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The k-node connected subgraph problem: Polyhedral analysis and Branch-and-Cut

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Abstract

In this paper we consider the k-node connected subgraph problem. We propose an integer linear programming formulation for the problem and investigate the associated polytope. We introduce further classes of valid inequalities and discuss their facial aspect. We also devise separation routines and discuss some reduction operations that can be used in a preprocessing phase for the separation. Using those results, we devise a Branch-and-Cut algorithm and present some preliminary computational results.

Keywords: k-node connected graph, polytope, separation, branch-and-cut.

1 Introduction

A graph $G = (V, E)$ is called k-node (resp. k-edge) connected ($k \geq 0$) if for every pair of nodes i, $j \in V$, there are at least k node-disjoint (resp. edge-

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disjoint) paths between i and j. Given a graph $G = (V, E)$ and a weight function c on E that associates with an edge $e \in E$ the weight $c(e) \in \mathbb{R}$, the *k-node connected subgraph problem* (kNCSP for short) is to find a k-node connected spanning subgraph $H = (V, F)$ of G such that $\sum_{e \in F} c(e)$ is minimum. This problem has applications to the design of reliable communication and transportation networks [1,7].

In this article, we consider the kNCSP from a polyhedral point of view. We introduce further classes of valid inequalities for the associated polytope, discuss their facial aspect and devise a Branch-and-Cut algorithm.

The kNCSP is NP-hard for $k \geq 2$ [6]. The edge version of the problem has been widely studied in the literature [1,2,3,7]. However, the kNCSP has been particulary considered for $k = 2$ [8]. A little attention has been given for the high connectivity case where $k \geq 3$.

We will denote a graph by $G = (V, E)$ where V is the *node set* and E is the *edge set*. Given $F \subseteq E$, $c(F)$ will denote $\sum_{e \in F} c(e)$. For $W \subseteq V$, we let $\overline{W} = V \backslash W$. If $W \subset V$ is a node subset of G, then $\delta_G(W)$ will denote the set of edges in G having one node in W and the other in $\overline{W}$. For $W \subset V$, we denote by $E(W)$ the set of edges of G having both endnodes in W and by $G[W]$ the subgraph induced by W. If x^F is the incidence vector of the edge set F of a k-node connected spanning subgraph of G, then x^F satisfies the following inequalities (see [7]):

$$x(e) \geq 0 \qquad e \in E, \tag{1}$$
$$x(e) \leq 1 \qquad e \in E, \tag{2}$$
$$x(\delta_G(W)) \geq k \qquad \emptyset \neq W \subseteq V, \tag{3}$$
$$x(\delta_{G \backslash Z}(W)) \geq k - |Z| \qquad \begin{array}{l} \emptyset \neq Z \subseteq V;\ |Z| \leq k-1, \\ \emptyset \neq W \subseteq V \backslash Z. \end{array} \tag{4}$$

Conversely, any integer solution of the system above is the incidence vector of the edge set of a k-node connected subgraph of G. Constraints (3) and (4) are called cut inequalities and node cut inequalities, respectively. We will denote by kNCSP(G) the convex hull of all the integer solutions of (1)-(4). It can be shown that it suffices to suppose that $|Z| = k - 1$ for inequalities (4).

The kNCSP has been studied by Grötschel et al. [7] within a more general survivability model. A polyhedral analysis is presented along with a cutting plane approach. In [5] Diarrassouba et al. consider the 2NCSP with bounded lengths. Here it is supposed that each path does not exceed L edges for a fixed integer $L \geq 1$. They investigate the polyhedral structure of the polytope and propose a Branch-and-Cut algorithm. In [8] Mahjoub and Nocq study the linear relaxation of the 2NCSP(G).

2 Valid inequalities

In this section, we describe two classes of valid inequalities for kNCSP(G). Given a partition $\pi = (V_1, ..., V_p)$, $p \geq 2$, we will denote by G_π the subgraph induced by π, that is, the graph obtained by contracting the sets V_i, $i = 1, ...p$. We will denote by $\delta_G(V_1, ..., V_p)$ the edge set of G_π.

