, v_3\} \subseteq S$, then (i) holds. Since $S \cap C = \{v_1, v_3\}$, we should have that $u \in S$; otherwise, $v_2 \in S$, which is a contradiction. Since $(S\backslash\{u\}) \cup \{v_2\}$ is a stable set, we have that $a(v_2) \leq a(u)$ and, from Lemma 5.5, we can deduce that $a(v_2) = a(u)$.

(b2) Since $a(v_2) = a(u) = a(y)$, the statement follows from Theorem 4.2. □

LEMMA 5.10. *Let $G$ be a graph as in Fig.* 5.6, *where the dashed lines represent paths with one or two edges and $u$ (respectively, $v$, $w$) is a node adjacent to $v_1$ (respectively, $v_2$, $v_3$) in the path that replaces the edge $v_1v_4$ (respectively, $v_2v_4$, $v_3v_4$) of $K_4$. If $ax \leq \alpha$ defines a facet of $P(G)$ whose support is $G$, then either*

(i) $a(v_1) = a(v_7) + a(v_8)$, $a(v_2) = a(v_5) + a(v_6)$, *and* $a(v_3) = a(v_{10}) + a(v_9)$, *or*

(ii) $a(v_1) = a(u)$, $a(v_2) = a(v)$, *and* $a(v_3) = a(w)$.

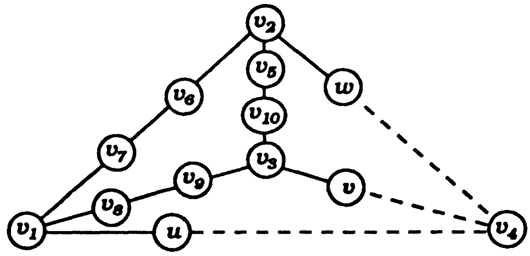


FIG. 5.6

*Proof.* Let $C$ denote the cycle $(v_1, v_8, v_9, v_3, v_{10}, v_5, v_2, v_6, v_7, v_1)$. Since $ax \le \alpha$ has as support the graph $G$ and since $C$ is an odd hole, there is a stable set $S$ in $G$ such that $|S \cap C| < 4$ and $ax^S = \alpha$.

*Case* 1. $S \cap C = \{v_1, v_2, v_3\}$.

Let

$$S_1 = (S \backslash \{v_1\}) \cup \{v_7, v_8\},$$
$$S_2 = (S \backslash \{v_2\}) \cup \{v_5, v_6\},$$
$$S_3 = (S \backslash \{v_3\}) \cup \{v_{10}, v_9\}.$$

Clearly, $S_1$, $S_2$, and $S_3$ are stable sets in $G$. Thus we should have

$$a(v_1) \ge a(v_7) + a(v_8),$$
$$a(v_2) \ge a(v_5) + a(v_6),$$
$$a(v_3) \ge a(v_{10}) + a(v_9).$$

From Lemma 5.8, it follows that these inequalities should be equations.

*Case* 2. $\{v_1, v_2, v_3\} \not\subseteq S$.

CLAIM 1. *We have that* $\{v_1, v_2, v_3\} \cap S = \emptyset$.

Let us assume, for instance, that $v_1 \in S$. If $\{v_2, v_3\} \cap S = \emptyset$, then $\{v_6, v_9\} \subseteq C$ and $|\{v_{10}, v_5\} \cap S| = 1$; hence $|S \cap C| = 4$, a contradiction.

Suppose that $\{v_2, v_3\} \cap S \ne \emptyset$ and let us assume, for instance, that $\{v_2, v_3\} \cap S = \{v_2\}$; then we should have that $\{v_{10}, v_9\} \subseteq S$ and $|S \cap C| = 4$, a contradiction, which proves Claim 1.

CLAIM 2. *We have that* $\{u, v, w\} \subseteq S$.

Assume, for instance, that $u \notin S$. Since we can assume that $S \cap C = \{v_6, v_9, v_5\}$, we would have that $v_1 \in S$, which is a contradiction. This proves Claim 2.

Let

$$S_1 = (S \backslash \{u, v_7, v_8\}) \cup \{v_1, v_6, v_9\},$$
$$S_2 = (S \backslash \{v, v_{10}, v_9\}) \cup \{v_3, v_5, v_8\},$$
$$S_3 = (S \backslash \{w, v_5, v_6\}) \cup \{v_2, v_{10}, v_7\}.$$

These node sets define stable sets of $G$. Since $a(v_7) + a(v_8) = a(v_6) + a(v_9)$, we have that $a(v_1) \le a(u)$. It follows from Lemma 5.5 that $a(v_1) = a(u)$.

