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# A branch-and-cut algorithm for the Multiple Steiner TSP with Order constraints

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**Abstract**

The paper deals with a problem motivated by survivability issues in multilayer IP-over-WDM telecommunication networks. Given a set of traffic demands for which we know a survivable routing in the IP layer, our purpose is to look for the corresponding survivable topology in the WDM layer. The problem amounts to Multiple Steiner TSPs with order constraints. We propose an integer linear programming formulation for the problem and investigate the associated polytope. We also present new valid inequalities and discuss their facial aspect. Based on this, we devise a Branch-and-cut algorithm and present preliminary computational results.

*Keywords:* IP-over-WDM networks, Steiner TSP, order constraint, Branch-and-cut algorithm.

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# 1 Introduction

Multilayer Network Design and survivability problems have recently seen a particular attention [1,2,3]. The problem that we are studying in this paper deals with survivability in multilayer telecommunication networks. Consider an IP-over-WDM network consisting of a logical IP layer over an optical WDM layer. The IP layer is composed of IP routers interconnected by logical links and the WDM layer consists of optical switches interconnected by optical links. To each router in the IP layer corresponds an optical switch in the WDM layer. The links between the IP routers are logical and are ensured by paths in the optical layer. We suppose given a set $K$ of demands such that for each demand we know two node-disjoint paths routing it in the IP layer. Besides, with each optical link in the WDM layer is associated a positive cost for its installation.

The *Multilayer Survivable Optical Network Design* problem (MSOND problem) is to find, for each demand, two node-disjoint optical paths routing it in the WDM layer. These paths must go in the same order through the optical switches corresponding to the routers visited in the logical paths of the IP layer, and such that the total cost is minimum. The optical switches that must be visited for a demand $k \in K$ are called *terminals* and the other nodes are called *Steiner nodes* for this demand. Consider a demand $k \in K$, looking for two paths respecting some order through the terminals of demand $k$ amounts to looking for a cycle visiting these terminals in a predefined order. This is in a close relationship with the Steiner Cycle problem or the Steiner TSP [4]. However, an additional constraint related to the order between the terminals is here considered. The MSOND problem is hence nothing but a succession of Steiner TSPs with a specific order on the terminals for each demand $k \in K$.

The paper is organized as follows. In section 2, we introduce further notations and propose a linear integer programming formulation for the problem. In section 3, we investigate the associated polytope. In section 4, we give new families of valid inequalities and discuss the facial aspects for some constraints. Finally, section 5 will be devoted to present preliminary computational results.

# 2 Notations and formulation

As previously mentioned, in the MSOND problem, optimization concerns only the WDM layer. Let us associate with this layer an undirected graph $G = (V, E)$ where $V$ corresponds to the optical switches and $E$ to the optical links between these switches. For each demand $k \in K$, let $T_k$ represent the set of its terminals and $S_k$ the set of its Steiner nodes. As the terminals of each demand must be visited in a predefined order, demand $k \in K$ can also be represented by a sequence of terminals $(w_1^k, w_2^k, ..., w_{l_k}^k)$. Two successive terminals, $w_j^k$ and $w_{j+1}^k$, where $j = 1, ..., l_k$ and $w_{l_k+1}^k = w_1^k$, define a *section* $q_j^k$ of demand $k$. For $W \subset V$, we denote by $\delta_G(W)$ (or if the context is clear $\delta(W)$) the set of edges in $G$ having exactly one node in $W$. $\delta(W)$ is called a *cut*. For $W \subset V$ such that $w_j^k \in W$ and $w_{j+1}^k \in V \setminus W$, $\delta_{G^{k,j}}(W)$ will denote the cut separating the terminals $w_j^k$ and $w_{j+1}^k$ in the graph $G^{k,j}$. Here, $G^{k,j}$ is the graph obtained from graph $G$ by deleting all the terminals of demand $k$ excepted $w_j^k$ and $w_{j+1}^k$.