F-partition inequalities:

Theorem 2.1 *Let $Z \subset V$ with $|Z| \leq k-1$. Consider a partition $(V_0, ..., V_p)$ of $V \backslash Z$, and let F be a subset of $\delta_{G\backslash Z}(V_0)$ such that $p(k-|Z|)-|F|$ is odd. Then the inequality*

$$x(\delta_{G\backslash Z}(V_0, ..., V_p)\backslash F) \geq \lceil \frac{p(k-|Z|)-|F|}{2} \rceil \tag{5}$$

is valid for kNCSP(G).

SP-partition inequalities:

A graph is called serie-parallele if it is not contractible to K_4, the complete graph of four nodes.

Theorem 2.2 *Consider a partition $\pi = (V_1, ..., V_p)$ of V. If G_π is serie-parallele then the inequality*

$$x(\delta_G(V_1, ..., V_p)) \geq \lceil \frac{k}{2} \rceil p - 1 \tag{6}$$

is valid for the kNCSP.

Inequalities of type (6) are called *SP-partition inequalities.*

3 Facial aspect

In this section, we discuss the facial aspect for some of the inequalities presented above. Let $G = (V, E)$ be a graph. We can prove that kNCSP(G) is full-dimensional if and only if G is $(k+1)$-node connected. In the following we assume that G is $(k+1)$-node connected. We have the following results.

Theorem 3.1 *i) Inequalities*(2) *define facets for kNCSP(G) for every* $e \in E$. *ii) Inequalities*(1) *define facets for kNCSP(G) if and only if* e *does not belong to a cut* $\delta_{G\backslash Z}(W)$ *for some* $Z \subset V$ *containing exactly* $k+1-|Z|$ *edges.*

Proof. i) Easy.
ii) Suppose that e belongs to a cut $\delta_{G\backslash Z}(W)$ containing exactly $k+1-|Z|$ edges. Let $\delta_{G\backslash Z}(W)$={e,$f_1$,...,$f_q$} be that cut, with $q = k-|Z|$. Then we have that $x(e) + x(f_1) + x(f_2) + ... + x(f_q) \geq k+1-|Z|$. By adding the valid inequalities $-x(f_i) \geq -1$, $i = 1, ..., q$, we obtain that $x(e) \geq 0$. So inequality (1) is redundant and cannot, therefore, define a facet. Let us suppose now that e does not belong to a cut $\delta_{G\backslash Z}(W)$ of cardinality exactly $k+1-|Z|$. As G is $(k+1)$-node connected, any cut $\delta_{G\backslash Z}(W)$ of $G\backslash Z$ containing e must consist of at least $k+2-|Z|$ edges. Consider the edge sets $T_f = E\backslash\{e, f\}$ for $f \in E\backslash\{e\}$. We claim that the graphs $H = (V, T_f)$ are k-node connected. Indeed, suppose, on the contrary, that this is not the case. Then there must exist $Z \subset V$ and $W \subset V\backslash Z$ such that $|\delta_{H\backslash Z}(W)| \leq k-1-|Z|$. By adding e and f to H two cases may arise. If $e \notin \delta_{G\backslash Z}(H)$, then $|\delta_{G\backslash Z}(W)| \leq k-|Z|$. But this contradicts the fact that G is $(k+1)$-node connected. If $e \in \delta_{G\backslash Z}(H)$, then $|\delta_{G\backslash Z}(H)| \leq k+1-|Z|$. But this contradicts the assumption that any cut $\delta_{G\backslash Z}(W)$ containing e must have at least $k+2-|Z|$ edges. □

The following theorem is given without proof.

