# Composition of Graphs and the Triangle-Free Subgraph Polytope 

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#### Abstract

In this paper, we study a composition (decomposition) technique for the triangle-free subgraph polytope in graphs which are decomposable by means of 3 -sums satisfying some property. If a graph $G$ decomposes into two graphs $G_{1}$ and $G_{2}$, we show that the triangle-free subgraph polytope of $G$ can be described from two linear systems related to $G_{1}$ and $G_{2}$. This gives a way to characterize this polytope on graphs that can be recursively decomposed. This also gives a procedure to derive new facets for this polytope. We also show that, if the systems associated with $G_{1}$ and $G_{2}$ are TDI, then the system characterizing the polytope for $G$ is TDI. This generalizes previous results in R. Euler and A.R. Mahjoub (Journal of Comb. Theory series B, vol. 53, no. 2, pp. 235-259, 1991) and A.R. Mahjoub (Discrete Applied Math., vol. 62, pp. 209-219, 1995).


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## 1. Introduction

Given a graph $G=(V, E)$ and two subgraphs $G_{1}=\left(V_{1}, E_{1}\right)$ and $G_{2}=\left(V_{2}, E_{2}\right), G$ is called a $k$-sum of $G_{1}$ and $G_{2}$ if $V=V_{1} \cup V_{2},\left|V_{1} \cap V_{2}\right|=k$ and the subgraph ( $V_{1} \cap V_{2}, E_{1} \cap E_{2}$ ) is complete. The set $V_{1} \cap V_{2}$ is called a $k$-node cutset of $G$. In this paper we study a composition (decomposition) technique for the triangle-free subgraph polytope in graphs which are decomposable by 3-node cutsets. If $G$ decomposes into $G_{1}$ and $G_{2}$, then we derive a system of inequalities that defines the triangle-free subgraph polytope from systems related to $G_{1}$ and $G_{2}$. As a consequence, we obtain a procedure to construct this polytope in graphs that can be decomposed by means of 3-sums satisfying some property. This also gives a way to construct facets for the triangle subgraph polytope by composition of facets from the pieces. We also show that if the systems associated with the pieces are TDI, then the system describing the polytope for $G$ is TDI.

Developing composition (decomposition) techniques for NP-hard combinatorial optimization problems has been a motivating subject for many researchers along the past two decades (Barahona et al., 1994; Barahona, 1983; Bouchakour and Mahjoub, 1997; Burlet and Fonlupt, 1995; Euler and Mahjoub, 1991; Hadjar, 1996; Margot, 1994). In fact, for an

NP-hard combinatorial optimization problem, it is sometimes difficult to give a complete description of the associated polytope in some graph. Netherless, if the graph decomposes into pieces (with respect to certain decomposition operations), it may be possible to give a complete description of the polytope from polytopes related to the pieces. This approach has been studied for different combinatorial optimization problems such as the max-cut problem (Barahona, 1983), the stable set problem (Burlet and Fonlupt, 1995; Margot, 1994), the acyclic subdigraph problem (Barahona et al., 1994), the dominating set problem (Bouchakour and Mahjoub, 1997). Margot (1994) studied a general composition (decomposition) approach for combinatorial optimization polytopes using projection. This permitted him to generalize known results related to independence systems. Hadjar (1996) examined an approach of composition based on dynamic programming that he applied to the travelling salesman problem.

Given a graph $G=(V, E)$ and a weight system associated with the edges of $E$, the triangle-free subgraph problem (TFSP for short), is to find a triangle-free subgraph, that is a subgraph containing no $K_{3}$, whose weight is maximum. The TFSP has applications to the maximum weight clique problem (Balas et al., 1987). It may also be seen as a relaxation of the maximum bipartite subgraph problem (Conforti et al., 1986). The TFSP is NP-hard in general (Yannakakis, 1981). However, it has been shown to be polynomially solvable in some special classes of graphs. Euler and Mahjoub (1991) showed that the TFSP can be solved in polynomial time in the graphs noncontractible to $K_{5} \backslash e$ (the complete graph on five nodes minus one edge). Conforti et al. (1986) studied the maximum triangle-free subgraph problem within the framework of a more general model. They showed that the TFSP is NP-complete on chordal graphs (a graph is chordal (or triangulated) if every cycle of length greater than three has a chord). They also proved that the TFSP can be solved in polynomial time in chordal graphs with a fixed clique size.

If $G=(V, E)$ is a graph and $T \subset E$ an edge subset of $G$, then the $0-1$ vector $x^{T} \in \mathfrak{R}^{E}$ with $x(e)=1$ if $e \in T$ and $x(e)=0$ if not, is called the incidence vector of $T$. The convex hull of the incidence vectors of all the triangle-free edge sets of $G$, denoted by $P_{\Delta}(G)$, is called the triangle-free subgraph polytope of $G$, i.e.,

$$
P_{\Delta}(G)=\operatorname{conv}\left\{x^{T} \in \mathfrak{R}^{E} \mid(V, T) \text { is triangle-free }\right\} .
$$

The polytope $P_{\Delta}(G)$ is full dimensional. This implies that (up to multiplication by a positive scalar) there is a unique nonredundant inequality system $A x \leq b$ such that $P_{\Delta}(G)=$ $\left\{x \in \mathfrak{R}^{E} \mid A x \leq b\right\}$. These inequalities define facets of $P_{\Delta}(G)$.

Given a weight vector $w \in \mathfrak{R}^{E}$, the triangle-free subgraph problem can then be formulated as the linear program

$$
\operatorname{maximize}\left\{w x, \quad x \in P_{\Delta}(G)\right\} .
$$

If $T \subset E$ is a triangle-free edge set, then its incidence vector, $x^{T}$, satisfies the inequalities:

$$
\begin{array}{ll}
0 \leq x(e) \leq 1 & \text { for all } e \in E, \\
\sum_{e \in \Delta} x(e) \leq 2 & \text { for all triangle } \Delta \subset E . \tag{2}
\end{array}
$$

Inequalities (1) are called trivial inequalities and inequalities (2) triangle inequalities. Moreover, every $0-1$ solution of (1)-(2) is the incidence vector of a triangle-free edge set of $G$. Conforti et al. (1986) discussed several classes of facet-defining inequalities for the polytope $P_{\Delta}(G)$. Euler and Mahjoub (1991) studied $P_{\Delta}(G)$ in the graphs decomposable by means of 1 and 2-sums. They showed that, if $G$ decomposes into $G_{1}$ and $G_{2}$, then a system defining $P_{\Delta}(G)$ can be obtained from the union of the systems that define $P_{\Delta}\left(G_{1}\right)$ and $P_{\Delta}\left(G_{2}\right)$ and by identifying the variables associated with the edges of $E_{1} \cap E_{2}$. As a consequence, they obtained that, when $G$ is noncontractible to $K_{5} \backslash e$, the polytope $P_{\Delta}(G)$ is completely described by the inequalities (1)-(2) together with the inequalities:

$$
\begin{equation*}
\sum_{e \in W_{2 k+1}} x(e) \leq 3 k+1 \quad \text { for all } W_{2 k+1} \subset E \tag{3}
\end{equation*}
$$

Here $W_{n}$, for a given positive integer $n$, is the wheel on $n+1$ nodes, that is the graph that consists of a cycle on $n$ nodes and a node adjacent to all the vertices of the cycle. Inequalities (3) are called odd wheel inequalities. These inequalities have been introduced by Conforti et al. (1986) where they also showed that they are facet defining.

The paper is organized as follows. In the next section, we discuss some structural properties of $P_{\Delta}(G)$. In Section 3 we study compositions of polyhedra. In Section 4 we study compositions of TDI systems and in Section 5 we give some concluding remarks.

The remainder of this section is devoted to more definitions and notations.
We assume that the reader is familiar with the basic definitions and concepts of polyhedral combinatorics. Refer to Pulleyblank (1989) for necessary background. The graphs we consider are finite, undirected and without loops. We describe a graph by $G=(V, E)$ where $V$ is the node set and $E$ the edge set of $G$. If $e$ is an edge with endnodes $u$ and $v$, then we write $e=u v$. If $G=(V, E)$ is a graph and $X \subset V$ is a subset of nodes, we denote by $G(X)$ the subgraph of $G$ induced by $X$. Given a constraint $a x \leq b, a \in \Re^{E}$ and a solution $x^{*} \in \mathfrak{R}^{E}$, we will say that $a x \leq b$ is tight for $x^{*}$ if $a x^{*}=b$.

## 2. Structural properties

In this section, we shall discuss some structural properties of the facets of $P_{\Delta}(G)$. Let $G=(V, E)$ be a graph and $a x \leq b$ a facet-defining inequality of $P_{\Delta}(G)$ different from inequalities (1) and (2). Thus $a(e) \geq 0$ for all $e \in E$. Let us denote by $E_{a}$ the support of $a$, that is the set of edges $e \in E$ such that $a(e)>0$. Let $G_{a}=\left(V_{a}, E_{a}\right)$ be the graph induced by $E_{a}$. A graph is said to be $k$-connected if, for any pair of nodes $\{u, v\}$, there are at least $k$ node-disjoint paths between $u$ and $v$.

Let

$$
\tau(G)=\{T \subseteq E \mid(V, T) \text { is } \Delta-\text { free }\}
$$

and

$$
\tau_{a}=\left\{T \in \tau(G) \mid a x^{T}=b\right\} .
$$

We have the following.

## Lemma 2.1.

(i) Every edge $e \in E_{a}$ belongs to at least one triangle of $E_{a}$;
(ii) $G_{a}$ does not contain two edges that both belong to the same unique triangle of $E_{a}$;
(iii) $G_{a}$ is 3-connected.

## Proof:

(i) Suppose there is an edge $e_{0}$ of $E_{a}$ that does not belong to any triangle of $E_{a}$. As $a x \leq b$ is different from a trivial constraint, there must exist an edge set $T \in \tau_{a}$ that does not contain $e_{0}$. As $\left(T \cap E_{a}\right) \cup\left\{e_{0}\right\}$ is still triangle-free, this implies that $a_{e_{0}} \leq 0$, a contradiction.
(ii) Suppose there are two edges $e_{1}, e_{2} \in E_{a}$ that belong to a unique triangle, say $\Delta=$ $\left\{e_{1}, e_{2}, f\right\}$, of $E_{a}$. We claim that every edge set $T \in \tau_{a}$ contains two edges of $\Delta$. In fact, if $T \cap \Delta=\emptyset$, then $\left(T \cap E_{a}\right) \cup\left\{e_{1}\right\}$ is triangle-free and thus, $a\left(e_{1}\right) \leq 0$, a contradiction. If $|T \cap \Delta|=1$, then $\left(T \cap E_{a}\right) \cup\left\{e_{i}\right\}$ would be triangle-free for some $i \in\{1,2\}$ and, as a consequence, $a\left(e_{i}\right) \leq 0$, which yields again a contradiction. Hence, $|T \cap \Delta|=2$ for all $T \in \tau_{a}$. Since $a x \leq b$ is not a triangle inequality, this is impossible.
(iii) This can also be derived from Euler and Mahjoub (1991). For the sake of completeness, we give here a proof. Suppose that $G_{a}$ is not 3-connected. Without loss of generality, we may suppose that $G_{a}$ is 2-connected. Thus, $G_{a}$ decomposes into two graphs $G_{a}^{1}=$ $\left(V_{a}^{1}, E_{a}^{1}\right)$ and $G_{a}^{2}=\left(V_{a}^{2}, E_{a}^{2}\right)$ with respect to a 2-node cutset $\{u, v\}$.

First, suppose that $u v \notin E_{a}$. Thus, for every $T \in \tau_{a}$, the following holds

$$
\sum_{e \in E_{a}} a(e) x^{T}(e)=\sum_{e \in E_{a}^{1}} a(e) x^{T}(e)+\sum_{e \in E_{a}^{2}} a(e) x^{T}(e)=b .
$$

Let $\alpha_{k}=\max \left\{\sum_{e \in E_{a}^{k}} a(e) x^{T}(e), T \in \tau_{a}\right\}$, for $k=1,2$. It then follows that $b=\alpha_{1}+\alpha_{2}$, which implies that every set $T \in \tau_{a}$ satisfies the equation $\sum_{e \in E_{a}^{1}} a(e) x^{T}(e)=\alpha_{1}$. Since $a x=b$ is not a positive multiple of this equality, we get a contradiction.