Let $y_e$, $e \in E$ be a variable such that $y_e$ is equal to 1 if edge $e$ is taken and 0 otherwise. Given a demand $k \in K$ and an edge $e \in E$, $x_e^k$ will define a variable which takes 1 if demand $k$ is routed using edge $e$ and 0 otherwise.

The MSOND problem is equivalent to the following ILP.

$$min \sum_{e \in E} c(e) y_e$$

$$\sum_{e \in \delta_{G^{k,j}}(W)} x_e^k \geq 1 \qquad for\, all\, k \in K,\ q_j^k = (w_j^k, w_{j+1}^k),\ W \subset V,\ w_j^k \in W\, and\, w_{j+1}^k \in V \setminus W \tag{1}$$

$$\sum_{e \in \delta(w)} x_e^k \leq 2 \qquad for\, all\, w \in V,\ k \in K \tag{2}$$

$$x_e^k \leq y_e \qquad for\, all\, e \in E,\ k \in K \tag{3}$$

$$0 \leq x_e^k,\ y_e \leq 1 \qquad for\, all\, e \in E,\ k \in K \tag{4}$$

$$x_e^k \in \{0,1\},\ y_e \in \{0,1\} \qquad for\, all\, e \in E,\ k \in K \tag{5}$$

Inequalities (1) ensure for each section $q_j^k$, $j = 1, ..., l_k$ of a demand $k \in K$ a path in the reduced graph $G^{k,j}$. This guarantees for each demand two paths passing in order through its terminals. Inequalities (2) ensure the node-disjunction between these paths. Inequalities (3) are the linking constraints. Inequalities (4) and (5) are the trivial and integrity constraints, respectively.

## 3 Associated polytope

Denote by MSOND(G,K,T) the convex hull of the incidence vectors of the solutions of (1)-(5) associated with graph $G$, the demands' set $K$ and the set $T = \bigcup_{k \in K} T_k$ of terminals of the different demands. In the sequel, we suppose that $G$ is complete and that each demand $k \in K$ has at lest 2 Steiner nodes ($|S_k| \geq 2$). We have the following remarks.

**Remark 3.1** Given a demand $k \in K$ and a terminal node $w_j^k \in T_k$, the following equation is valid for MSOND(G,K,T).

$$\sum_{e \in \delta(w_j^k)} x_e^k = 2 \tag{6}$$

**Remark 3.2** Consider a demand $k \in K$ and two non-successive terminals $w_i^k, w_j^k \in T_k$. Let $e' = w_i^k w_j^k$. We then have

$$x_{e'}^k = 0 \tag{7}$$

Now, we can state the dimension of MSOND(G,K,T).

**Theorem 3.3** *$dim(MSOND(G,K,T)) = (|K|+1)|E| - \sum_{k \in K} \frac{|T_k|(|T_k|-1)}{2}$*

**Proof (Sketch)** Consider an equation $ax + by = \beta$ of MSOND(G,K,T), we prove that $b = 0$, and $ax = \beta$ is a linear combination of equations (6) and (7), which implies that (6) and (7) are the only equations of MSOND(G,K,T). Denote by $M$ the matrix of equations of MSOND(G,K,T). $M$ looks as follows $M = \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_K \end{pmatrix}$ where $M_k$ is the matrix of equations (6) and (7) for demand $k \in K$. Since for $k \in K$, there are $|T_k|$ equations of (6) and $\frac{|T_k|(|T_k|-1)}{2} - |T_k|$ equations of (7), it follows that $rank(M_k) = |T_k| + (\frac{|T_k|(|T_k|-1)}{2} - |T_k|) = \frac{|T_k|(|T_k|-1)}{2}$. By construction of $M$, we deduce that $rank(M) = \sum_k \frac{|T_k|(|T_k|-1)}{2}$. As $dim(\text{MSOND(G,K,T)}) = N - rank(M)$, the result follows. Here $N = (|K|+1)|E|$ represents the total number of variables. □

## 4 Valid inequalities and facial aspect

In this section, we describe some classes of valid inequalities for MSOND(G,K,T). These are given in the following theorems.

