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Capacitated Network Design using Bin-Packing

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Abstract

In this paper, we consider the *Capacitated Network Design* (CND) problem. We investigate the relationship between CND and the *Bin-Packing* problem. This is exploited for identifying new classes of valid inequalities for the CND problem and developing a branch-and-cut algorithm to solve it efficiently.

Keywords: Capacitated Network Design, Bin-Packing, Facets, Branch-and-Cut.

1 Introduction

Given an optical network having a set of optical devices interconnected by optical fibers, and a set of traffic demands, this problem consists in finding

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the number of modular capacities (modules) that have to be installed over each fiber so that the traffic is routed and the cost is minimum. In practice, several modules can be installed on one fiber. Each one can carry many traffic demands but one demand has to be routed on a unique path.

More formally, the problem can be presented as follows. Consider a bidirected graph $G = (V, A)$ that represents an optical network. Each node $v \in V$ corresponds to an optical device and each arc $a = ij \in A$ corresponds to an optical link. Let K be a set of commodities or traffic demands. Each commodity $k \in K$ has an origin node $o_k \in V$, a destination node $d_k \in V$ and a traffic D^k that has to be routed between o_k and d_k. We suppose that we can install a set of modules on each link. The set of available modules is denoted by W and a module $w \in W$ installed between nodes i and j is a copy of the arc $a = ij \in A$. Every module w can carry one or many commodities, but a commodity can not be splitted on many modules. We denote by C and c_{ij}, the capacity and the cost of each module, respectively. The CND problem consists in determining the number of modules to install on each arc of G so that the commodities can be routed and the total cost is minimum.

In this work, we mainly focus on the restriction of CND to one edge. This approach is similar to one presented by Bienstock and Muratore in [2], for survivable network design problems restricted to a cut. In [1], authors have exploited the relationship of Network Design problem with several problems studied earlier, like the binary knapsack problem.

We are interested in the relationship between CND and Bin-Packing problem [3]. Our primary motivation comes from the structure of CND whose restriction on one edge reduces to study a variant of the Bin-Packing problem. In particular, our contribution concerns this relationship and how it can be exploited to identify valid inequalities for CND problem and develop a branch-and-cut algorithm to solve it efficiently. The paper is organized as follows. In section 2, we give a compact formulation for the CND problem. We introduce an *aggregated* model for the problem in section 3, and highlight the relationship between CND and the Bin-Packing problem. We study the basic properties of the polyhedron associated with the latter problem in section 4. Finally, we show some experiments in section 5.

2 Compact formulation for CND

In this section, we give a compact flow-based formulation for the CND problem. Let y^w_{ij} be a variable that takes value 1 if the module $w \in W$ is installed on the arc $ij \in A$, and 0 otherwise. And let x^{kw}_{ij} be a variable that takes value

1 if the commodity $k \in K$ is routed on the module $w \in W$ of the arc $ij \in A$, and 0 otherwise. The CND problem is then equivalent to the following ILP:

$$\min \sum_{ij \in A} \sum_{w \in W} c_{ij} y_{ij}^w$$

$$\sum_{w \in W} \sum_{j \in V} x_{ji}^{kw} - \sum_{w \in W} \sum_{j \in V} x_{ij}^{kw} = \begin{cases} 1, & if\ i = d_k, \\ -1, & if\ i = o_k, \\ 0, & otherwise, \end{cases} \quad \forall k \in K, \forall i \in V, \tag{1}$$

$$\sum_{k \in K} D^k x_{ij}^{kw} \leq C y_{ij}^w, \quad \forall w \in W, \forall ij \in A, \tag{2}$$

$$x_{ij}^{kw} \in \{0,1\}, y_{ij}^w \in \{0,1\}, \quad \forall k \in K, \forall w \in W, \forall ij \in A. \tag{3}$$

Equalities (1) are the flow conservation constraints, they require that a unique path between o_k and d_k is associated with each commodity k. Inequalities (2) are the capacity constraints for each installed module.