Theorem 3.2 *Inequality* (3) *defines facets for kNCSP(G) only if* $G[W]$ *and* $G[\overline{W}]$ *are* $(\lceil \frac{k+1}{2} \rceil)$*-edge connected.*

A *matching* of G is a set of pair with nonadjacent edges.

Remark 3.3 Let W and $\overline{W}$ be a partition of G such that $|W| \geq k$, $|\overline{W}| \geq k+1$ and $G[W]$ and $G[\overline{W}]$ are both k-node-connected. Let $\{e_1, ..., e_k\}$ be edges of $\delta_G(W)$ forming a matching of G such that every edge e_i has ends $u_i \in W$ and $v_i \in \overline{W}$. Let $S = E(W) \cup E(\overline{W}) \cup \{e_1, ..., e_k\}$. Then S is a solution of kNCSP(G).

Theorem 3.4 *Inequality* (3) *defines a facets for kNCSP(G) if the following hold.*

i) $G[W]$ *and* $G[\overline{W}]$ *are* $(k+1)$*-node connected,*

ii) there exists a matching M *containing* k *edges in* $\delta_G(W)$*,*

iii) $|\overline{W}| \geq k+1$*, and there exists a node* s *in* $\overline{W}$ *such that* s *is not incident to the matching* M *and it is adjacent to all the nodes of* M *in* W*.*

Proof.

Let us denote by $ax \geq \alpha$ the cut inequality induced by W, and let $\mathscr{F} = \{x \in kNCSP(G) | ax = \alpha\}$. Suppose there exists a defining facet inequality $bx \geq \beta$ such that $\mathscr{F} \subseteq F = \{x \in kNCSP(G) | bx = \beta\}$. We will prove that there is a scalar ρ such that $b = \rho a$. By ii) there exists a matching $M = \{e_1, ..., e_k\}$ in $\delta_G(W)$ of k edges such that $e_i = u_i v_i$, $i = 1, ..., k$, with u_i in W and v_i in $\overline{W}$. Let $U_1 = \{u_1, ..., u_k\}$ and $V_1 = \{v_1, ..., v_k\}$. Let $T_1 = E(W) \cup E(\overline{W}) \cup M$. As by i) $G[W]$ and $G[\overline{W}]$ are $(k+1)$-node-connected, by Remark 3.3, T_1 is a solution of kNCSP(G). We will show in what follows that the coefficient b_e are equal for all $e \in \delta_G(W)$. Let $f_i = u_i s$, $i = 1, ..k$. Such edges exist by iii). Let $S_i = (T_1 - e_i) + f_i$, $i = 1, ...k$. Note that S_i contains a matching of k edges between W and $\overline{W}$. Hence S_i is a solution of the kNCSP(G). Moreover x^{T_1}, $x^{S_i} \in \mathscr{F} \subseteq F$. Hence $bx^{T_1} = bx^{S_i}$, implying that $b_{e_i} = b_{f_i} = \rho$, for $i = 1, ...k$, for some $\rho \in \mathbb{R}$. By symetry, we also obtain that $b_g = \rho$ for all $g \in [W \backslash U_1, V_1] \cup [U_1, \overline{W} \backslash V_1] \cup M$. Now consider an edge $e = uv$ such that $u \in W \backslash U_1$ and $v \in \overline{W} \backslash V_1$. It is clear that $T_2 = (T_1 \backslash \{e_1\}) \cup \{e\}$ is a solution of the kNCSP(G). Moreover $x^{T_2} \in \mathscr{F} \subseteq F$. Hence $bx^{T_1} = bx^{T_2}$, yielding $b_{e_1} = b_e = \rho$. Finally consider an edge $h = u_i v_j$, $i, j \in \{1, ..., k\}$, with $i \neq j$. Consider the subset $T_3 = (T_1 \backslash \{e_i, e_j\}) \cup \{h, u_j s\}$. We have that T_3 is a solution of kNCSP(G), and $x^{T_3} \in \mathscr{F} \subseteq F$. Which implies that $b_{e_i} + b_{e_j} = b_h + b_{u_j s}$. As $b_{e_i} = b_{e_j} = b_{u_j s} = \rho$, it follows that $b_h = \rho$. Thus we obtain that $b_e = \rho$ for all $e \in \delta_G(W)$. Now we will show that $b_e = 0$ for all $e \in E \backslash \delta_G(W)$. As $G[W]$ and $G[\overline{W}]$ are $(k+1)$-node-connected, we have that $T_4 = T_1 \backslash \{e\}$ induces a k-node connected graph for all edge $e \in E(W) \cup E(\overline{W})$. Moreover $x^{T_4} \in \mathscr{F} \subseteq F$. Hence $b_e = 0$. Consequently, we have that $b_e = \rho$ for all $e \in \delta_G(W)$, and $b_e = 0$ for all $e \in E \backslash \delta_G(W)$. Thus $b = \rho a$. □