In a similar way, we can prove that $a(v_2) = a(w)$ and $a(v_3) = a(v)$. □

**6. The stable set polytope of a subdivision of $K_4$.** The purpose of this section is to prove Theorem 4.4.

First, we can apply Lemma 5.6 to prove our claim for the graphs of Cases 1–9 in Fig. 5.2. In each case, there is a unique facet whose support is the whole graph, obtained by applying Theorem 4.1 starting from the clique inequality of $K_4$. Now consider the graph of Case 10. It has seven nodes and seven maximal stable sets. Then there is a unique facet that may have this graph as support; this is the facet obtained by applying Theorem 4.3 to the clique inequality of $K_4$. Starting from this graph, we can apply Lemma 5.7 to prove our claim for the graphs in Cases 11 and 12. Now we must study the graphs of Cases 13–16. In all these cases, we denote by $ax \leq \alpha$ a facet whose support is the whole graph. We begin with the graph of Case 13; see Fig. 6.1.

Lemma 5.9 implies that either (i) $a(v_2) = a(v_5) + a(v_6)$ and $a(v_3) = a(v_8) + a(v_9)$, or (ii) $a(v_1) = a(v_5) + a(v_9)$ and $a(v_2) = a(v_7)$. It also implies that either (iii) $a(v_2) = a(v_6) + a(v_7)$ and $a(v_3) = a(v_8) + a(v_{10})$, or (iv) $a(v_4) = a(v_7) + a(v_{10})$, $a(v_2) = a(v_5)$, and $a(v_3) = a(v_9)$. Thus we have that either (i) and (iii) hold or (ii) and (iv) hold.

Consider the cycle $C = (v_1, v_4, v_7, v_2, v_5, v_1)$. Since $ax \leq \alpha$ has as support the graph $G$ and $C$ is an odd hole, there is a stable set $S$ in $G$ such that $|S \cap C| < 2$ and $ax^S = \alpha$. Then $S \cap C = \{v_2\}$; therefore $S = \{v_2, v_8, v_9, v_{10}\}$. Since $(S \backslash \{v_9\}) \cup \{v_1\}$ and $(S \backslash \{v_{10}\}) \cup \{v_4\}$ are stable sets, we have that $a(v_1) \leq a(v_9)$ and $a(v_4) \leq a(v_{10})$. Lemma 5.5 implies that (v) $a(v_1) = a(v_9)$ and (vi) $a(v_4) = a(v_{10})$.

In the same way, by considering the cycle $C = (v_3, v_9, v_1, v_4, v_{10}, v_3)$, we can prove that (vii) $a(v_1) = a(v_5)$ and (viii) $a(v_4) = a(v_7)$.

Consider (i), (iii), and (v)–(viii). This implies that

$$a(v_1) = a(v_4) = a(v_5) = a(v_7) = a(v_9) = a(v_{10})$$

and

$$a(v_2) = a(v_3) = a(v_6) + a(v_7).$$

Since the inequality $x(v_2) + x(v_6) + x(v_8) + x(v_3) \leq 2$ is valid for $P(G)$, there is a stable set $T$ such that $ax^T = \alpha$ and $x^T$ satisfies $x(v_2) + x(v_6) + x(v_8) + x(v_3) < 2$. We can choose $T = \{v_5, v_7, v_8, v_9, v_{10}\}$.

Consider the set $S$ defined above. Since $ax^S = ax^T$, we have that $a(v_5) = a(v_8) = a(v_6)$. Therefore $ax \leq \alpha$ represents the inequality

$$2x(v_2) + 2x(v_3) + x(v_1) + \sum_{j=4}^{10} x(v_j) \leq 5.$$

Condition (v) implies that (ii) and (iv) cannot hold.

Now we study the graph of Case 14; see Fig. 6.2.