Next, suppose that the edge $e_{0}=u v$ belongs to $E_{a}$. Without loss of generality, we may suppose that $a\left(e_{0}\right)=1$. Let us denote by $\tau_{e_{0}}(G)$ the set of edge sets of $\tau(G)$ that contain $e_{0}$, and let $\bar{\tau}_{e_{0}}(G)=\tau(G) \backslash \tau_{e_{0}}(G)$.
Let

$$
\beta_{k}=\max \left\{\sum_{e \in E_{a}^{k}, e \neq e_{0}} a(e) x^{T}(e), T \in \tau_{e_{0}}(G)\right\},
$$

and

$$
\gamma_{k}=\max \left\{\sum_{e \in E_{a}^{k}} a(e) x^{T}(e), T \in \bar{\tau}_{e_{0}}(G)\right\},
$$

for $k=1,2$. Since $a x \leq b$ is not a trivial inequality and, therefore, there exists a set of $\tau_{a}$ that contains (does not contain) $e_{0}$, one should have $\beta_{1}+\beta_{2}+1=b$ and $\gamma_{1}+\gamma_{2}=b$. As $\beta_{1}+\gamma_{2} \leq b$ and $\gamma_{1}+\beta_{2} \leq b$, it then follows that:

$$
\beta_{k} \leq \gamma_{k} \leq \beta_{k}+1, \quad \text { for } k=1,2
$$

Let $\theta=\gamma_{1}-\beta_{1}$. Now it is easy to see that the inequality

$$
\sum_{e \in E_{a}^{1}, e \neq e_{0}} a(e) x^{T}(e)+\theta x\left(e_{0}\right) \leq \gamma_{1}
$$

is verified at equality by the incidence vectors of all the sets of $\tau_{a}$. But this yields again a contradiction.

## 3. Composition of polyhedra

In this section, we show that a system of inequalities that describes $P_{\Delta}(G)$ can be derived provided that $G$ is a 3-sum of two graphs $G_{1}$ and $G_{2}$ satisfying a certain property and $P_{\Delta}\left(G_{1}\right)$ and $P_{\Delta}\left(G_{2}\right)$ are known.

### 3.1. 3-sums

Let $G=(V, E)$ be a graph that is the 3 -sum of $G_{1}=\left(V_{1}, E_{1}\right)$ and $G_{2}=\left(V_{2}, E_{2}\right)$ such that
(i) $V=V_{1} \cup V_{2}$,
(ii) $E=E_{1} \cup E_{2}$,
(iii) $\Delta_{0}=E_{1} \cap E_{2}=\left\{e_{1}, e_{2}, e_{3}\right\}$,
(iv) $G_{1}$ (resp. $G_{2}$ ) contains four edges $f_{1}, f_{1}^{\prime}, g_{1}, g_{1}^{\prime}$ (resp. $f_{2}, f_{2}^{\prime}, g_{2}, g_{2}^{\prime}$ ) such that $\Delta_{1}^{1}=$ $\left\{e_{1}, f_{1}, f_{1}^{\prime}\right\}, \Delta_{2}^{1}=\left\{e_{3}, g_{1}, g_{1}^{\prime}\right\}$ (resp. $\left.\Delta_{1}^{2}=\left\{e_{2}, f_{2}, f_{2}^{\prime}\right\}, \Delta_{2}^{2}=\left\{e_{3}, g_{2}, g_{2}^{\prime}\right\}\right)$ are triangles and $\Delta_{0}, \Delta_{1}^{1}, \Delta_{2}^{1}$ (resp. $\Delta_{0}, \Delta_{1}^{2}, \Delta_{2}^{2}$ ) are the only triangles of $G_{1}$ (resp. $G_{2}$ ) that intersect $\left\{e_{1}, e_{2}, e_{3}, f_{1}, g_{1}\right\}$ (resp. $\left\{e_{1}, e_{2}, e_{3}, f_{2}, g_{2}\right\}$ ) (see figure 1).

Let $A_{k}=\left\{e_{1}, e_{2}, e_{3}, f_{k}, g_{k}\right\}, \hat{E}_{k}=E_{k} \backslash A_{k}$ and $\bar{E}_{k}=E_{k} \backslash \Delta_{0}$, for $k=1$, 2. Let $A=A_{1} \cup$ $A_{2}, m_{1}=\left|E_{1}\right|$ and $m_{2}=\left|E_{2}\right|$.


Figure 1. 3-sum of $G_{1}$ and $G_{2}$.

Lemma 3.1. If $a x \leq b$ is a facet-defining inequality of $P_{\Delta}\left(G_{k}\right)$ different from inequalities (1) and (2), then $a(e)=a\left(e^{\prime}\right)$, for all $e, e^{\prime} \in A_{k}, k=1,2$.

Proof: We will give the proof for $P_{\Delta}\left(G_{1}\right)$ (the proof is similar for $P_{\Delta}\left(G_{2}\right)$ ). If $a(e)=0$ for some $e \in A_{1}$, it follows from Lemma 2.1 that $a(e)=0$, for all $e \in A_{1}$.

So, consider the case where $a(e)>0$, for all $e \in A_{1}$. As $a x \leq b$ is not a trivial inequality, there must exist a set $T_{1} \in \tau_{a}$ such that $e_{2} \notin T_{1}$. Hence, $e_{1}, e_{3} \in T_{1}$. For otherwise, if, for instance, $e_{1}$ does not belong to $T_{1}$, then $T_{1} \cup\left\{e_{2}\right\}$ would be triangle-free, which implies that $a\left(e_{2}\right) \leq 0$, a contradiction. Let $T_{1}^{\prime}=\left(T_{1} \backslash\left\{e_{1}\right\}\right) \cup\left\{e_{2}\right\}$ and $T_{1}^{\prime \prime}=\left(T_{1} \backslash\left\{e_{3}\right\}\right) \cup\left\{e_{2}\right\}$. Since, $T_{1}^{\prime}, T_{1}^{\prime \prime} \in \tau(G)$, the following hold

$$
\begin{equation*}
a\left(e_{2}\right) \leq a\left(e_{1}\right), \quad a\left(e_{2}\right) \leq a\left(e_{3}\right) \tag{4}
\end{equation*}
$$

Similarly, we obtain that

$$
\begin{array}{r}
a\left(f_{1}\right) \leq a\left(e_{1}\right), \quad a\left(f_{1}\right) \leq a\left(f_{1}^{\prime}\right), \\
a\left(g_{1}\right) \leq a\left(e_{3}\right), \quad a\left(g_{1}\right) \leq a\left(g_{1}^{\prime}\right) . \tag{6}
\end{array}
$$

Furthermore, as $a x \leq b$ is not a triangle inequality, there exists a triangle-free subgraph, $T_{2} \in \tau_{a}$, such that $\left|T_{2} \cap \Delta_{0}\right|<2$. This implies that neither $e_{1}$ nor $e_{3}$ is in $T_{2}$. In consequence, $f_{1}, f_{1}^{\prime}, g_{1}, g_{1}^{\prime}$ must belong to $T_{2}$. Now, as the edge sets $\left(T_{2} \backslash\left\{f_{1}\right\}\right) \cup\left\{e_{1}\right\}$ and $\left(T_{2} \backslash\left\{g_{1}\right\}\right) \cup\left\{e_{3}\right\}$ are triangle-free, we have that

$$
\begin{equation*}
a\left(e_{1}\right) \leq a\left(f_{1}\right), \quad a\left(e_{3}\right) \leq a\left(g_{1}\right) \tag{7}
\end{equation*}
$$

By combining (5)-(7) we obtain that

$$
\begin{equation*}
a\left(e_{1}\right)=a\left(f_{1}\right), \quad a\left(e_{3}\right)=a\left(g_{1}\right) \tag{8}
\end{equation*}
$$

On the other hand, there is a triangle-free subgraph $T_{3} \in \tau_{a}$ such that $\left|T_{3} \cap \Delta_{1}\right|<2$. Hence, $e_{1}, f_{1}^{\prime} \notin T_{3}$ and therefore $e_{2}, e_{3} \in T_{3}$. Since the edge sets $\left(T_{3} \backslash\left\{e_{2}\right\}\right) \cup\left\{e_{1}\right\}$ and $\left(T_{3} \backslash\left\{e_{3}\right\}\right) \cup\left\{e_{1}\right\}$ contain no triangles, we get

$$
\begin{equation*}
a\left(e_{1}\right) \leq a\left(e_{2}\right), \quad a\left(e_{1}\right) \leq a\left(e_{3}\right) . \tag{9}
\end{equation*}
$$

From (4), (8) and (9), it follows that $a\left(e_{1}\right)=a\left(e_{2}\right)=a\left(f_{1}\right)$.
By symmetry, we also obtain that $a\left(e_{3}\right)=a\left(e_{2}\right)=a\left(g_{1}\right)$ and then $a(e)=a\left(e^{\prime}\right)$, for all $e$, $e^{\prime} \in A_{1}$.

From Lemma 3.1, a minimal system defining $P_{\Delta}\left(G_{k}\right), k=1,2$ canbe written as follows:

$$
\begin{equation*}
\sum_{e \in \hat{E}_{k}} a_{i}^{k}(e) x(e) \leq b_{i}^{k}, \quad i \in I_{1}^{k}, \tag{10}
\end{equation*}
$$

$$
\begin{align*}
& \sum_{e \in \hat{E}_{k}} a_{j}^{k}(e) x(e)+\sum_{e \in A_{k}} x(e) \leq b_{j}^{k}, \quad j \in I_{2}^{k}  \tag{11}\\
& \sum_{e \in \Delta} x(e) \leq 2, \quad \text { for all triangle } \Delta \subset E_{k} \\
& \quad x(e) \leq 1, \quad \text { for all } e \in E_{k}, \\
& \quad x(e) \geq 0, \quad \text { for all } e \in E_{k},
\end{align*}
$$

where $I_{1}^{k}$ (resp. $I_{2}^{k}$ ) is the set of the indices corresponding to the constraints of $P_{\Delta}(G)$ different from (1) and (2) whose support does not intersect (resp. intersects) the set $A_{k}$.

### 3.2. 3-sums and the polytope $P_{\Delta}(G)$

In what follows, we are going to show that facets of $P_{\Delta}(G)$ can be obtained from facets of $P_{\Delta}\left(G_{1}\right)$ and $P_{\Delta}\left(G_{2}\right)$. First, we introduce new constraints that are valid for $P_{\Delta}(G)$.

Definition 3.1. Given two constraints

$$
\begin{gather*}
\sum_{e \in \hat{E}_{1}} a(e) x(e)+\sum_{e \in A_{1}} x(e) \leq b  \tag{12}\\
\sum_{e \in \hat{E}_{2}} a^{\prime}(e) x(e)+\sum_{e \in A_{2}} x(e) \leq b^{\prime} \tag{13}
\end{gather*}
$$

valid for $P_{\Delta}\left(G_{1}\right)$ and $P_{\Delta}\left(G_{2}\right)$, respectively, we call mixed constraint, the inequality

$$
\begin{equation*}
\sum_{e \in \hat{E}_{1}} a(e) x(e)+\sum_{e \in \hat{E}_{2}} a^{\prime}(e) x(e)+\sum_{e \in A} x(e) \leq b+b^{\prime}-2 . \tag{14}
\end{equation*}
$$

We have the following.
Lemma 3.2. Mixed constraints are valid for $P_{\Delta}(G)$.
Proof: Let $T \in \tau(G)$. Let $T_{k}$ be the restriction of $T$ on $E_{k}, k=1,2$.