4 Branch-and-Cut algorithm

Let $P_E(G)$ (resp. $P_N(G)$) be the polytope given by inequalities (1)-(3) (resp. (1)-(4)). In what follows, we are going to describe first some graph reduction operations which will be utile for our Branch-and-Cut algorithm.

In [4] Didi Biha and Mahjoub introduce the following reduction operations with respect to a solution $\overline{x}$ of $P_E(G)$.

θ_1: Delete an edge $e \in E$ such that $\overline{x}(e) = 0$.

θ_2: Contract a node subset $W \subseteq V$ such that $G[W]$ is k-edge connected and $\overline{x}(e) = 1$ for all $e \in E(W)$.

θ_3: Contract a node subset $W \subseteq V$ such that $|W| \geq 2$, $|\overline{W}| \geq 2$, $|\delta_G(W)| = k$, and $E(\overline{W})$ contains at least one edge with fractional value.

θ_4**:** Contract a node subset $W \subseteq V$ such that $|W| \geq 2$, $|\overline{W}| \geq 2$, $G[W]$ is $\lceil \frac{k}{2} \rceil$-edge connected, $|\delta_G(W)| = k+1$, and $\overline{x}(e) = 1$ for all $e \in E(W)$.

Starting from a graph G and a solution $\overline{x} \in P_E(G)$ and applying θ_1, θ_2, θ_3, θ_4, we obtain a reduced graph G' and a solution $\overline{x}' \in P_E(G')$. Didi Biha and Mahjoub [4] show that $\overline{x}'$ is an extreme point of $P_E(G')$ if and only if $\overline{x}$ is an extreme point of $P_E(G)$. We can show that this result also applies for $P_N(G)$.

We will use operations θ_1, θ_2, θ_3, θ_4 as a preprocessing for the separation procedures in our Branch-and-Cut algorithm.

We now present our Branch-and-Cut algorithm for the kNCSP. The algorithm has been implemented in C++ using CPLEX 12.5 with the default settings. All experiments were run on a 2.10GHzx4 Intel Core(TM) i7-4600U running linux with 16 GB of RAM. We have tested our approach on several instances derived from SNDlib[3] and TSPlib[4] topologies. To start the optimization we consider the following linear program

$$
\begin{aligned}
&\min \sum_{e \in E} c(e)x(e) && \\
&x(\delta_G(u)) \geq k && \text{for all } u \in V, \\
&x(\delta_{G \setminus Z}(u)) \geq 1 && \text{for all } u \in V; Z \subseteq V; |Z| = k-1, \\
&0 \leq x(e) \leq 1 && \text{for all } e \in E.
\end{aligned}
$$