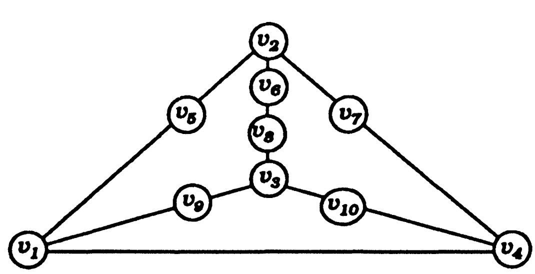


FIG. 6.1

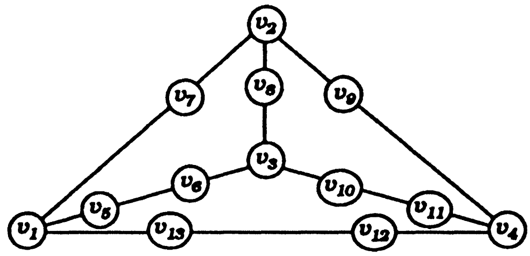


FIG. 6.2

Lemma 5.9 implies that either (i) $a(v_1) = a(v_5) + a(v_7)$, $a(v_3) = a(v_6) + a(v_8)$, and $a(v_2) = a(v_9)$, or (ii) $a(v_2) = a(v_7) + a(v_8)$, $a(v_3) = a(v_{10})$, and $a(v_1) = a(v_{13})$. It also implies that either (iii) $a(v_4) = a(v_{11}) + a(v_9)$, $a(v_3) = a(v_8) + a(v_{10})$, and $a(v_2) = a(v_7)$, or (iv) $a(v_2) = a(v_8) + a(v_9)$, $a(v_3) = a(v_6)$, and $a(v_4) = a(v_{12})$, and that either (v) $a(v_4) = a(v_{12}) + a(v_9)$, $a(v_1) = a(v_7) + a(v_{13})$, and $a(v_2) = a(v_8)$, or (vi) $a(v_2) = a(v_7) + a(v_9)$, $a(v_1) = a(v_5)$, and $a(v_4) = a(v_{11})$.

We have that either (i), (iii), and (v) hold or (ii), (iv), and (vi) hold. Consider (i), (iii), and (v). This implies that

$$\beta = a(v_{10}) = a(v_6) = a(v_{11}) = a(v_5) = a(v_{12}) = a(v_{13}),$$
$$\gamma = a(v_2) = a(v_7) = a(v_8) = a(v_9),$$
$$\beta + \gamma = a(v_1) = a(v_4) = a(v_3).$$

Lemma 5.8 implies that $\beta \geq \gamma$. Consider the cycle $C = (v_1, v_5, v_6, v_3, v_{10}, v_{11}, v_4, v_{12}, v_{13}, v_1)$. Since $ax \leq \alpha$ has as support the graph $G$ and $C$ is an odd hole, there is a stable set $S$ in $G$ such that $|S \cap C| < 4$ and $ax^S = \alpha$. Then $S = \{v_1, v_2, v_3, v_4\}$. Since $(S\backslash\{v_4\}) \cup \{v_{11}, v_{12}\}$ is a stable set, it follows that $\beta \leq \gamma$. Therefore we have the inequality

$$2x(v_1) + 2x(v_3) + 2x(v_4) + x(v_2) + \sum_{j=5}^{13} x(v_j) \leq 7.$$

Now consider (ii), (iv), and (vi). This implies that

$$\beta = a(v_1) = a(v_5) = a(v_6) = a(v_3) = a(v_{10}) = a(v_{11}) = a(v_4) = a(v_{12}) = a(v_{13}),$$
$$\gamma = a(v_7) = a(v_8) = a(v_9),$$
$$2\gamma = a(v_2).$$

Lemma 5.5 implies that $\beta \geq \gamma$. Consider the cycle $C = (v_1, v_5, v_6, v_3, v_{10}, v_{11}, v_4, v_{12}, v_{13}, v_1)$. Since $ax \leq \alpha$ has as support the graph $G$ and $C$ is an odd hole, there is a stable set $S$ in $G$ such that $|S \cap C| < 4$ and $ax^S = \alpha$. Then $S = \{v_5, v_7, v_8, v_9, v_{10}, v_{13}\}$. Since $(S\backslash\{v_9\}) \cup \{v_4\}$ is a stable set, it follows that $\beta \leq \gamma$. Therefore we have the inequality

$$2x(v_2) + \sum_{j \neq 2} x(v_j) \leq 6.$$

Now we study Case 15; see Fig. 6.3.