- If $\left|T \cap \Delta_{0}\right|=0$, then the edge set $T_{1} \cup\left\{e_{2}\right\}$ (resp. $T_{2} \cup\left\{e_{1}\right\}$ ) induces a triangle-free subgraph of $G_{1}$ (resp. $G_{2}$ ). Thus

$$
\begin{aligned}
& \sum_{e \in \hat{E}_{1}} a(e) x^{T_{1}}(e)+\sum_{e \in A_{1} \backslash \Delta_{0}} x^{T_{1}}(e)+1 \leq b, \\
& \sum_{e \in \bar{E}_{2}} a^{\prime}(e) x^{T_{2}}(e)+\sum_{e \in A_{2} \backslash \Delta_{0}} x^{T_{2}}(e)+1 \leq b^{\prime}
\end{aligned}
$$

then

$$
\sum_{e \in \hat{E}_{1}} a(e) x^{T}(e)+\sum_{e \in \hat{E}_{2}} a^{\prime}(e) x^{T}(e)+\sum_{e \in A} x^{T}(e) \leq b+b^{\prime}-2 .
$$

- If $\left|T \cap \Delta_{0}\right|=2$, then, since inequalities (12) and (13) are satisfied by $x^{T_{1}}$ and $x^{T_{2}}$, respectively, we obtain that $x^{T}$ verifies (14).
- Now, suppose that $\left|T \cap \Delta_{0}\right|=1$. If $T \cap \Delta_{0}=\left\{e_{3}\right\}$, as the edge sets $T_{1} \cap\left\{e_{2}\right\}$ and $T_{2} \cup\left\{e_{1}\right\}$ are triangle-free, it is easy to see that $x^{T}$ also verifies the mixed constraint. Thus suppose, for instance, that $T \cap \Delta_{0}=\left\{e_{1}\right\}$ (the case where $T \cap \Delta_{0}=\left\{e_{2}\right\}$ is similar). We have that $T_{1} \cup\left\{e_{2}\right\} \in \tau\left(G_{1}\right)$ and, therefore,

$$
\begin{aligned}
& \sum_{e \in \hat{E}_{1}} a(e) x^{T_{1}}(e)+\sum_{e \in A_{1}} x^{T_{1}}(e)+1 \leq b \\
& \sum_{e \in \hat{E}_{2}} a^{\prime}(e) x^{T_{2}}(e)+\sum_{e \in A_{2} \backslash \Delta_{0}} x^{T_{2}}(e)+x^{T_{2}}\left(\Delta_{0}\right) \leq b^{\prime}
\end{aligned}
$$

Hence, (14) is satisfied by $x^{T}$.
In all cases, we obtain that $x^{T}$ satisfies inequality (14). Therefore, mixed constraints are valid for $P_{\Delta}(G)$.

Let $Q(G)$ be the polytope given by the following inequalities

$$
\begin{array}{ll}
\sum_{e \in \hat{E}_{k}} a_{i}^{k}(e) x(e) \leq b_{i}^{k} & i \in I_{1}^{k}, k=1,2, \\
\sum_{e \in \hat{E}_{k}} a_{j}^{k}(e) x(e)+\sum_{e \in A_{k}} x(e) \leq b_{j}^{k} & j \in I_{2}^{k}, k=1,2, \\
\sum_{e \in \hat{E}_{1}} a_{j}^{1}(e) x(e)+\sum_{e \in A} x(e)+\sum_{e \in \hat{E}_{2}} a_{l}^{2}(e) x(e) \leq b_{j}^{1}+b_{l}^{2}-2 \\
& j \in I_{2}^{1}, l \in I_{2}^{2}, \\
\sum_{e \in \Delta} x(e) \leq 2 & \text { for all triangle } \Delta \subset E, \\
x(e) \leq 1 & \text { for all } e \in E, \\
x(e) \geq 0 & \text { for all } e \in E . \tag{20}
\end{array}
$$

Let us denote an inequality of type (16) by $[j, k], j \in I_{2}^{k}, k \in\{1,2\}$ and of type (17) by $[j, l], j \in I_{2}^{1}, l \in I_{2}^{2}$. We can now state our main result.

Theorem 3.3. $\quad P_{\Delta}(G)=Q(G)$.
Proof: Since all the constraints of $Q(G)$ are valid for $P_{\Delta}(G)$, we have $P_{\Delta}(G) \subseteq Q(G)$. Moreover, any integer solution of $Q(G)$ corresponds to a triangle-free edge set of $G$. Let us suppose, on the contrary, that $P_{\Delta}(G) \neq Q(G)$. Thus, there exists an extreme point $x$ of $Q(G)$ which is fractional. We shall denote by $S(x), S_{0}(x), S_{1}(x), S_{f}(x)$ the set of edges $e$ such that $x(e)>0, x(e)=0, x(e)=1,0<x(e)<1$, respectively.

Let $x^{1}$ (resp. $x^{2}$ ) be the restriction of $x$ on $E_{1}$ (resp. $E_{2}$ ). As $x^{1}$ (resp. $x^{2}$ ) belongs to $P_{\Delta}\left(G_{1}\right)$ (resp. $P_{\Delta}\left(G_{2}\right)$ ), there must exist $q_{1}$ (resp. $q_{2}$ ) extreme points $y^{1}, \ldots, y^{q_{1}}$ (resp.
$\left.z^{1}, \ldots, z^{q_{2}}\right)$ of $P_{\Delta}\left(G_{1}\right)\left(\right.$ resp. $\left.P_{\Delta}\left(G_{2}\right)\right)$ such that

$$
x^{1}=\sum_{i} \alpha_{i} y^{i}, \alpha_{i}>0, \sum_{i} \alpha_{i}=1 \text { and } x^{2}=\sum_{j} \beta_{j} z^{j}, \beta_{j}>0, \sum_{j} \beta_{j}=1 .
$$

Note here that all the vectors $y^{1}, \ldots, y^{q_{1}}, z^{1}, \ldots, z^{q_{2}}$ are integer and that every constraint of $P_{\Delta}\left(G_{1}\right)\left(\right.$ resp. $\left.P_{\Delta}\left(G_{2}\right)\right)$ tight for $x^{1}$ (resp. $x^{2}$ ) is also tight for $y^{i}, i \in\left\{1, \ldots, q_{1}\right\}$ (resp. $\left.z^{j}, j \in\left\{1, \ldots, q_{2}\right\}\right)$.

Let us denote by $B\left(\Delta_{0}\right)$ the set $2^{\Delta_{0}} \backslash \Delta_{0}$. For $F \in B\left(\Delta_{0}\right)$, let $r(F)=\sum_{\Delta_{0} \cap S_{1}\left(y^{i}\right)=F} \alpha_{i}$ and $s(F)=\sum_{\Delta_{0} \cap S_{1}\left(z^{j}\right)=F} \beta_{j}$.

We have the following claims.
Claim 1. There is a set $F_{0} \in B\left(\Delta_{0}\right)$ such that $r\left(F_{0}\right) \neq s\left(F_{0}\right)$.
Proof: Clearly, the vector $t_{i j}=\left(y^{i}, \tilde{z}^{j}\right)$ induces a triangle-free subgraph of $G$ for every $i, j$ such that $\Delta_{0} \cap S_{1}\left(y^{i}\right)=\Delta_{0} \cap S_{1}\left(z^{j}\right)$, where $\tilde{z}^{j}$ is the restriction of $z^{j}$ onto $E_{2} \backslash \Delta_{0}$. If $r(F)=s(F)$ for all $F \in B\left(\Delta_{0}\right)$, then we would have

$$
x=\sum_{i, j} \gamma_{i j} t_{i j}
$$

where $\gamma_{i j}=\left(\alpha_{i} \cdot \beta_{j}\right) / r(F)$ for $F=\Delta_{0} \cap S_{1}\left(y^{i}\right)=\Delta_{0} \cap S_{1}\left(z^{j}\right)$.
Since $\sum_{i, j} \gamma_{i j}=1$ and $\gamma_{i j}>0$ for all $i, j$, this contradicts the extremality of $x$.

## Claim 2.

(i) $x(e)<1$ for all $e \in \Delta_{0}$.
(ii) $x\left(\Delta_{0}\right)<2$.
(iii) $\Delta_{0} \subset S(x)$.

## Proof:

(i) Suppose, for instance, that $x\left(e_{1}\right)=1$ (the other cases are similar). Since $y^{i}\left(e_{1}\right)=$ $z^{j}\left(e_{1}\right)=1$, for all $i, j$, we have

$$
x\left(e_{2}\right)=r\left(\left\{e_{1}, e_{2}\right\}\right)=s\left(\left\{e_{1}, e_{2}\right\}\right) \quad \text { and } \quad x\left(e_{3}\right)=r\left(\left\{e_{1}, e_{3}\right\}\right)=s\left(\left\{e_{1}, e_{3}\right\}\right)
$$

As $x\left(e_{1}\right)=r\left(\left\{e_{1}, e_{2}\right\}\right)+r\left(\left\{e_{1}, e_{3}\right\}\right)+r\left(\left\{e_{1}\right\}\right)=s\left(\left\{e_{1}, e_{2}\right\}\right)+s\left(\left\{e_{1}, e_{3}\right\}\right)+s\left(\left\{e_{1}\right\}\right)=1$, $r\left(\left\{e_{1}\right\}\right)=s\left(\left\{e_{1}\right\}\right)$. Since $r\left(\left\{e_{2}\right\}\right)=s\left(\left\{e_{2}\right\}\right)=0$ and $r\left(\left\{e_{3}\right\}\right)=s\left(\left\{e_{3}\right\}\right)=0$, we obtain that $r(F)=s(F)$ for all $F \in B\left(\Delta_{0}\right)$. But this contradicts Claim 1.
(ii) Let us suppose that $x\left(\Delta_{0}\right)=2$.

First, if $\left|\Delta_{0} \cap S(x)\right|=2$, then $x(e)=1$ for two edges of $\Delta_{0}$. But this contradicts (i). Thus, $\left|\Delta_{0} \cap S(x)\right|=3$. Since $y^{i}\left(\Delta_{0}\right)=z^{j}\left(\Delta_{0}\right)=2$, for all $i$, $j$, we obtain that

$$
\begin{aligned}
& x\left(e_{1}\right)=r\left(\left\{e_{1}, e_{2}\right\}\right)+r\left(\left\{e_{1}, e_{3}\right\}\right)=s\left(\left\{e_{1}, e_{2}\right\}\right)+s\left(\left\{e_{1}, e_{3}\right\}\right) \\
& x\left(e_{2}\right)=r\left(\left\{e_{1}, e_{2}\right\}\right)+r\left(\left\{e_{2}, e_{3}\right\}\right)=s\left(\left\{e_{1}, e_{2}\right\}\right)+s\left(\left\{e_{2}, e_{3}\right\}\right) \\
& x\left(e_{3}\right)=r\left(\left\{e_{1}, e_{3}\right\}\right)+r\left(\left\{e_{2}, e_{3}\right\}\right)=s\left(\left\{e_{1}, e_{3}\right\}\right)+s\left(\left\{e_{2}, e_{3}\right\}\right)
\end{aligned}
$$

As $r(F)=s(F)=0$ for all $F \in B\left(\Delta_{0}\right)$ with $|F| \neq 2$, this system implies that $r(F)=$ $s(F)$ for all $F \in B\left(\Delta_{0}\right)$. This contradicts again Claim 1.
(iii) First, if $\Delta_{0} \cap S(x)=\emptyset$, then $r(\emptyset)=s(\emptyset)=1$ and $r(F)=s(F)=0$, for all $F \in B\left(\Delta_{0}\right) \backslash \emptyset$. Therefore this contradicts Claim 1.