**Theorem 4.1** *Consider a demand $k \in K$ and let $W \subset V$ such that $W \cap T_k \neq \emptyset \neq (V \setminus W) \cap T_k$. Then the following inequality is valid for MSOND(G,K,T).*

$$\sum_{e \in \delta(W)} x_e^k \geq 2 \tag{8}$$

Inequalities (8) are a straight consequence related to the connectivity requirements of the problem and will be called the *Steiner 2-connectivity inequalities.*

**Theorem 4.2** *Consider a demand $k \in K$. Let $w_j \in T_k$ be a terminal node and $S \subseteq S_k$. Denote $E_j = [S, \{w_j\}]$ and $F_j = [S, \{w_{j+2}, ..., w_{j-2}\}]$. Then*

$$\sum_{e \in \delta(S) \setminus \{E_j, F_j\}} x_e^k \geq \sum_{e \in E_j} x_e^k \tag{9}$$

*is valid for MSOND(G,K,T).*

Inequalities (9) are called the *Steiner non-successive terminals inequalities.* These can be seen as flow inequalities and are saying the following. The flow going from $w_j$ to a subset of Steiner nodes $S \subseteq S_k$ (corresponding to the flow circulating through edges $E_j$) must be used to route only sections that are adjacent to $w_j$ (corresponding to $\delta(S) \setminus \{E_j, F_j\}$).

**Theorem 4.3** *Consider a demand $k \in K$ and let $V_0, ..., V_p$ be a partition of $V$ such that $|V_i \cap T^k| \geq 1, i = 1, ..., p$ and $V_0 \cap T^k = \emptyset$. Let $F \subseteq \delta(V_0)$ such that $|F|$ is odd. Then*

$$x^k(\delta(V_0, ..., V_p) \setminus F) \geq p - \lfloor \frac{|F|}{2} \rfloor \tag{10}$$

*is valid for MSOND(G,K,T).*

These inequalities are called the *Steiner F-partition inequalities.*

**Proof.** Clearly, the following inequalities are valid for MSOND(G,K,T),

$$\begin{array}{ll} x^k(\delta(V_i)) \geq 2 & for\, all\, i = 1, ..., p \\ -x^k(f) \geq -1 & for\, all\, f \in F \\ x^k(g) \geq 0 & for\, all\, g \in \delta(V_0) \setminus F \end{array}$$

The result follows by summing these inequalities, dividing by 2 and rounding up the right-hand side. □

**Theorem 4.4** *Consider a demand $k \in K$ and let $V_1, ..., V_p$ be a partition of $V$ such that $|V_i \cap T^k| \geq 1, i = 1, ..., p$. Suppose that $r \leq p$ subsets in the partition contain respectively $q_i$, $i = 1, ..., r$ non-successive terminals (or sequences of terminals). Let $S \subseteq S_k$ be a subset of Steiner nodes of demand $k$. Then*

$$x^k(\delta_{G \setminus S}(V_1, V_2, ..., V_p)) \geq (p + \sum_{i=1}^{r} q_i - r) - |S| \tag{11}$$

*is valid for MSOND(G,K,T).*

These inequalities are called the *Steiner partition inequalities.*

**Proof (Sketch)** The idea of the proof is to replace each subset $V_i$, $i = 1, ..., r$ which contains $q_i$ non-successive terminals (or sequences of terminals) by $q_i$ subsets each one containing either a unique terminal or a sequence of successive terminals. The proof is by induction on $r$. □

We have investigated the facial aspect for all the above valid inequalities. Because of the space limit of the paper, we will give the results only for inequalities (2) and (8).

**Theorem 4.5** *Inequalities* (2) *define facets for MSOND(G,K,T) if and only if* $w$ *is not a terminal for demand* $k \in K$.