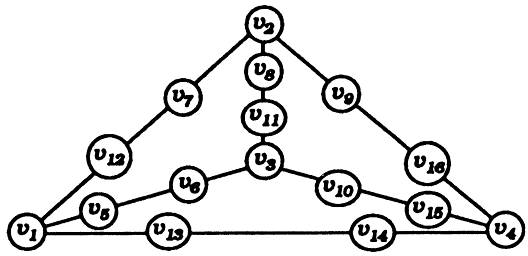


FIG. 6.3

Lemma 5.10 implies that either (i) $\beta = a(v_j)$ for $5 \leq j \leq 16$ and $2\beta = a(v_j)$ for $1 \leq j \leq 4$ or (ii) $\beta = a(v_j)$, for $1 \leq j \leq 16$. In the first case, we have the inequality

$$2 \sum_{j=1}^{4} x(v_j) + \sum_{j=5}^{16} x(v_j) \leq 9.$$

In the second case, we have the inequality

$$\sum x(v_j) \leq 7.$$

Now we study Case 16; see Fig. 6.4.

Part (b1) of Lemma 5.9 implies that either (i) $a(v_2) = a(v_7) + a(v_8)$, $a(v_3) = a(v_6) + a(v_{11})$, and $a(v_1) = a(v_5)$ or (ii) $a(v_1) = a(v_6) + a(v_7)$, $a(v_3) = a(v_{10})$, and $a(v_2) = a(v_9)$. It also implies that either (iii) $a(v_1) = a(v_5) + a(v_7)$, $a(v_4) = a(v_9) + a(v_{12})$, and $a(v_2) = a(v_8)$ or (iv) $a(v_2) = a(v_7) + a(v_9)$, $a(v_1) = a(v_6)$, and $a(v_4) = a(v_{10})$. We have that either (i) and (iv) hold or (ii) and (vii) hold.

Consider (i) and (iv). This implies that $a(v_1) = a(v_5)$. It follows from Lemma 5.11 (b2) that the inequality $ax \leq \alpha$ is obtained from a facet of $P(G')$, using the procedure of Theorem 4.1, where $G'$ is obtained from $G$ by contracting the edges $v_5v_{12}$ and $v_{12}v_4$. The graph $G'$ is that of Case 13, which has already been studied. For this, $P(G')$ has only one facet whose support is $G'$, which is the inequality

$$2x(v_2) + 2x(v_3) + x(v_1) + \sum_{j=4}^{10} x(v_j) \leq 5.$$

Then $ax \leq \alpha$ should be

$$x(v_1) + 2x(v_2) + 2x(v_3) + \sum_{j=4}^{12} x(v_j) \leq 6.$$

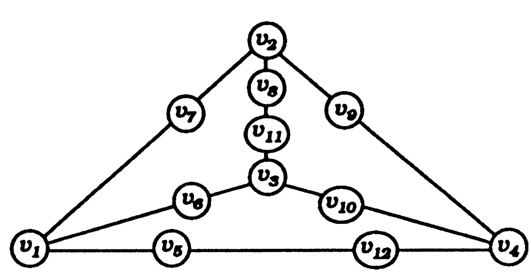


FIG. 6.4

If (ii) and (iii) hold, then $a(v_2) = a(v_8)$. In a similar way, we can prove that $ax \leq \alpha$ is

$$2x(v_1) + x(v_2) + x(v_3) + 2x(v_4) + \sum_{j=5}^{12} x(v_j) \leq 6.$$

It is easy to see that all the inequalities derived in this section can be obtained by applying Theorems 4.1 and 4.3, starting from the clique inequality of $K_4$. This completes the proof of Theorem 4.4.

The 18 inequalities derived in this section, together with the clique inequality of $K_4$, form a family of 19 $K_4$ inequalities, referred to in Theorem 4.5.

**7. Some examples.** In this section, we apply the combinatorial procedure that describes the facets of $P(G)$ for graphs with no $W_4$ minor. Consider the graphs of Figs. 7.1(a) and 7.1(b).

The constraint

$$2x(u_1) + x(u_2) + 2x(u_3) + 2x(u_4) + \sum_{j \geq 5} x(u_j) \leq 7$$

defines a facet for the polytope of the first graph, and the constraint

$$x(v_1) + 2x(v_2) + 2x(v_3) + \sum_{j \geq 4} x(v_j) \leq 5$$

defines a facet for the second one.

By identifying the nodes $\{u_1, v_1\}$ and $\{u_4, v_2\}$ and deleting a 5-cycle, we obtain the graph of Fig. 7.2.