Now, let us suppose that $\left|\Delta_{0} \cap S(x)\right|=1$. W.l.o.g., we may assume that $\Delta_{0} \cap S(x)=\left\{e_{1}\right\}$. Thus, by (i), we have $0<x\left(e_{1}\right)<1$. Hence, $r(F)=s(F)=0$ for all $F \in B\left(\Delta_{0}\right) \backslash\left\{\emptyset,\left\{e_{1}\right\}\right\}$. Moreover, $x\left(e_{1}\right)=r\left(\left\{e_{1}\right\}\right)=s\left(\left\{e_{1}\right\}\right)$. Since $r(\emptyset)+r\left(\left\{e_{1}\right\}\right)=s(\emptyset)+s\left(\left\{e_{1}\right\}\right)=1$, it follows that $r(\emptyset)=s(\emptyset)$ and hence, by Claim 1, we get a contradiction.

Thus, $\left|S(x) \cap \Delta_{0}\right|=2$. Suppose, w.l.o.g., that $S(x) \cap \Delta_{0}=\left\{e_{1}, e_{3}\right\}$ (the proof for the other cases is similar). So

$$
\begin{aligned}
& x\left(e_{1}\right)=r\left(\left\{e_{1}\right\}\right)+r\left(\left\{e_{1}, e_{3}\right\}\right)=s\left(\left\{e_{1}\right\}\right)+s\left(\left\{e_{1}, e_{3}\right\}\right), \\
& x\left(e_{3}\right)=r\left(\left\{e_{3}\right\}\right)+r\left(\left\{e_{1}, e_{3}\right\}\right)=s\left(\left\{e_{3}\right\}\right)+s\left(\left\{e_{1}, e_{3}\right\}\right) .
\end{aligned}
$$

As $r(\emptyset)+r\left(\left\{e_{1}\right\}\right)+r\left(\left\{e_{3}\right\}\right)+r\left(\left\{e_{1}, e_{3}\right\}\right)=s(\emptyset)+s\left(\left\{e_{1}\right\}\right)+s\left(\left\{e_{3}\right\}\right)+s\left(\left\{e_{1}, e_{3}\right\}\right)=1$, we obtain that $s\left(\left\{e_{1}, e_{3}\right\}\right)-r\left(\left\{e_{1}, e_{3}\right\}\right)=s(\emptyset)-r(\emptyset)$.

If $r\left(\left\{e_{1}, e_{3}\right\}\right)=s\left(\left\{e_{1}, e_{3}\right\}\right)$, then it follows that $r(F)=s(F)$ for all $F \in B\left(\Delta_{0}\right)$, a contradiction.

Hence, suppose, w.l.o.g., that $s\left(\left\{e_{1}, e_{3}\right\}\right)-r\left(\left\{e_{1}, e_{3}\right\}\right)>0$. Let us associate a variable $\gamma_{i, j}$ with every pair $\left(y^{i}, z^{j}\right)$ if either $\Delta_{0} \cap S_{1}\left(y^{i}\right)=\Delta_{0} \cap S_{1}\left(z^{j}\right)$ or $\Delta_{0} \cap S_{1}\left(y^{i}\right)=\left\{e_{1}\right\}$ and $\Delta_{0} \cap$ $S_{1}\left(z^{j}\right)=\emptyset$ or $\Delta_{0} \cap S_{1}\left(y^{i}\right)=\left\{e_{3}\right\}$ and $\Delta_{0} \cap S_{1}\left(z^{j}\right)=\left\{e_{1}, e_{3}\right\}$. Note that for each of these cases, the vector $t_{i, j}=\left(y^{i}, \tilde{z}^{j}\right)$, where $\tilde{z}^{j}$ is the restriction of $z^{j}$ on $E_{2} \backslash \Delta_{0}$, induces a triangle-free subgraph of $G$. Let $\left(\gamma_{i, j}^{*}\right)$ be a solution of the system

$$
\begin{array}{ll}
\sum_{j} \gamma_{i, j}=\alpha_{i}, & i=1, \ldots, q_{1} \\
\sum_{i} \gamma_{i, j}=\beta_{j}, & j=1, \ldots, q_{2}
\end{array}
$$

Note that the above system is a transportation problem and hence it has a solution. Also note that $\sum_{i, j} \gamma_{i, j}^{*}=1$. Since, $x=\sum \gamma_{i, j}^{*} t_{i, j}$, this is impossible.

By Claim 2(i) and (iii), we have that $0<x(e)<1$ for all $e \in \Delta_{0}$.
Claim 3. The inequalities

$$
\begin{aligned}
& \sum_{e \in \hat{E}_{k}} a_{j}^{k}(e) x(e)+x\left(e_{3}\right)+x\left(g_{k}\right) \leq b_{j}^{k}-2 \\
& \sum_{e \in \hat{E}_{k}} a_{j}^{k}(e) x(e)+x\left(e_{k}\right)+x\left(f_{k}\right) \leq b_{j}^{k}-2
\end{aligned}
$$

are valid for $P_{\Delta}\left(G_{k}\right)$, for all $j \in I_{2}^{k}, k=1,2$.
Proof: Easy.

Now, we shall consider two cases.
Case 1. $x$ does not satisfy any mixed constraint at equality.
Claim 4. $x(e)+x\left(e^{\prime}\right) \leq 1$, for all $e, e^{\prime} \in \Delta_{0}$.
Proof: Suppose, for instance, that $x\left(e_{1}\right)+x\left(e_{2}\right)>1$ (The proof is similar if either $x\left(e_{1}\right)+$ $x\left(e_{3}\right)>1$ or $\left.x\left(e_{2}\right)+x\left(e_{3}\right)>1\right)$. So, there are $i \in\left\{1, \ldots, q_{1}\right\}$ and $j \in\left\{1, \ldots, q_{2}\right\}$ such that $y^{i}\left(e_{1}\right)+y^{i}\left(e_{2}\right)>1$ and $z^{j}\left(e_{1}\right)+z^{j}\left(e_{2}\right)>1$. Hence, $y^{i}\left(e_{1}\right)=y^{i}\left(e_{2}\right)=1$ and $z^{j}\left(e_{1}\right)=z^{j}\left(e_{2}\right)=$ 1 and, as a consequence, the vector $x^{\prime}=\left(y^{i}, z^{j}\right)$ induces a triangle-free subgraph of $G$. Moreover, any constraint of $Q(G)$ that is tight for $x$ is also tight for $x^{\prime}$. Since $x \neq x^{\prime}$, this is a contradiction.

By Claim 4, we have $x(e)+x\left(e^{\prime}\right) \leq 1$, for all $e, e^{\prime} \in \Delta_{0}$. Since $x\left(e_{2}\right)<1$, there exists $i \in\left\{1, \ldots, q_{1}\right\}$ such that $y^{i}\left(e_{2}\right)=0$.

Suppose first that there is $p \in I_{2}^{1}$ such that the constraint [ $p, 1$ ] is tight for $x$ (the case where $p \in I_{2}^{2}$ is similar). Then, $y^{i}\left(e_{1}\right)=y^{i}\left(e_{3}\right)=1$. Moreover, as $[p, 1]$ is tight for $x$, by Claim 3, it follows that $x\left(f_{1}\right)+x\left(e_{1}\right)+x\left(e_{2}\right) \geq 2$ and $x\left(g_{1}\right)+x\left(e_{2}\right)+x\left(e_{3}\right) \geq 2$. As $x\left(f_{1}\right), x\left(g_{1}\right) \leq 1, x\left(e_{1}\right)+x\left(e_{2}\right) \leq 1, x\left(e_{2}\right)+x\left(e_{3}\right) \leq 1$, we obtain that

$$
\begin{align*}
& x\left(e_{1}\right)+x\left(e_{2}\right)=1  \tag{21}\\
& x\left(e_{2}\right)+x\left(e_{3}\right)=1
\end{align*}
$$

Next, let $\tilde{x}^{2} \in \Re^{m_{2}}$ be defined as $\tilde{x}^{2}(e)=x^{2}(e)$, for all $e \in E_{2}, e \neq e_{1}$, and $\tilde{x}^{2}\left(e_{1}\right)=1$. We claim that $\tilde{x}^{2} \in P_{\Delta}\left(G_{2}\right)$. Indeed, first note that, by (21), we have $\tilde{x}^{2}\left(\Delta_{0}\right)=2$. Moreover, since all mixed inequalities [ $p, l$ ] are satisfied by $x$, it follows that $\sum_{e \in \bar{E}_{2}} a_{l}^{2}(e) \tilde{x}(e) \leq b_{l}^{2}-2$, for all $l \in I_{2}^{2}$. So, $\tilde{x}^{2}$ satisfies constraints $[l, 2], l \in I_{2}^{2}$, implying that $\tilde{x}^{2} \in P_{\Delta}\left(G_{2}\right)$. Now, as $\tilde{x}^{2}\left(e_{2}\right)\left(=x\left(e_{2}\right)\right)<1$, there must exist an integer solution $z$ of $P_{\Delta}\left(G_{2}\right)$ such that $z\left(e_{2}\right)=0$. As $\tilde{x}^{2}\left(\Delta_{0}\right)=2$, it follows that $z\left(e_{1}\right)=z\left(e_{3}\right)=1$.

Let $\hat{x} \in \mathfrak{R}^{E}$ be such that

$$
\hat{x}(e)= \begin{cases}y^{i}(e) & \text { if } e \in \bar{E}_{1} \\ z(e) & \text { if } e \in \bar{E}_{2} \\ 1 & \text { if } e=e_{1}, e_{3} \\ 0 & \text { if } e=e_{2}\end{cases}
$$

It is not hard to see that $\hat{x}$ satisfies at equality all the constraints that are tight for $x$, a contradiction.

Now, suppose that $x$ does not satisfy at equality any of the constraints $[p, 1], p \in I_{2}^{1}$ and $[l, 2], l \in I_{2}^{2}$. As $x\left(e_{1}\right)<1$, there is $i \in\left\{1, \ldots, q_{1}\right\}$ such that $y^{i}\left(e_{1}\right)=0$. Let $\bar{x}^{2} \in \mathfrak{R}^{m_{2}}$ be defined as $\bar{x}^{2}(e)=x^{2}(e)$, for all $e \in E_{2} \backslash\left\{e_{1}\right\}$, and $\bar{x}^{2}\left(e_{1}\right)=0$. Clearly, $\bar{x}^{2} \in P_{\Delta}\left(G_{2}\right)$. Moreover, from Claim 2(ii) and our assumption above, the inequalities that are tight for $x^{2}$ are also tight for $\bar{x}^{2}$.

Suppose $y^{i}\left(e_{3}\right)=1$. As $\bar{x}^{2}\left(e_{3}\right)>0$, there must exist an integer solution $\bar{z} \in P_{\Delta}\left(G_{2}\right)$ that satisfies at equality all the constraints that are tight for $x^{2}$, with $\bar{z}\left(e_{3}\right)=1$. Let $x^{\prime} \in \Re^{m_{2}}$ be
such that

$$
x^{\prime}(e)= \begin{cases}y^{i}(e) & \text { if } e \in \bar{E}_{1}, \\ \bar{z}(e) & \text { if } e \in \bar{E}_{2}, \\ 1 & \text { if } e=e_{3}, \\ 0 & \text { if } e=e_{1}, \\ \bar{z}\left(e_{2}\right) & \text { if } e=e_{2}\end{cases}
$$

Since all the constraints $[l, k], l \in I_{2}^{k}, k=1,2$ as well as the mixed constraints of $P_{\Delta}(G)$ are not tight for $x$, we have that the inequalities of $P_{\Delta}(G)$ that are tight for $x$ are also tight for $x^{\prime}$. As $x^{\prime} \neq x$, this yields a contradiction.

If $y^{i}\left(e_{3}\right)=0$, we obtain a contradiction along the same way.
Case 2. There is at least one mixed constraint, say $\left[j_{0}, l_{0}\right], j_{0} \in I_{2}^{1}, l_{0} \in I_{2}^{2}$, that is tight for $x$.
Let $M=\left\{(j, l) \in I_{2}^{1} \times I_{2}^{2} \mid[j, l]\right.$ is tight for $\left.x\right\}$. Let $M_{k}=\left\{j \in I_{2}^{k} \mid\right.$ there is $l \in I_{2}^{\bar{k}}$ with $(j, l) \in M\}$ where $\bar{k}=\{1,2\} \backslash\{k\}$. In other words, $M$ represents the set of mixed constraints that are tight for $x$ and $M_{k}$, the set of constraints of $P_{\Delta}\left(G_{k}\right)$ of type (16) that give rise to mixed constraints of $P_{\Delta}(G)$ tight for $x$.