3 Aggregated formulation and Bin-Packing

Suppose now that G consists of nodes i, j connected by a single edge ij. Then the CND problem here, is to determine the number of modules to install over ij, in such a way that each commodity using ij is assigned to at most one module and the total cost is minimum. Consider the polyhedron:

$$P_{ij} := conv\{(x, y) \in \{0,1\}^{|K| \times |W|} \times \{0,1\}^{|W|} :$$

$$\sum_{k \in K} D^k x_{ij}^{kw} \leq C y_{ij}^w \ \forall w \in W, \sum_{w \in W} x_{ij}^{kw} \leq 1 \ \forall k \in K\}$$

P_{ij} is the convex hull of CND problem restricted to ij. If K and W are a set of objects and a set of bins, respectively, then P_{ij} corresponds to the Bin-Packing polytope. Note that the polyhedron P_{ij} has many symmetric solutions. To overcome this difficulty, we will introduce a new *aggregated* model that does not specify which copy of the arc ij is used for the routing of a commodity k. Indeed, the idea is just to determine the number of modules that have to be installed, so that each commodity can be assigned to one of these modules. We will define the additional integer design variable y_{ij} as the number of modules installed on ij. We also define the variables x_{ij}^k that takes the value 1, if k uses some copy of the arc ij for its routing and 0 otherwise. The CND problem can then be formulated using the following ILP:

$$\min \sum_{ij \in A} c_{ij} y_{ij}$$

$$\sum_{j \in V} x^k_{ji} - \sum_{j \in V} x^k_{ij} = \begin{cases} 1, & if\ i = d_k, \\ -1, & if\ i = o_k, \\ 0, & otherwise, \end{cases} \qquad \forall k \in K, \ \forall i \in V, \tag{4}$$

$$\sum_{k \in K} D^k x^k_{ij} \le C y_{ij}, \qquad \forall ij \in A, \tag{5}$$

$$(x^k_{ij}, y_{ij}) \in Q_{ij}, \qquad \forall ij \in A, \forall k \in K. \tag{6}$$

where

$$Q_{ij} := conv\{(x,y) \in \{0,1\}^{|K|} \times \mathbb{Z}^+ : x^k_{ij} = \sum_{w \in W} x^{kw}_{ij}, y_{ij} \ge \sum_{w \in W} y^w_{ij},$$
$$\sum_{k \in K} D^k x^{kw}_{ij} \le C y^w_{ij} \ \forall w \in W, x^{kw}_{ij} \in \{0,1\}, y^w_{ij} \in \{0,1\}, \forall k \in K, \forall w \in W\}$$

As in formulation (1)-(3), equalities (4) are the flow conservation constraints for each commodity of K. Constraints (5) and (6) express the capacity constraints over the polyhedron Q_{ij}.

Q_{ij} is the projection on (x^k_{ij}, y_{ij}) of the polyhedron P_{ij}. Observe that the symmetric solutions of P_{ij} will project on a single point. We denote by $BP(S)$ the solution of the Bin-Packing problem for a subset S of K. In other words, $BP(S)$ is the minimum number of bins needed to carry the objects of S. We also introduce $S(x)$ that denotes the subset of objects corresponding to incidence vector x. Then we provide an alternative definition of Q_{ij}:

$$Q_{ij} := conv\{(x,y) \in \{0,1\}^{|K|} \times \mathbb{Z}^+ : y_{ij} \ge BP(S(x))\}$$

Q_{ij} would then be more suitable to investigate. This polyhedron is associated with a problem that will be referred to as *Bin-Packing Function* (BPF). Since polyhedra Q_{ij} are identical for every $ij \in A$, in the remaining of this article, we ommit the indices ij. We then refer to Q_{ij} as Q, x^k_{ij} as x^k and y_{ij} as y.

4 Polyhedral analysis and valid inequalities

The purpose of this section is to discuss the polyhedron Q. We will describe its dimension, identify some valid inequalities and give necessary and sufficient conditions for these inequalities to be facet defining.

Proposition 4.1 *Q is full dimensionnal.*

Proof. We will exhibit $|K|+2$ solutions whose incidence vectors are affinely independent. Let us introduce the $|K|$ solutions S_k, $k \in K$, defined such that one module is used to satisfy the commodity k, while the other commodities are not satisfied. Consider the incidence vector associated with each S_k, given by $(0, \ldots, x^k = 1, 0, \ldots, y = 1)$. We denote by S_{k_1,k_2}, the solution defined as follows. Suppose that three modules are installed and two commodities, say k_1 and k_2, are satisfied. The incidence vector of S_{k_1,k_2} is given by $(0, \ldots, x^{k_1} = 1, x^{k_2} = 1, 0, \ldots, y = 3)$. Consider now the solution S_0 where no commodity is satisfied and no module is installed. The associated incidence vector is then given by $(0, .., 0)$. It is clear that S_0, S_{k_1,k_2}, and S_k, $k \in K$, are feasible solutions and their incidence vectors are affinely independent. □

Theorem 4.2 *$x^k \geq 0$ and $x^k \leq 1$ define facets for Q.*

We will not give the proofs for these inequalities as they are very similar to proof of Proposition 4.1. We will however focus in the following sections on introducing new classes of facet defining inequalities.