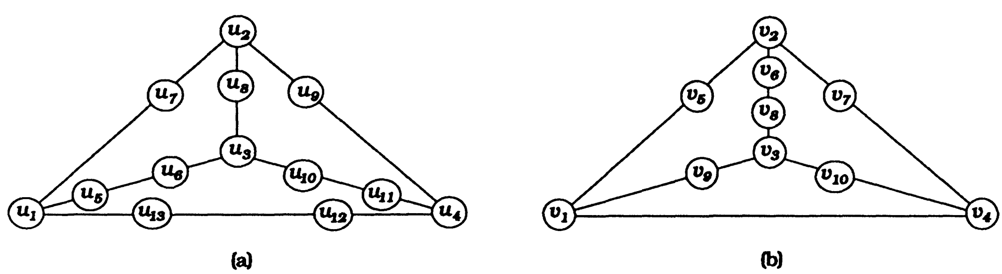


FIG. 7.1

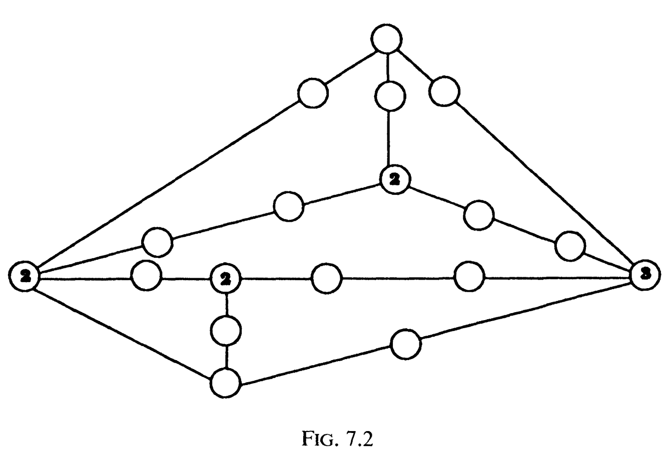


FIG. 7.2

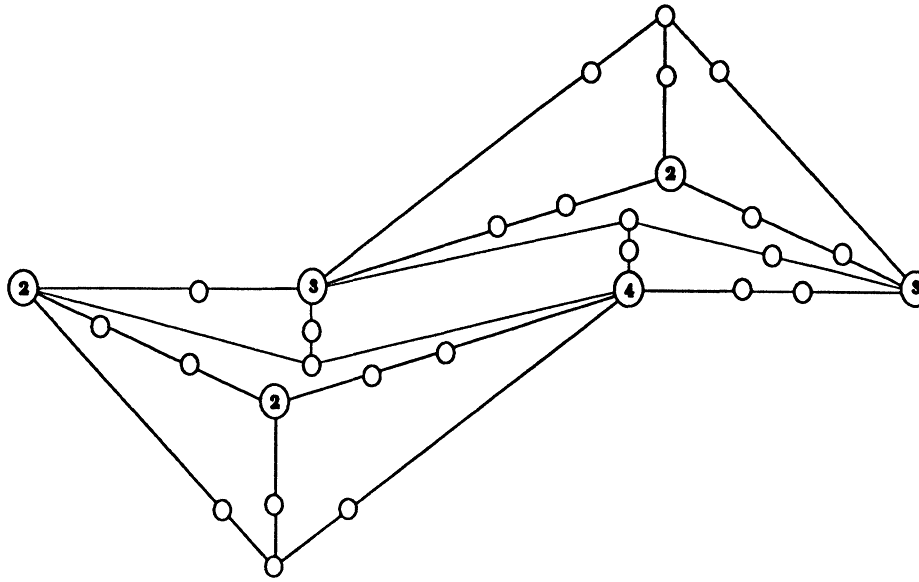


FIG. 7.3

Theorem 2.5 gives a facet-defining inequality, whose support is the whole graph, and its coefficients different from 1 appear in the figure. The value of the right-hand side is 10. Here $v_5$ plays the role of $w_1$, and $u_{12}$ and $u_{13}$ play the roles of $w_2$ and $w_3$, respectively.

Now consider the graph of Fig. 7.3. It has been obtained by composing the graph of Fig. 7.2 with itself. Again, Theorem 2.5 shows that there is a facet-defining inequality whose support is the whole graph and whose coefficients different from 1 are in the figure. We can state the following.

*Remark* 7.1. Given any integer $p > 0$, there exists a graph $G$ with no $W_4$ minor, such that $P(G)$ has a facet with coefficients $1, 2, \ldots, p$.

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