Case 2.1. $\quad x$ satisfies at equality at least one of the constraints (16), say $\left[j_{1}, 1\right], j_{1} \in I_{2}^{1}$, (The case where a constraint $\left[l_{1}, 2\right]$ is tight for some $l_{1} \in I_{2}^{2}$ can be treated similarly).
Since, by Claim 2 (ii), $x\left(\Delta_{0}\right)<2$, none of the inequalities $[l, 2], l \in I_{2}^{2}$ is tight for $x$. Moreover, as $\left[j_{1}, l\right], l \in I_{2}^{2}$ is satisfied by $x, \sum_{e \in \bar{E}_{2}} a_{l}^{2}(e) x(e) \leq b_{l}^{2}-2$, for all $l \in I_{2}^{2}$. Hence, if $[j, l]$ is tight for $x$ for $j \in I_{2}^{1}, l \in I_{2}^{2}$, then $[j, 1]$ is tight for $x$ and $\sum_{e \in \bar{E}_{2}} a_{l}^{2}(e) x(e)=$ $b_{l}^{2}-2$.

On the other hand, as $\left[j_{1}, 1\right]$ is tight for $x$, by Claim 3, it follows that $x\left(e_{2}\right)+x\left(e_{3}\right) \geq 1$. Let $\tilde{x}^{2} \in \Re^{m_{2}}$ be given by

$$
\tilde{x}^{2}(e)= \begin{cases}x^{2}(e) & \text { if } e \in E_{2} \backslash\left\{e_{1}\right\} \\ 2-\left(x\left(e_{2}\right)+x\left(e_{3}\right)\right) & \text { if } e=e_{1}\end{cases}
$$

We have $0<\tilde{x}^{2}\left(e_{1}\right) \leq 1$ and $\tilde{x}^{2}\left(\Delta_{0}\right)=2$. Thus, as $\sum_{e \in \bar{E}_{2}} a_{l}^{2}(e) \tilde{x}^{2}(e) \leq b_{l}^{2}-2$, for all $l \in I_{2}^{2}$, it follows that $\tilde{x}^{2} \in P_{\Delta}\left(G_{2}\right)$. Note that every constraint $[l, 2], l \in M_{2}$ is tight for $\tilde{x}^{2}$.

Since $x\left(e_{2}\right)<1$, there must exist an integer solution $\hat{z}$ of $P_{\Delta}\left(G_{2}\right)$ with $\hat{z}\left(e_{2}\right)=0$. Moreover, $\hat{z}$ satisfies at equality every constraint that is tight for $\tilde{x}^{2}$. Hence, $\hat{z}\left(\Delta_{0}\right)=2$ and, in consequence, $\hat{z}\left(e_{1}\right)=\hat{z}\left(e_{3}\right)=1$. Furthermore, there exists an $i \in\left\{1, \ldots, q_{1}\right\}$ such that $y^{i}\left(e_{2}\right)=0$. As $\left[j_{1}, 1\right]$ is satisfied at equality by $y^{i}$, we have that $y^{i}\left(e_{1}\right)=$ $y^{i}\left(e_{3}\right)=1$.

Consider now the solution $\hat{x} \in \mathfrak{R}^{E}$ such that

$$
\hat{x}(e)= \begin{cases}y^{i}(e) & \text { if } e \in \bar{E}_{1}, \\ \hat{z}(e) & \text { if } e \in \bar{E}_{2}, \\ 1 & \text { if } e=e_{1}, e_{3}, \\ 0 & \text { if } e=e_{2}\end{cases}
$$

We can see that $\hat{x}$ satisfies at equality all the constraints that are tight for $x$, which yields a contradiction.

Case 2.2. $x$ does not satisfy at equality any constraint of type (16).
Let

$$
\delta_{j}^{k}=b_{j}^{k}-\left(\sum_{e \in \bar{E}_{k}} a_{j}^{k}(e) x(e)+x\left(\Delta_{0}\right)\right), \quad \text { for all } j \in I_{2}^{k}, k=1,2 .
$$

Note that $\delta_{j}^{k}>0$, for all $j \in I_{2}^{k}, k=1,2$. Since $\left[j_{0}, l_{0}\right]$ is tight for $x, x\left(\Delta_{0}\right)+\delta_{j_{0}}^{1}+\delta_{l_{0}}^{2}=2$. Thus, as $x$ also satisfies $\left[j, l_{0}\right]$, we obtain that $\delta_{j_{0}}^{1} \leq b_{j}^{1}-\left(\sum_{e \in \bar{E}_{1}} a_{j}^{1}(e) x(e)+x\left(\Delta_{0}\right)\right)$, for all $j \in I_{2}^{1}$. Hence, the following hold

$$
\begin{array}{ll}
\delta_{j_{0}}^{1} \leq \delta_{j}^{1}, & \text { for all } j \in I_{2}^{1} \\
\delta_{j_{0}}^{1}=\delta_{j}^{1}, & \text { for all } j \in M_{1}
\end{array}
$$

Similarly, we get

$$
\begin{array}{ll}
\delta_{l_{0}}^{2} \leq \delta_{l}^{2}, & \text { for all } l \in I_{2}^{2} \\
\delta_{l_{0}}^{2}=\delta_{l}^{2}, & \text { for all } l \in M_{2}
\end{array}
$$

Moreover, by Claim 3, $x\left(e_{2}\right)+\delta_{j_{0}}^{1}+x\left(e_{3}\right)+x\left(g_{1}\right) \geq 2$. Hence, $x\left(e_{2}\right)+\delta_{j_{0}}^{1}+x\left(e_{3}\right) \geq 1$. Therefore, as $x\left(\Delta_{0}\right)+\delta_{j_{0}}^{1}+\delta_{l_{0}}^{2}=2, x\left(e_{1}\right)+\delta_{l_{0}}^{2} \leq 1$. Similarly, we have $x\left(e_{2}\right)+\delta_{j_{0}}^{1} \leq 1$.

Let $\tilde{x}^{1} \in \mathfrak{R}^{m_{1}}$ and $\tilde{x}^{2} \in \mathfrak{R}^{m_{2}}$ be given by

$$
\tilde{x}^{1}(e)=\left\{\begin{array}{lll}
x^{1}(e) & \text { if } e \in E \backslash\left\{e_{2}\right\}, \\
x\left(e_{2}\right)+\delta_{j_{0}}^{1} & \text { if } e=e_{2}
\end{array} \quad \tilde{x}^{2}(e)= \begin{cases}x^{2}(e) & \text { if } e \in E \backslash\left\{e_{1}\right\} \\
x\left(e_{1}\right)+\delta_{l_{0}}^{2} & \text { if } e=e_{1}\end{cases}\right.
$$

It is easy to see that $\tilde{x}^{1} \in P_{\Delta}\left(G_{1}\right)$ and $\tilde{x}^{2} \in P_{\Delta}\left(G_{2}\right)$.
As $\tilde{x}^{1}\left(\Delta_{0}\right)<2$, there exists an integer solution $y$ of $P_{\Delta}\left(G_{1}\right)$ that satisfies at equality all the constraints that are tight for $\tilde{x}^{1}$ and such that $y\left(\Delta_{0}\right)<2$. Since the constraint $\left[j_{0}, 1\right]$ is tight for $\tilde{x}_{1}$ and thus for $y$, it follows that $y\left(e_{2}\right)=1$ and $y\left(e_{1}\right)=y\left(e_{3}\right)=0$. Similarly, there is an integer solution $z$ of $P_{\Delta}\left(G_{2}\right)$ that satisfies at equality all the constraints that are tight for $\tilde{x}^{2}$ and such that $z\left(e_{1}\right)=1, z\left(e_{2}\right)=z\left(e_{3}\right)=0$.

Let $x^{\prime} \in \mathfrak{R}^{E}$ be the solution such that

$$
x^{\prime}(e)= \begin{cases}y(e) & \text { if } e \in \bar{E}_{1} \\ z(e) & \text { if } e \in \bar{E}_{2} \\ 0 & \text { if } e \in \Delta_{0}\end{cases}
$$

Clearly, $x^{\prime} \in P_{\Delta}(G)$. Moreover, all the constraints of $Q(G)$ that are tight for $x$ are also tight for $x^{\prime}$. As $x \neq x^{\prime}$, this is a contradiction, which finishes the proof of our theorem.

Theorem 3.3 provides a complete description of $P_{\Delta}(G)$ provided that such descriptions are known for $P_{\Delta}\left(G_{1}\right)$ and $P_{\Delta}\left(G_{2}\right)$. The following theorem shows that if the systems defining $P_{\Delta}\left(G_{1}\right)$ and $P_{\Delta}\left(G_{2}\right)$ are minimal then the system defining $P_{\Delta}(G)$ is as well.

Theorem 3.4. If inequalities (12) and (13) are facet-defining inequalities for $P_{\Delta}\left(G_{1}\right)$ and $P_{\Delta}\left(G_{2}\right)$, respectively, then the mixed constraint (14) is facet-defining for $P_{\Delta}(G)$.

Proof: It suffices to show that there exists a point $z$ in $P_{\Delta}(G)$ that satisfies inequality (14) at equality and all other constraints of $P_{\Delta}(G)$ with strict inequality.
Suppose that inequality (12) (resp. (13)) is facet-defining for $P_{\Delta}\left(G_{1}\right)$ (resp. $P_{\Delta}\left(G_{2}\right)$ ). Then, there are $m_{1}$ (resp. $m_{2}$ ) triangle-free edge sets $R_{1}, \ldots R_{m_{1}}$ (resp. $S_{1}, \ldots, S_{m_{2}}$ ) of $G_{1}$ (resp. $G_{2}$ ) such that $x^{R_{1}}, \ldots, x^{R_{m_{1}}}$ (resp. $x^{S_{1}}, \ldots, x^{S_{m_{2}}}$ ) satisfy (12) (resp. (13)) at equality and are affinely independent. Note that $R_{i} \cap \Delta_{0} \neq \emptyset$ for $i=1, \ldots, m_{1}$ and that $R_{i} \cap \Delta_{0}$ is one of the sets $\left\{e_{2}, e_{3}\right\},\left\{e_{1}, e_{3}\right\},\left\{e_{1}, e_{2}\right\}$ and $\left\{e_{2}\right\}$. Similarly, we have that $S_{j} \cap \Delta_{0} \neq$ $\emptyset$ for $j=1, \ldots, m_{2}$ and that $S_{j} \cap \Delta_{0}$ is one of the sets $\left\{e_{2}, e_{3}\right\},\left\{e_{1}, e_{3}\right\},\left\{e_{1}, e_{2}\right\}$ and $\left\{e_{1}\right\}$. Let $I_{1}, I_{2}, I_{3}, I_{4}$ be the sets of $i \in\left\{1, \ldots, m_{1}\right\}$ such that $R_{i} \cap \Delta_{0}=\left\{e_{2}, e_{3}\right\}, R_{i} \cap \Delta_{0}=$ $\left\{e_{1}, e_{3}\right\}, R_{i} \cap \Delta_{0}=\left\{e_{1}, e_{2}\right\}$ and $R_{i} \cap \Delta_{0}=\left\{e_{2}\right\}$, respectively. Let $J_{1}, J_{2}, J_{3}, J_{4}$ be similarly defined for $S_{1}, \ldots, S_{m_{2}}$. Note that $I_{t} \neq \emptyset \neq J_{t}$, for $t=1, \ldots, 4$. Thus, for every set $R_{i}, i \in$ $I_{t}, t=1,2,3$ (resp. $i \in I_{4}$ ) there is a set $S_{j_{i}}, j_{i} \in J_{t}$ (resp. $j_{i} \in J_{4}$ ) such that $T_{i}=R_{i} \cup S_{j_{i}}$
 equality. Similarly, we can construct $m_{2}$ triangle-free edge sets $T_{j}^{\prime}, j=1, \ldots, m_{2}$ with respect to $S_{1}, \ldots, S_{m_{2}}$.