4.1 Valid inequalities

Proposition 4.3 *For each subset $S \subseteq K$ and a non negative integer $p \in \mathbb{Z}^+$, inequality*

$$\sum_{k \in S} x^k \leq y + p, \tag{7}$$

is valid for Q if and only if $BP(S) \geq |S| - p$.

Proof. ($\Leftarrow$) Suppose that the solution of the BPF problem for a subset $S \subseteq K$ verifies $BP(S) \geq |S| - p$, for some $p \in \mathbb{Z}^+$. Then by definition of the polyhedron Q, we have $y \geq BP(S) \geq |S| - p$. Hence, we have $|S| \leq y + p$. In consequence $\sum_{k \in S} x^k \leq |S| \leq y + p$. Thus inequality (7) is valid for Q,
($\Rightarrow$) Suppose now that $BP(S) < |S| - p$, then the solution $(1, 1, 1, \ldots, x^{|S|} = 1, 0, \ldots, y = BP(S))$ is cut off by (7). □

Theorem 4.4 *For each $S \subseteq K$ and a non negative integer p, inequalities (7) define facets for Q if and only if the following conditions hold*

(i) $BP(S) = |S| - p$,

(ii) $BP(S \cup \{\widetilde{s}\}) = |S| - p$, *where $\widetilde{s}$ is the largest element of $K \setminus S$,*

(iii) $BP(S \setminus \{\overline{s}\}) < |S| - p$, *where $\overline{s}$ is the smallest element of S.*

Proof. ($\Rightarrow$) We show that (i), (ii) and (iii) are necessary conditions for (7) to define facets.

(i) Consider a subset S and a non negative integer p such that the inequality (7) induced by S and p defines a facet. Then, there exists at least one solution (x^*, y^*) such that $\sum_{k\in S} x^{*k} = y^* + p$. We have by definition of polyhedron Q that $y^* \geq BP(S)$. Thus, $BP(S) \leq \sum_{k\in S} x^{*k} - p$, and then

$$BP(S) \leq |S| - p \tag{8}$$

We also have by the validity condition of (7) that

$$BP(S) \geq |S| - p \tag{9}$$

Hence, by (8) and (9), we conclude that $BP(S) = |S| - p$,

(ii) Suppose that there exists an element $\widetilde{s}$ of $K \setminus S$ such that $BP(S \cup \{\widetilde{s}\}) \leq |S| - p$. Then the inequality (7) is dominated by another constraint

$$\sum_{k\in S\cup\{\widetilde{s}\}} x^k \leq y + p$$

In consequence, (7) can not be a facet of Q.

(iii) If $BP(S\backslash\{\overline{s}\}) \geq |S| - p$, we can see that (7) is dominated by

$$\sum_{k\in S\backslash\{\overline{s}\}} x^k \leq y + p$$

and $x^k \leq 1, \ for \ k = \overline{s}$, Thus (7) can not define facets for Q.

($\Leftarrow$) Let $\overline{S}$ be a subset of K and $\overline{p}$ a given non negative integer. Suppose that inequality (7) induced by $\overline{S}$ and $\overline{p}$, is valid for Q. We will exhibit $|K|+1$ solutions denoted by S_k, $k \in \{1, ..., |K|+1\}$ of BPF that satisfy this constraint with equality. M_1 denotes a $(|K|+1)\times(|K|+1)$ matrix containing the incidence vectors of solutions S_k, $k \in \{1, ..., |K|+1\}$.

$$M_1 = \begin{array}{c} \\ S_1 \\ S_2 \\ \vdots \\ S_{|K\setminus\overline{S}|+1} \\ S_{|K\setminus\overline{S}|+2} \\ \vdots \\ S_{|K|+1} \end{array} \begin{array}{c} \begin{array}{cccccccc} x^1 & x^2 & x^3 & \dots & x^{|\overline{S}|} & x^{|\overline{S}|+1} & \dots & x^{|K|} & y \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 0 & \dots & 0 & |\overline{S}|-\overline{p} \\ 1 & 1 & 1 & \dots & 1 & 1 & \dots & 0 & |\overline{S}|-\overline{p} \\ & & & & & & & & \\ 1 & 1 & 1 & \dots & 1 & 0 & \dots & 1 & |\overline{S}|-\overline{p} \\ 1 & 1 & 1 & \dots & 0 & 0 & \dots & 0 & |\overline{S}|-\overline{p}-1 \\ & & & & & & & & \\ 0 & 1 & 1 & \dots & 1 & 0 & \dots & 0 & |\overline{S}|-\overline{p}-1 \end{pmatrix} \end{array}$$