The Branch-and-Cut algorithm uses the inequalities previously described and their separations are performed in the following order: cut inequalities (3), node cut inequalities (4), SP-partition inequalities (6), F-partition inequalities (5). Generated inequalities are added by sets of 200 or less inequalities at a time. The test set consists in complete graphs whose edge weights are the rounded euclidian distance between the edge's vertices. The tests were performed for $k = 3, 4, 5$. In all our experiments, we have used the reduction operations described above. A part of our experimental results is presented in Table 1. Each instance is given by its name followed by an extension representing the number of nodes of the graph. The other entries of the table are: The connectivity (k), the number of generated cuts, for inequalities (3) (#EC) and (4) (#NC), respectively, the number of generated SP-partition inequalities (6) (#SPC), the number of generated F-partition inequalities (5) (#FPC), the weight of the optimal solution obtained (COpt), the Gap, that is the relative error between the best upper bound (the optimal solution if the

[3] http://sndlib.zib.de/home.action.

[4] http://elib.zib.de/pub/mp-testdata/tsp/tsplib/tsp/index.html

problem has been solved to optimality) and the lower bound obtained at the root node of the Branch-and-Cut tree (Gap), the number of subproblems in the Branch-and-Cut tree (NSub), and the total CPU time in h:min:sec (CPU). The maximum CPU time is fixed to 2 hours.

Instance	k	#EC	#NC	#SPC	#FC	COpt	Gap(%)	NSub	CPU
france25	3	250	8	8	17	3229.93	0.39	14	0:00:25
pioro40	3	11	0	0	2	5606.39	0.00	1	0:00:07
berlin52	3	95	14	14	83	12549.1	0.09	6	0:14:23
eil76	3	85	1	9	142	884.722	0.12	6	0:57:57
gr96	3	135	31	18	6	-	4.32	2	2:00:00
france25	4	0	0	-	0	4662.47	0.39	14	0:00:01
pioro40	4	0	5	-	6	8095.94	0.00	1	0:01:59
berlin52	4	4	6	-	0	18267.3	0.00	1	0:01:13
st70	4	14	33	-	4	1624.64	0.00	1	0:38:13
kroA100	4	22	161	-	0	51113.9	0.00	1	1:01:34
france25	5	25	1	0	25	6440.4	0.00	1	0:02:11
pioro40	5	0	23	0	4	10899.3	0.00	1	0:08:43
berlin52	5	0	0	0	0	24673	0.00	1	0:00:01
st70	5	0	0	0	0	2201.31	0.00	1	0:00:06
kroA100	5	55	152	0	31	84123.2	0.00	1	0:05:32

Table 1

Our first series of experiments concerns the kNCSP for $k = 3$. The instances we have considered have graphs with up to 96 nodes. It appears that all the instances have been solved to optimality within the time limit except the instance gr96. For most of the instances, the gap is less than 1%. We also observe that our separation procedures detect a large enough number of SP-partition and F-partition inequalities and seem to be quite efficient. Our next series of experiments concerns the kNCSP with $k = 4, 5$. The instances considered have graphs with up to 100 nodes. Note that when k is even, the SP-partition inequalities are redundant with respect to the cut inequalities (3). Thus, they are included in the resolution process only when k is odd. We can observe that the F-partition inequalities also play an important role for k even. For $k = 5$, we may observe that the CPU time for all the instances is relatively small. Moreover most of the instances are solved in the cutting plane phase. We can observe that the kNCSP becomes easier to solve when k increases. In fact the instance gr96 has been solved for $k = 5$ in 1 second, whereas it could not be solved to optimality for $k = 3$ after 2 hours. In order to evaluate the impact of the reduction operations θ_1, θ_2, θ_3, θ_4 on the separation procedures, we tried to solve the kNCSP without using them. The CPU time increased for the majority of the instances when the reduction operations are not used. This implies that our Branch-and-Cut algorithm is less efficient without the reduction operations.

5 Concluding remarks

In this paper we have studied the k-node connected subgraph problem. We have proposed an integer linear programming formulation for the problem and studied the associated polytope. We have introduced new valid inequalities and discussed some facial aspects. Using this, we devised a Branch-and-Cut algorithm that has been tested on SNDlib and TSPlib based istances. A deeper facial investigation and more significant computational results will be further presented.

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