Now, it is easy to see that for every constraint of $P_{\Delta}(G)$ different from (14), there exists at least one edge set among $T_{1}, \ldots, T_{m_{1}}, T_{1}^{\prime}, \ldots, T_{m_{2}}^{\prime}$ that satisfies this constraint with strict inequality.

Now, by considering $z=\frac{1}{m_{1}+m_{2}}\left(\sum_{i} x^{T_{i}}+\sum_{j} x^{T_{j}^{\prime}}\right)$, we have the required vector.
As a consequence of Theorem 3.4, we have the following.
Corollary 3.5. $Q(G)$ is a minimal system defining $P_{\Delta}(G)$.

## 4. Total dual integrality

A system $A x \leq b$ is called total dual integral (TDI) if the dual of $\max \{w x, A x \leq b\}$ has an integer optimal solution for every integer vector $w$ such that the maximum exists. In this
section, we are going to show that the system defining $P_{\Delta}(G)$ is total dual integral if the systems defining $P_{\Delta}\left(G_{1}\right)$ and $P_{\Delta}\left(G_{2}\right)$ so are.

In Euler and Mahjoub (1991), showed the following
Lemma 4.1 Euler and Mahjoub (1991). Let $G=(V, E)$ be the 2-sum of two graphs $G_{1}=\left(V_{1}, E_{1}\right)$ and $G_{2}=\left(V_{2}, E_{2}\right)$. Then the following hold

1. A system that defines $P_{\Delta}(G)$ can be obtained as the union of the systems defining $P_{\Delta}\left(G_{1}\right)$ and $P_{\Delta}\left(G_{2}\right)$ and by identifying the variables associated with the edge of $E_{1} \cap E_{2}$.
2. If the systems defining $P_{\Delta}\left(G_{1}\right)$ and $P_{\Delta}\left(G_{2}\right)$ are TDI then the system defining $P_{\Delta}(G)$, as described in (1), is TDI.

Theorem 4.2. If the systems defining $P_{\Delta}\left(G_{1}\right)$ and $P_{\Delta}\left(G_{2}\right)$ are TDI, then the system (15)(20) defining $P_{\Delta}(G)$ is also TDI.

Proof: Let us denote by $P_{G}(w)$ the linear program $\max \{w x$ subject to (15)-(20)\} where $w$ is an integer weight function. Let $D_{G}(w)$ be the dual of $P_{G}(w)$ and $\lambda_{G}(w)$ its optimal value. If $F \subset E$, we shall denote by $G_{F}=(V, F)$ the subgraph induced by $F$ and by $w^{F}$ the restriction of $w$ on the set $F$. For the sake of commodity, if $F=E_{k}$, then we let $w^{E_{k}}=w^{k}, k=1,2$.

Claim 1. Let $e_{0}$ be an edge of $E$ that does not belong to any triangle of $G$. Let $\tilde{G}=$ $\left(V, E \backslash\left\{e_{0}\right\}\right)$ and $\tilde{A} x \leq \tilde{b}$ be a system defining $P_{\Delta}(\tilde{G})$. If $\tilde{A} x \leq \tilde{b}$ is TDI, then the system

$$
\left\{\begin{array}{r}
\tilde{A} x \leq \tilde{b} \\
x\left(e_{0}\right) \leq 1 \\
x\left(e_{0}\right) \geq 0
\end{array}\right.
$$

defines $P_{\Delta}(G)$ and is TDI.
Proof: Easy.
Since, by Theorem 3.3, $Q(G)$ is integral, the problem $P_{G}(w)$ always has an integral optimal solution and, in consequence, $\lambda_{G}(w)$ is integer. For $F \subset E$, let $\kappa(F)$ denote the set of triangles of $F$.

Let us associate with a constraint of type (15) ((16)), the dual variable $y_{i}^{k}\left(y_{j}^{k}\right)$, with a mixed constraint (17), the dual variable $z_{j, l}$, with a constraint (18) corresponding to a triangle $\Delta \in \kappa\left(\bar{E}_{k}\right)\left(\Delta=\Delta_{0}\right)$, the dual variable $\gamma_{\Delta}^{k}\left(\gamma_{\Delta_{0}}\right)$ and with a constraint $x(e) \leq 1$ corresponding to an edge $e \in \bar{E}_{k}\left(e \in\left\{e_{1}, e_{2}, e_{3}\right\}\right)$, the variables $\delta_{e}^{k}\left(\delta_{e}\right)$, for $k=1,2$.

The dual $D_{G}(w)$ can then be written as

$$
\begin{gathered}
\min \sum_{k=1}^{2}\left(\sum_{i \in I_{1}^{k}} b_{i}^{k} y_{i}^{k}+\sum_{j \in I_{2}^{k}} b_{j}^{k} y_{j}^{k}+\sum_{\Delta \in \kappa\left(\bar{E}_{k}\right)} 2 \gamma_{\Delta}^{k}+\sum_{e \in \bar{E}_{k}} \delta_{e}^{k}\right) \\
+\sum_{i=1}^{3} \delta_{e_{i}}+2 \gamma_{\Delta_{0}}+\sum_{j \in I_{2}^{1}, l \in I_{2}^{2}}\left(b_{j}^{1}+b_{l}^{2}-2\right) z_{j, l}
\end{gathered}
$$

s.t.

To show the theorem, we shall use ideas similar to those of Barahona et al. (1994) and Mahjoub (1995) i.e., we shall proceed by induction on $w$.

If $w^{A} \leq 0$, then clearly $\lambda_{G}(w)=\lambda_{G_{1}}\left(w^{1}\right)+\lambda_{G_{2}}\left(w^{2}\right)$. Thus an optimal solution of $D_{G}(w)$ can be obtained as the union of the optimal solutions of $D_{G_{1}}\left(w^{1}\right)$ and $D_{G_{2}}\left(w^{2}\right)$ and by setting $z_{j, l}=0$ for all $j \in I_{2}^{1}, l \in I_{2}^{2}, \gamma_{\Delta_{0}}=0$ and $\delta_{e}=0$ for all $e \in \Delta_{0}$. As $P_{\Delta}\left(G_{1}\right)$ and $P_{\Delta}\left(G_{2}\right)$ are TDI, the optimal solutions of $D_{G_{1}}\left(w^{1}\right)$ and $D_{G_{2}}\left(w^{2}\right)$ can be considered integer and thus $D_{G}(w)$ has an integer optimal solution.

Now, assume that $D_{G}(w)$ has an integer optimal solution for every integer vector $w$, $w \leq t, w \neq t$, and let us show that $D_{G}(w)$ has an integer optimal solution for $w=t$.

Let us denote by $\hat{G}_{k}=\left(\hat{V}_{k}, \hat{E}_{k}\right)$ the graph obtained from $G_{k}$ by removing the set of edges $A_{k}=\left\{f_{k}, g_{k}, e_{1}, e_{2}, e_{3}\right\}$ for $k=1,2$.

Let $Q\left(\widehat{G}_{k}\right)$ be the system given by the inequalities of $P_{\Delta}\left(G_{k}\right)$ that do not contain variables $x(e)$ where $e \in A_{k}$.

Claim 2. $P_{\Delta}\left(\hat{G}_{k}\right)=Q\left(\hat{G}_{k}\right)$.

Proof: Clearly, every triangle-free edge set of $\hat{G}_{k}$ corresponds to an integer solution of $Q\left(\hat{G}_{k}\right)$, thus $P_{\Delta}\left(\hat{G}_{k}\right) \subset Q\left(\hat{G}_{k}\right)$.
If $P_{\Delta}\left(\hat{G}_{k}\right) \neq Q\left(\hat{G}_{k}\right)$ then there exists a facet-defining inequality of $P_{\Delta}\left(\hat{G}_{k}\right)$, say $a x \leq \alpha$, different from the ones describing $Q\left(\hat{G}_{k}\right)$. Thus, there are $l=m_{k}-5$ triangle-free edge sets $T_{1}, \ldots, T_{l}$ of $\hat{G}_{k}$ whose incidence vectors satisfy $a x \leq \alpha$ at equality and are affinely
independent. Consider the edges sets

$$
\begin{aligned}
T_{l+1} & =T_{1} \cup\left\{f_{k}\right\}, \\
T_{l+2} & =T_{1} \cup\left\{g_{k}\right\}, \\
T_{l+i} & =T_{1} \cup\left\{e_{i}\right\}, \quad i=1, \ldots, 3 .
\end{aligned}
$$

These sets together with $T_{1}, \ldots, T_{l}$ constitute $m_{k}$ triangle-free edge sets whose incidence vectors satisfy $a x \leq \alpha$ at equality and are affinely independent. As the inequality $a x \leq \alpha$ is valid for $P_{\Delta}\left(G_{k}\right)$, this implies that the inequality $a x \leq \alpha$ is facet-defining for $P_{\Delta}\left(G_{k}\right)$, a contradiction.

Claim 3. $Q\left(\hat{G}_{k}\right)$ is TDI.
Proof: Let $\hat{w}^{k}$ be an integer vector associated with the edges of $\hat{E}_{k}$. Let $w^{k} \in \mathfrak{R}^{E_{k}}$ be such that $w^{k}(e)=\hat{w}^{k}(e)$ if $e \in \hat{E}_{k}$ and $w^{k}(e)=0$ if not.

Since $P_{\Delta}\left(G_{k}\right)$ is TDI, there is an integer optimal solution $(y, s)$ of $D_{G_{k}}\left(w^{k}\right)$ where $y$ (resp. $s)$ is the dual vector associated with the inequalities of $P_{\Delta}\left(G_{k}\right)$ that do not contain (resp. contain) variables $x(e)$ with $e \in A_{k}$. We claim that $s=0$. Indeed, as, by Claim 2, $Q\left(\hat{G}_{k}\right)$ is integral, there is an integer optimal solution $\hat{x}$ to the program $P_{\hat{G}_{k}}\left(\hat{w}^{k}\right)=\max \left\{\hat{w}^{k} x \mid x \in\right.$ $\left.Q\left(\hat{G}_{k}\right)\right\}$. As $w^{k}(e)=0$ for all $e \in A_{k}$, the solution $x \in \mathfrak{R}^{E_{k}}$ such that $x(e)=\hat{x}(e)$ if $e \in \hat{E}^{k}$, $x(e)=0$ otherwise, is optimal for $P_{G_{k}}\left(w^{k}\right)$. Moreover, we have that any constraint of $P_{\Delta}\left(G_{k}\right)$ that involves variables $x(e)$ with $e \in A_{k}$ and different from $x(e) \geq 0$ is not tight for $x$. This is clear for the trivial and the triangle inequalities. Now, for the constraints of type (11), let $x^{*} \in \mathfrak{R}^{E_{k}}$ such that $x^{*}(e)=x(e)$ for $e \in E_{k} \backslash f_{k}$ and $x^{*}(e)=1$ for $e=f_{k}$. Clearly, $x^{*}$ induces a triangle-free edge set of $G_{k}$. Consequently, $x^{*}$ satisfies inequality (11), implying that these inequalities are not tight for $x$. As a consequence, by the complementary slackness theorem, it follows that $s=0$. This also implies that $y$ is an integer optimal solution for the dual program of $P_{\hat{G}_{k}}\left(\hat{w}^{k}\right)$.