We can easily check that the incidence vectors of S_k, $k \in \{1, ..., |K\setminus\overline{S}|+1\}$ verify conditions (i) and (ii). Indeed, the value of y in incidence vector of S_1 ensures $BP(\overline{S}) = |\overline{S}| - \overline{p}$, and adding to $\overline{S}$ the greatest commodity of $K \setminus \overline{S}$ does not change the value of $BP(\overline{S})$. On an other hand, incidence vectors of $S_{|K\setminus\overline{S}|+2}$ to $S_{|K|+1}$ verify the condition (iii), as the removal of any commodity of $\overline{S}$ yields the decreasing of $BP(\overline{S})$ value. Moreover, the incidence vectors of S_k, $k \in \{1, 2, ..., |K|+1\}$ are affinely independent. □

Proposition 4.5 *For $S \subseteq K$ and a parameter $q \in \mathbb{Z}^+_*$, inequality*

$$\sum_{k\in S} x^k \le qy, \tag{10}$$

is valid for Q if and only if $BP(S') \ge \lceil \frac{|S'|}{q} \rceil$, $\forall S' \subseteq S$.

Theorem 4.6 *For $S \subseteq K$ and a given parameter $q \in \mathbb{Z}^+_*$, $q \ge 2$, inequalities (10) are facets defining for Q if and only if $BP(S') \ge \lceil \frac{|S'|}{q} \rceil$, $\forall S' \subseteq S$.*

5 Computational results

Based on the theoretical results described above, we devised a branch-and-cut algorithm that has been implemented in C++ using CPLEX 12.0 with the default settings. We have proposed a heuristic separation procedure for inequalities (7), that uses a greedy algorithm to find S, strengthened by an exact evaluation of $BP(S)$. We have tested our approach on several instances derived from SNDlib topologies (*http://sndlib.zib.de*), with a restricted subset of commodities. The obtained results are presented in Table 1 with the following entries. The first three columns contain the size of each instance.

In column four, we can find the value of the solution given by the aggregated formulation Z_{AF}, by only considering constraints (4) and (5). Those solutions ignore the Bin-Packing structure of the problem and usually are not feasible. Z_{BC} is an upper bound found by the branch-and-cut over the aggregated formulation including the cuts from the Bin-Packing Polyhedron, optimum if the instance is solved to optimality. Z_{CF} is an upper bound found by the compact formulation, optimum if the instance is solved to optimality. The remaining columns are the number of constraints (7) separated, and the CPU time for branch-and-cut and the compact formulation, denoted T_{BC} and T_{CF}, respectively (given in days:hours:min:sec). In Z_{CF} column, values were written in italics if the optimal solution could not be found within 1 hour. Notice that for most of the instances, adding only few cuts helped to find the optimal solution in a short time. These results are very promising and show the efficiency of facets (7). We are now implementing the separation of (10) and some other facets that were not presented in this paper.

Table 1
Results for instances with $|W|$=4

$\|V\|$	$\|A\|$	$\|K\|$	Z_{AF}	Z_{BC}	Z_{CF}	#CutsI	T_{BC}	T_{CF}
12	36	20	24000.00	25000.00	25000.00	3	0:00:02.27	0:00:32.31
15	44	10	17256.00	17720.00	17720.00	3	0:00:00.97	0:00:12.00
15	44	20	32806.00	32806.00	32806.00	1	0:00:11.41	0:10:22.31
17	52	15	4692.10	5105.90	5105.90	11	0:00:21.99	0:01:30.05
17	52	20	6165.00	6416.40	*6603.80*	15	0:06:47.49	1:00:00.00
22	72	15	35332.00	35942.00	35942.00	4	0:01:26.16	0:49:06.23
22	72	20	38172.00	38172.00	*38481.00*	0	0:04:06.69	1:00:00.00
28	82	15	10528.00	10528.00	10528.00	0	0:00:24.70	0:08:27.72
51	160	20	5700.00	5800.00	*5800.00*	1	0:02:37.35	1:00:00.00

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