**Proof.** Let $F = \{(x, y) \in \text{MSOND(G,K,T)} : x^k(\delta(w)) = 2\}$ be the facet induced by inequalities (2). If $w \in T_k$ then $F = \text{MSOND(G,K,T)}$. Consequently, $F$ is not a proper face and hence it is not facet defining. Assume now that $w = s \in S_k$. We will exhibit $dim(\text{MSOND(G,K,T)})$ affinely independent solutions of $F$. To this end, we first suppose that $s$ behaves like a terminal node for demand $k$. Assume that $s$ behaves like a terminal between the terminals $w_1^k$ and $w_2^k$. Consider the new polytope MSOND(G,K,T') where $T' = (T \setminus T_k) \cup T_k^{'}$ and $T_k^{'} = T_k \cup \{s\}$. By theorem 3.3, there are $(|K| + 1)|E| - \sum_{h \in K} \frac{|T_h|(|T_h|-1)}{2} - |T_k| + 1$ affinely independent solutions in MSOND(G,K,T'). These solutions are in $F$ as well. Now we complete these solutions by $|T_k| - 1$ additional ones obtained as follows. For all the demands $h \neq k \in K$, we route the demand $h$ by considering the edges $\{w_j^h, w_{j+1}^h\}$, where $j = 1, ..., l_h$ and $w_{l_h+1}^k = w_1^h$, between the terminals of $T_h$. For demand $k$, a first solution is obtained by routing on the edges between the successive terminals for all the sections of the demand excepted section $(w_2^k, w_3^k)$. For this section, the routing is ensured by inserting the Steiner node $s$ between $(w_2^k, w_3^k)$, which gives a feasible solution for MSOND(G,K,T) that is in $F$. The same procedure is applied to sections $(w_3^k, w_4^k)$,...,$(w_{l_k}^k, w_1^k)$. And this leads to exactly $|T_k| - 1$ new solutions belonging to $F$. By construction, all the previous solutions are affinely independent in $F$ and the result follows. □

**Theorem 4.6** *Inequalities* (8) *define facets for MSOND(G,K,T) if and only if the two following conditions hold:*

(i) *either* $|W \cap T_k| = 1$ *or* $W \cap T_k$ *is a sequence of successive terminals,*

(ii) $W \cap S_k \neq \emptyset$ *and* $(V \setminus W) \cap S_k \neq \emptyset$.

**Proof (Sketch)** Let $F = \{(x, y) \in \text{MSOND(G,K,T)} : x^k(\delta(W)) = 2\}$ be the face induced by inequality (8). If $W$ contains non-successive terminals or non-successive sequences of terminals, then $x^k(\delta(W)) \geq 4$. This implies that $F = \emptyset$

and hence $F$ does not define a facet. Now, assume that $W$ contains either only one terminal or a sequence of successive terminals. In this case, if $W \cap S_k = \emptyset$ then $F$ = MSOND(G,K,T) and hence $F$ is not a facet defining. Assume now that conditions (1) and (2) are satisfied. Consider a valid inequality $ax + by \leq \beta$ for MSOND(G,K,T) and let $F'$ be the corresponding induced face $F' = \{(x, y) \in$ MSOND(G,K,T) $: ax + by = \beta\}$. Suppose that $F \subseteq F'$. The result follows by proving that $ax + by = \beta$ is a linear combination of equations $x^k(\delta(W)) = 2$, (6) and (7). □

## 5 Computational results

Using the previous results, we devise a Branch-and-Cut algorithm. This is tested on three SNDlib-based instances (*polska*, *newyork* and *pioro*40) with a number of demands ranging from 8 to 30. The maximum CPU time is fixed to 2 hours. The results are reported in Table 1. The columns of the table represent: the name of the instance, the number of nodes ($V$), the number of demands ($K$), the average number of terminals ($T$), the number of generated cuts for inequalities (1) (#C), (8) (#S2C) and (9) (#SNST) respectively, the relative error between the best upper bound and the lower bound obtained at the root node (gap) and finally the total time of execution in hours:min:sec (CPU). We can observe that all the instances *polska* where solved to optimality within less than 6 minutes, which means that our algorithm performs well for relatively small instances. The resolution of the problem becomes harder when the size of instances grows. In fact, instances *newyork* and *pioro*40 took much more time to be solved. In particular, for *pioro*40 with 12 demands, no feasible solution could be found within the time limit. We can also note that inequalities (8) and (9) are quite efficient, they permit to strengthen the linear relaxation. In fact, as it can be seen, all the solved instances have a gap not exceeding 10%. However, for bigger instances, the use of further valid inequalities would be necessary. In this perspective, more significant computational results, details about separation routines and a deeper facial investigation will be presented.