Now suppose that $w\left(e_{0}\right) \leq 0$ for some edge $e_{0} \in A$. Let $\tilde{G}$ be the graph obtained from $G$ by removing $e_{0}$. Note that $\tilde{G}$ is a 1 -sum (2-sum) of two graphs, say $\tilde{G}_{1}$ and $\tilde{G}_{2}$, with, possibly, one or two edges that do not belong to any triangle in $\tilde{G}$. The graph $\tilde{G}_{k}$, for $k=1,2$, may be either the graph $G_{k}$ itself or a graph obtained as a 2 -sum of $\hat{G}_{k}$ and a triangle of $G_{k}$. For a triangle $T$, the system given by the trivial inequalities and inequality $x(T) \leq 2$ completely describes $P_{\Delta}(T)$ and is TDI (Mahjoub, 1995). So, from Claims 1, 3 and Lemma 4.1, it follows that $P_{\Delta}(\tilde{G})$ is TDI.

In what follows, we shall assume that $w(e)>0$, for all $e \in A$. Let $I_{w}$ be the set of inequalities of $P_{G}(w)$ that are satisfied at equality by every optimal solution of $P_{G}(w)$.

We shall distinguish three cases:
Case 1. $x\left(e_{0}\right) \leq 1$ is in $I_{w}$ for some edge $e_{0}$ of $A$. Let $w^{\prime}$ be the vector given by

$$
w^{\prime}(e)= \begin{cases}w(e) & \text { if } e \in E \backslash\left\{e_{0}\right\} \\ w\left(e_{0}\right)-1 & \text { if } e=e_{0}\end{cases}
$$

We claim that $\lambda_{G}\left(w^{\prime}\right)=\lambda_{G}(w)-1$. Indeed, it is clear that $\lambda_{G}(w)-1 \leq \lambda_{G}\left(w^{\prime}\right) \leq \lambda_{G}(w)$. If $\lambda_{G}\left(w^{\prime}\right)=\lambda_{G}(w)$, then every optimal solution of $P_{G}\left(w^{\prime}\right)$ is at the same time optimal for $P_{G}(w)$. But this contradicts the fact that $x\left(e_{0}\right) \leq 1$ is in $I_{w}$ and our claim is proved. Now, by the induction hypothesis, $D_{G}\left(w^{\prime}\right)$ has an integer optimal solution. Increasing by one the value of the dual variable associated with $x\left(e_{0}\right) \leq 1$ in that solution gives an integer optimal solution of $D_{G}(w)$.

Case 2. $x(\Delta) \leq 2$ is in $I_{w}$ for some triangle $\Delta \subset A \cup\left\{f_{1}^{\prime}, g_{1}^{\prime}, f_{2}^{\prime}, g_{2}^{\prime}\right\}$. Let $w^{\prime}$ be the vector given by

$$
w^{\prime}(e)= \begin{cases}w(e) & \text { if } e \in E \backslash \Delta, \\ w(e)-1 & \text { if } e \in \Delta\end{cases}
$$

Claim 4. $\quad \lambda_{G}\left(w^{\prime}\right)=\lambda_{G}(w)-2$
Proof: As for every triangle-free edge set $T$ of $E, w^{\prime} x^{T}=w x^{T}-|T \cap \Delta|$, we have that

$$
\begin{equation*}
\lambda_{G}(w)-2 \leq \lambda_{G}\left(w^{\prime}\right) \leq \lambda_{G}(w) . \tag{22}
\end{equation*}
$$

Let $T^{\prime} \subset E$ be a maximum triangle-free edge set with respect to $w^{\prime}$. Hence, $w^{\prime} x^{T^{\prime}}=$ $\lambda_{G}\left(w^{\prime}\right)$.

If $\left|T^{\prime} \cap \Delta\right|=2$, then $w^{\prime} x^{T^{\prime}} \leq \lambda_{G}(w)-2$ and thus $\lambda_{G}\left(w^{\prime}\right)=\lambda_{G}(w)-2$.
If $\left|T^{\prime} \cap \Delta\right|=1$, then, as $x(\Delta) \leq 2$ is in $I_{w}, T^{\prime}$ cannot be of maximum weight with respect to $w$. So, $w x^{T^{\prime}} \leq \lambda_{G}(w)-1$, implying that $w^{\prime} x^{T^{\prime}} \leq \lambda_{G}(w)-2$. By (22), we then have $\lambda_{G}\left(w^{\prime}\right)=\lambda_{G}(w)-2$.

Now, suppose that $T^{\prime} \cap \Delta=\emptyset$. If $\Delta \neq \Delta_{0}$, then there exists $f \in \Delta \cap\left\{f_{1}, g_{1}, f_{2}, g_{2}\right\}$ such that the edge set $T=T^{\prime}+\{f\}$ is triangle-free. As $x(\Delta) \leq 2$ is in $I_{w}, T$ is not an optimal solution of $P_{G}(w)$. Thus, $w x^{T^{\prime}}+w(f) \leq \lambda_{G}(w)-1$. As $w(f)>0$, by (22), we have that $\lambda_{G}\left(w^{\prime}\right)=\lambda_{G}(w)-2$.

So suppose that $\Delta=\Delta_{0}$. We may assume that there is a maximum triangle-free edge set $T_{1} \subset E$, with respect to $w$, such that $f_{1} \notin T_{1}$ (otherwise we would be in case 1). As $w\left(f_{1}\right)>0$, we have that $\left\{f_{1}^{\prime}, e_{1}\right\} \subset T_{1}$ and, as $T_{2}=\left(T_{1} \backslash\left\{e_{1}\right\}\right)+\left\{f_{1}\right\}$ is triangle-free and $\left|T_{2} \cap \Delta_{0}\right| \neq 2, T_{2}$ cannot be an optimal triangle-free edge set. Hence, $w x^{T_{2}} \leq \lambda_{G}(w)-1$ and $w\left(f_{1}\right) \leq w\left(e_{1}\right)-1$. Similarly, we can show that $w\left(f_{2}\right) \leq w\left(e_{2}\right)-1$.

Moreover, as $T^{\prime} \cap \Delta=\emptyset$ and $w(e)>0$, for all $e \in \Delta_{0}$, we have that $\left\{f_{1}, f_{1}^{\prime}, f_{2}, f_{2}^{\prime}\right\} \subset$ $T^{\prime}$. The edge set $T=\left(T^{\prime} \backslash\left\{f_{1}, f_{2}\right\}\right)+\left\{e_{1}, e_{2}\right\}$ is thus triangle-free and hence, $w^{\prime} x^{T^{\prime}}-w\left(f_{1}\right)-$ $w\left(f_{2}\right)+w\left(e_{1}\right)+w\left(e_{2}\right) \leq \lambda_{G}(w)$. By the above inequalities, this yields $w^{\prime} x^{T^{\prime}} \leq \lambda_{G}(w)-2$. In consequence, $\lambda_{G}\left(w^{\prime}\right)=\lambda_{G}(w)-2$, which finishes the proof of the claim.

Now, consider the solution obtained from an integer solution of $D_{G}\left(w^{\prime}\right)$ by increasing by one the value of the dual variable associated with $x(\Delta) \leq 2$. We have that this solution is integer and optimal for $D_{G}(w)$.

For the rest of the proof, we may suppose that neither a constraint of type $x(e) \leq 1$ nor one of type $x(\Delta) \leq 2$ belongs to $I_{w}$. Otherwise we are either in Case 1 or Case 2.

Remark 4.1. We have that $y_{j}^{k}=0$ for all $j \in I_{2}^{k}$ for some $k \in\{1,2\}$. In fact, if $y_{j}^{1}>0$ for some $j \in I_{2}^{1}$, and $y_{l}^{2}>0$ for some $l \in I_{2}^{2}$, then every optimal solution $x$ for $P_{G}(w)$ satisfies inequalities $[j, 1]$ and $[l, 2]$ at equality. As $x$ must satisfy the corresponding mixed constraint $[j, l]$, one should have $x\left(\Delta_{0}\right)=2$, a contradiction.

## Claim 5.

(i) All the constraints of $D_{G}(w)$ corresponding to the variables $x(e), e \in A$, are tight for every dual optimal solution.
(ii) $I_{w}$ contains an inequality of type (17).
(iii) If two mixed constraints $\left[j_{1}, l_{1}\right]$ and $\left[j_{2}, l_{2}\right]$ are in $I_{w}$, for some $\left(j_{1}, l_{1}\right),\left(j_{2}, l_{2}\right)$, then [ $j_{1}, l_{2}$ ] and $\left[j_{2}, l_{1}\right]$ are also in $I_{w}$.
(iv) If an inequality $\left[j_{3}, 1\right], j_{3} \in I_{2}^{1}$ belongs to $I_{w}$, then the inequalities $\left[j_{3}, l\right]$ and $[j, 1]$ belong to $I_{w}$ for all $(j, l)$ such that $[j, l]$ belongs to $I_{w}$.

## Proof:

(i) As $x\left(f_{1}\right) \leq 1$ is not in $I_{w}$ there exists an integer optimal solution $x_{1}$ for $P_{G}(w)$ such that $x_{1}\left(f_{1}\right)=0$. As $w\left(f_{1}\right)>0$, this implies that $x_{1}\left(e_{1}\right)=x_{1}\left(f_{1}^{\prime}\right)=1$. So, the dual constraint corresponding to the variable $x\left(e_{1}\right)$ is tight for every dual optimal solution.

Since $x\left(\Delta_{1}^{1}\right) \leq 2$ does not belong to $I_{w}$, there is an integer optimal solution $x_{2}$ of $P_{G}(w)$, such that $x_{2}\left(\Delta_{1}^{1}\right)<2$. As $w(e)>0$ for all $e \in A$, it follows that $x_{2}\left(f_{1}\right)=$ $x_{2}\left(e_{2}\right)=x_{2}\left(e_{3}\right)=1$. Therefore, the dual constraints corresponding to the variables $x\left(f_{1}\right), x\left(e_{2}\right)$ and $x\left(e_{3}\right)$ are all tight for every optimal solution of $D_{G}(w)$. Now by considering the triangles $\Delta_{2}^{1}, \Delta_{1}^{2}, \Delta_{2}^{2}$, we show similarly that the dual constraints associated with $x\left(g_{1}\right), x\left(f_{2}\right), x\left(g_{2}\right)$ are also tight for every dual optimal solution of $D_{G}(w)$.
(ii) By our hypothesis together with the complementary slackness theorem, it follows that $\delta_{e}=0$ for all $e \in A$ and $\gamma_{\Delta}=0$ for all triangles $\Delta \in A \cup\left\{f_{1}^{\prime}, f_{2}^{\prime}, g_{1}^{\prime}, g_{2}^{\prime}\right\}$. This together with (i) imply that the constraint of $D_{G}(w)$ corresponding to the edge $f_{1}$ (resp. $f_{2}$ ), can be written as

$$
\begin{align*}
\sum_{j \in I_{2}^{1}} y_{j}^{1}+\sum_{j \in I_{2}^{1}, l \in I_{2}^{2}} z_{j, l}=w\left(f_{1}\right)  \tag{23}\\
\left(\text { resp. } \sum_{j \in I_{2}^{2}} y_{j}^{2}+\sum_{j \in I_{2}^{1}, l \in I_{2}^{2}} z_{j, l}=w\left(f_{2}\right)\right) \tag{24}
\end{align*}
$$

So, by Remark 4.1, we may suppose that, for any optimal dual solution, $y_{j}^{2}=0$ for all $j \in I_{2}^{2}$. By (24), we then have $\sum_{j \in I_{2}^{1}, l \in I_{2}^{2}} z_{j, l}=w\left(f_{2}\right)>0$. This implies that $z_{j, l}>0$ for some $j \in I_{2}^{1}, l \in I_{2}^{2}$ and consequently the corresponding mixed constraint of type (17) is in $I_{w}$.
(iii) As the sum of inequalities $\left[j_{1}, l_{1}\right]$ and $\left[j_{2}, l_{2}\right]$ is the same as the sum of $\left[j_{1}, l_{2}\right]$ and [ $\left.j_{2}, l_{1}\right]$, then, if the former inequalities are tight, the latter ones are so.
(iv) Suppose there is a mixed constraint $[j, l]$ in $I_{w}$. Suppose that the constraint $\left[j_{3}, l\right]$ is satisfied with strict inequality for some optimal solution $x$ for $P_{G}(w)$. Since $x$ satisfies at equality $\left[j_{3}, 1\right]$, it follows that $\sum_{e \in \bar{E}_{2}} a_{l}^{2}(e) x_{3}(e)<b_{l}^{2}-2$. But this implies that $x$ does not satisfy $[j, l]$ at equality, a contradiction. Thus $\left[j_{3}, l\right]$ is in $I_{w}$. As $\left[j_{3}, 1\right],\left[j_{3}, l\right] \in I_{w}$, every optimal solution $x$ for $P_{G}(w)$ verifies $\sum_{e \in \bar{E}_{2}} a_{l}^{2}(e) x(e)=b_{l}^{2}-2$. Since $[j, l]$ is tight for $x$, it follows that $[j, 1]$ is also tight for $x$ and as a consequence is in $I_{w}$.

Now, let $\left(y^{1}, y^{2}, \gamma, \delta, z\right)$ be an optimal solution of $D_{G}(w)$. By Remark 4.1, we may assume that $y_{j}^{2}=0$ for all $j \in I_{2}^{2}$. Also, as none of the inequalities $x(e) \leq 1$ for $e \in A$ and $x(\Delta) \leq 2$ for $\Delta \subset A \cup\left\{f_{1}^{\prime}, g_{1}^{\prime}, f_{2}^{\prime}, g_{2}^{\prime}\right\}$ belongs to $I_{w}$, it follows that $\delta_{e}^{k}=0$ for $e \in\left\{f_{k}, g_{k}\right\}$, $\gamma_{\Delta}^{k}=0$ for $\Delta \in\left\{\Delta_{1}^{k}, \Delta_{2}^{k}\right\}, k \in\{1,2\}, \delta_{e}=0$ for $e \in \Delta_{0}$ and $\gamma_{\Delta_{0}}=0$. So by Claim 5, we have

$$
\sum_{j \in I_{2}^{1}} y_{j}^{1}+\sum_{j \in I_{2}^{1}, l \in I_{2}^{2}} z_{j, l}=w(e), \quad \text { for all } e \in A \backslash\left\{f_{2}, g_{2}\right\},
$$

and

$$
\sum_{j \in I_{2}^{1}, l \in I_{2}^{2}} z_{j, l}=w(e), \text { for } e \in\left\{f_{2}, g_{2}\right\} .
$$

Therefore, $w(e)=w\left(e^{\prime}\right)$ for all $e, e^{\prime} \in A \backslash\left\{f_{2}, g_{2}\right\}$ and $w\left(f_{2}\right)=w\left(g_{2}\right)$. Let $\alpha=w(e)$, for $e \in\left\{f_{2}, g_{2}\right\}$ and $\bar{w}^{2} \in \Re^{E_{2}}$ be the weight vector on $E_{2}$ given by

$$
\bar{w}^{2}(e)= \begin{cases}w(e) & \text { for all } e \in \bar{E}_{2} \\ \alpha & \text { for all } e \in \Delta_{0}\end{cases}
$$

Claim 6. $\quad \lambda_{G}(w)=\lambda_{G_{1}}\left(w^{1}\right)+\lambda_{G_{2}}\left(\bar{w}^{2}\right)-2 \alpha$
Proof: First note that, as $x\left(\Delta_{1}^{1}\right) \leq 2$ does not belong to $I_{w}$, there exists an integer optimal solution for $P_{G}(w)$, say $x_{1}$, such that $x_{1}\left(\Delta_{1}^{1}\right)<2$. Thus, $x\left(f_{1}\right)=1$ and $x\left(f_{1}^{\prime}\right)=x\left(e_{1}\right)=0$. This implies that $x\left(e_{2}\right)=x\left(e_{3}\right)=1$ and hence $x_{1}\left(\Delta_{0}\right)=2$. Therefore, $\lambda_{G}(w)=$ $\sum_{e \in E_{1}} w^{1}(e) x_{1}(e)+\sum_{e \in E_{2}} \bar{w}^{2}(e) x_{1}(e)-2 \alpha$. So we have

$$
\begin{equation*}
\lambda_{G}(w) \leq \lambda_{G_{1}}\left(w^{1}\right)+\lambda_{G_{2}}\left(\bar{w}^{2}\right)-2 \alpha . \tag{25}
\end{equation*}
$$

On the other hand, let $s^{1}$ and $s^{2}$ be the solutions respectively defined as

$$
\left\{\begin{array} { l l } 
{ s _ { i } ^ { 1 } = y _ { i } ^ { 1 } } & { \text { for } i \in I _ { 1 } ^ { 1 } , } \\
{ s _ { j } ^ { 1 } = y _ { j } ^ { 1 } + \sum _ { l \in I _ { 2 } ^ { 2 } } z _ { j , l } } & { \text { for } j \in I _ { 2 } ^ { 1 } , } \\
{ s _ { e } ^ { 1 } = \delta _ { e } ^ { 1 } } & { \text { for } e \in E _ { 1 } , } \\
{ s _ { \Delta } ^ { 1 } = \gamma _ { \Delta } ^ { 1 } } & { \text { for } \Delta \in \kappa ( E _ { 1 } ) , }
\end{array} \quad \text { and } \quad \left\{\begin{array}{ll}
s_{i}^{2}=y_{i}^{2} & \text { for } i \in I_{1}^{2}, \\
s_{l}^{2}=\sum_{j \in I_{2}^{1}}^{z_{j, l}} & \text { for } l \in I_{2}^{2}, \\
s_{e}^{2}=\delta_{e}^{2} & \text { for } e \in E_{2}, \\
s_{\Delta}^{2}=\gamma_{\Delta}^{2} & \text { for } \Delta \in \kappa\left(E_{2}\right) .
\end{array}\right.\right.
$$

It is easy to verify that $s^{1}\left(\right.$ resp. $\left.s^{2}\right)$ is feasible for $D_{G_{1}}\left(w^{1}\right)\left(\right.$ resp. $\left.D_{G_{2}}\left(\bar{w}^{2}\right)\right)$, which yields

$$
\lambda_{G_{1}}\left(w^{1}\right)+\lambda_{G_{2}}\left(\bar{w}^{2}\right) \leq \lambda_{G}(w)+2 \alpha .
$$

Hence, the claim follows by (25).
Now, let $\bar{t}^{1}\left(\right.$ resp. $\left.\bar{s}^{2}\right)$ be an integer optimal dual solution of $D_{G}\left(w^{1}\right)\left(\right.$ resp. $D_{G}\left(\bar{w}^{2}\right)$ ).
Consider the system

$$
\begin{cases}\phi_{j}+\sum_{l \in M_{2}} \psi_{j, l}=\bar{s}_{j}^{1} & \text { for } j \in M_{1}, \\ \sum_{j \in M_{1}} \psi_{j, l}=\bar{s}_{l}^{2} & \text { for } l \in M_{2},\end{cases}
$$

where $M_{k}=\left\{j \in I_{2}^{k} \mid\right.$ there is $l \in I_{2}^{\bar{k}}$ with $\left.[j, l] \in I_{w}\right\}$ where $\bar{k} \in\{1,2\} \backslash\{k\}$. Since the corresponding matrix is a network flow matrix, the above system has an integer solution. Let $\left.\bar{y}=\left(\bar{y}^{1}, \bar{y}^{2}, \bar{\gamma}, \bar{\delta}, \bar{z}\right)\right)$ be such that

$$
\left\{\begin{array}{llll}
\bar{y}_{i}^{k}=\bar{s}_{i}^{k} & \text { for } i \in I_{1}^{k}, k=1,2, & & \\
\bar{\delta}_{e}^{k}=\bar{s}_{e}^{k} & \text { for } e \in \hat{E}^{k}, & \bar{\delta}_{e}^{k}=0 & \text { for } e \in\left\{f_{k}, g_{k}\right\}, k=1,2, \\
\bar{\delta}_{e}=0 & \text { for } e \in \Delta_{0}, & & \\
\bar{\gamma}_{\Delta}^{k}=\bar{s}_{\Delta}^{k} & \text { for } \Delta \in \kappa\left(\hat{E}^{k}\right), & \bar{\gamma}_{\Delta}^{k}=0 & \text { for } \Delta \in\left\{\Delta_{1}^{k}, \Delta_{2}^{k}\right\}, k=1,2, \\
\bar{\gamma}_{\Delta_{0}}=0, & & \\
\bar{y}_{j}^{1}=\phi_{j} & \text { for } j \in M_{1}, & \bar{y}_{j}^{1}=0 & \text { otherwise, } \\
\bar{z}_{j, l}=\psi_{j, l} & \text { for } j \in M_{1}, l \in M_{2}, & \bar{z}_{j, l}=0 & \text { otherwise, } \\
\bar{y}_{l}^{2}=0 & \text { for } l \in I_{2}^{2} . & &
\end{array}\right.
$$

The vector $\bar{y}$ is an integer feasible solution of $D_{G}(w)$ and its value is equal to $\lambda_{G_{1}}\left(w^{1}\right)+$ $\lambda_{G_{2}}\left(\bar{w}^{2}\right)-2 \alpha$. By Claim 6, $\bar{y}$ is an optimal solution of $D_{G}(w)$ and the proof of our theorem is complete.

## 5. Final remarks

We have studied a composition (decomposition) technique for the triangle-free subgraph polytope. We have shown that if $G$ decomposes into $G_{1}$ and $G_{2}$ by means of a 3 -sum satisfying a certain property, a system defining the triangle-free subgraph polytope for $G$ can be derived from two systems defining the polytopes for $G_{1}$ and $G_{2}$. Using this, we have described a procedure that permits to construct new facets for the triangle-free subgraph polytope from known ones. We have also shown that if the systems defining the triangle-free subgraph polytope for $G_{1}$ and $G_{2}$ are TDI, then the system defining the polytope for $G$ is also TDI.

Let $\Omega$ be the class of graphs obtained by means of $1-2$ and 3 -sums (as described above) from the graphs $K_{1}, K_{2}, K_{3}$, the prism, the even wheels and the odd wheels $W_{2 k+1}$ with


Figure 2. 3-sum of $W_{5}$ and $W_{7}$.
$k \geq 2$. A system describing the polytope $P_{\Delta}(G)$ for a graph $G \in \Omega$ can be easily obtained using the procedure described above. Moreover, as the systems characterizing the polytope $P_{\Delta}(G)$ in these graphs are all TDI (Mahjoub, 1995), by Theorem 4.2, it follows that the system defining $P_{\Delta}(G)$ is also TDI. This generalizes the results in Euler and Mahjoub (1991) and Mahjoub (1995).

To illustrate this, consider the graph $H=(V, E)$ shown in figure 2(c) which is the 3-sum of the two wheels $W_{5}$ and $W_{7}$, shown in figure 2(a) and (b) respectively. From Euler and Mahjoub (1991), together with Theorem 3.3, the following system completely describes $P_{\Delta}(H)$

$$
\begin{array}{rlrl}
\sum_{e \in W_{5}} x(e) & \leq 7, & \\
\sum_{e \in W_{7}} x(e) & \leq 10, & \\
\sum_{e \in E} x(e) \leq 15, & & \\
\sum_{e \in \Delta} x(e) & \leq 2 & & \text { for all triangle } \Delta \subset E, \\
x(e) & \leq 1 & & \text { for all } e \in E, \\
x(e) & \geq 0 & & \text { for all } e \in E .
\end{array}
$$

Moreover, this system is TDI.
We may consider a slightly different 3 -sum operation where the statement (iv) (of the 3 -sum given in Section 3.1) is modified in such a way that the triangle $\Delta_{1}^{2}$ contains the two edges $f_{2}, f_{2}^{\prime}$ and the edge $e_{1}$ instead of $e_{2}$. In this case, the composition, unfortunately, does not permit one to generate mixed constraints from the linear descriptions of the polytopes associated with the pieces. Further composition techniques are needed in this case.
Finally, let us note that an interesting question would be to study a generalization of the composition introduced in this paper within the framework of independence systems.

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