## 6 Concluding remarks

In this paper we studied a problem consisting of multiple Steiner TSPs with order constraints. We proposed an integer linear programming formulation for the problem and studied the associated polytope. We introduced new valid inequalities and discussed some facial aspects. Using this, we devised

| Instance | $V$ | $K$ | $T$ | #C | #S2C | #SNST | Gap(%) | CPU |
|---|---|---|---|---|---|---|---|---|
| polska | 12 | 8 | 4,88 | 329 | 863 | 135 | 4,38 | 0: 00: 08 |
| polska | 12 | 12 | 4,25 | 693 | 2808 | 202 | 8,65 | 0: 00: 58 |
| polska | 12 | 14 | 4,29 | 894 | 3494 | 202 | 8,65 | 0: 01: 18 |
| polska | 12 | 15 | 4,47 | 904 | 3627 | 202 | 8,65 | 0: 01: 25 |
| polska | 12 | 16 | 4,44 | 1029 | 4135 | 234 | 8,38 | 0: 01: 43 |
| polska | 12 | 18 | 4,28 | 723 | 2319 | 298 | 5,83 | 0: 00: 38 |
| polska | 12 | 20 | 4,35 | 764 | 2561 | 298 | 5,83 | 0: 00: 51 |
| polska | 12 | 25 | 3,92 | 1096 | 4497 | 330 | 5,83 | 0: 01: 44 |
| polska | 12 | 30 | 3,97 | 1610 | 7462 | 394 | 8,28 | 0: 06: 35 |
| newyork | 16 | 12 | 4,17 | 1536 | 6762 | 96 | 7,81 | 0: 14: 11 |
| newyork | 16 | 14 | 4,36 | 1372 | 5833 | 199 | 7,14 | 0: 10: 33 |
| newyork | 16 | 15 | 4,4 | 2319 | 10915 | 247 | 8,06 | 0: 31: 53 |
| newyork | 16 | 16 | 4,38 | 2072 | 11214 | 247 | 8,06 | 0: 36: 47 |
| newyork | 16 | 18 | 4,44 | 1111 | 4198 | 302 | 6,19 | 0: 03: 20 |
| newyork | 16 | 20 | 4,5 | 2279 | 11124 | 398 | 8,76 | 0: 39: 29 |
| pioro40 | 40 | 8 | 4,5 | 2963 | 262 | 576 | 0,72 | 0: 03: 17 |
| pioro40 | 40 | 10 | 4,5 | 2735 | 1123 | 720 | 1,4 | 0: 10: 52 |
| pioro40 | 40 | 12 | 4,67 | 7749 | 6680 | 864 | 4,34 | 2: 00: 00 |

Table 1
Preliminary results for the Branch-and-cut algorithm

a Branch-and-cut algorithm that has been tested on SNDlib-based instances. The first results show the efficiency of the valid inequalities to improve the linear relaxation of the formulation. A deeper facial investigation and more significant computational results will be further presented.

## References

[1] Borne, S., Gourdin, E., Liau, B. and A.R. Mahjoub, *Design of survivable IP-over-optical networks*, Ann. Oper. Res. **146** (2006), 41–73.

[2] Gouveia, L., Patrício, P. and A. de Sousa, *Hop-Constrained Node Survivable Network Design: An Application to MPLS over WDM*, Networks and Spatial Economics **8** (2008), 3–21.

[3] Orlowski, S., Raack, C., Koster, A. M. C. A., Baier, G., Engel, T. and P. Belotti, *Branch-and-Cut Techniques for Solving Realistic Two-Layer Network Design Problems*, In: Graphs and Algorithms in Communication Networks, (2010), 95–118, Springer.

[4] Salazar-González, J.J., *The steiner cycle polytope*, Eur. J. of Op. Res. **147** (2003), 671–679.