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Compositions in the bipartite subgraph polytope

J. Fonlupt, A.R. Mahjoub* and J.P. Uhry

Laboratoire ARTEMIS, Institute IMAG, Université Joseph Fourier, Grenoble, France

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Abstract

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In this paper we study the max-cut problem and the related bipartite subgraph polytope in graphs which are decomposable by clique-articulation having at most three nodes. If G decomposes into $G_1, \ldots, G_n$, we show that there exists a polynomial time algorithm to solve the max-cut problem in G provided that such an algorithm is known for appropriate graphs defined from $G_1, \ldots, G_n$. If G decomposes into G_1 and G_2, we derive a linear system of inequalities which defines the bipartite subgraph polytope of G from two linear systems related to G_1 and G_2. Using this, we show that if the two graphs defined from G_1 and G_2 are weakly bipartite then G is also weakly bipartite. By combining this and a theorem of Wagner, we show that graphs noncontractible to K_5 are weakly bipartite. Further classes of weakly bipartite graphs are also discussed.

Introduction

The graphs we consider are finite, undirected, loopless and without multiple edges. We denote a graph by $G=(V, E)$ where V is the *node set* and E is the *edge set* of G.

Given a graph $G=(V, E)$ and two subgraphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ of G, G is called a *k-sum* of G_1 and G_2 (*denoted by* $G=G_1(k)G_2$) if $V=V_1 \cup V_2$, $|V_1 \cap V_2|=k$, $0<k \in N$ and the subgraph $(V_1 \cap V_2, E_1 \cap E_2)$ is complete.

In this paper we study the max-cut problem and the related bipartite subgraph polytope in graphs which are decomposable by means of k-sums, $1 \leq k \leq 3$. If G decomposes into $G_1, \ldots, G_n$, we show that there exists a polynomial time algorithm to solve the max-cut problem in G provided that such an algorithm is known for appropriate graphs defined from $G_1, \ldots, G_n$. If G decomposes into G_1

* Present address: Department of Statistics, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia.

and G_2, we derive a linear system of inequalities which defines the bipartite subgraph polytope of G from two linear systems related to G_1 and G_2. Using this, we show that if the two graphs defined from G_1 and G_2 are weakly bipartite then G is also weakly bipartite. By combining this and a theorem of Wagner, we show that graphs noncontractible to K_5 are weakly bipartite. Further classes of weakly bipartite graphs are also discussed.

Given a graph $G=(V, E)$, if $e \in E$, is an edge with endnodes v_i and v_j, we also write $v_i v_j$ to denote the edge e. A graph $H=(U, F)$ is called subgraph of G if $U \subseteq V$ and $F \subseteq E$. A graph G is called bipartite if its node set may be partitioned into two non-empty disjoint sets V_1 and V_2 such that all edges have one node in V_1 and the other in V_2. The set of edges of a bipartite graph will be called a *bipartite edge set.*

Let $G=(V, E)$ be a graph. Given $W \subseteq V$ we denote by $\delta(W)$ the set of edges having exactly one end in W. The edge set $\delta(W)$ is called a *cut.*

Given a graph $G=(V, E)$ with edge weights $c(e)>0$ for all $e \in E$, then the *max-cut problem* (MCP for short) is to find a cut $\delta(U)$ in G such that

$$C(\delta(U)) := \sum_{e \in \delta(U)} C(e) \quad \text{is maximum.}$$

This problem has been extensively investigated in the past few years and has many real-world applications [1, 2, 7–9].

If $G=(V, E)$ is a graph and $F \subseteq E$ an edge set then the 0–1 vector $x^F \in \mathbb{R}^{|E|}$ with $x^F(e)=1$ if $e \in F$, and $x^F(e)=0$ if $e \notin F$ is called the *incidence vector* of F. The convex hull $P_B(G)$ of the incidence vectors of all edge sets of bipartite subgraphs of G is called the *bipartite subgraph polytope* of G, i.e.,

$$P_B(G) = \operatorname{conv}\{x^F \in \mathbb{R}^{|E|} \mid (V, F) \text{ is a bipartite subgraph of } G\}.$$

Clearly, every edge set of a bipartite subgraph of G is contained in a cut of G. This implies that the MCP in G is equivalent to the following linear program

$$\operatorname{Max}\{cx, x \in P_B(G)\}. \tag{1.1}$$

Hence whenever problem (1.1) can be solved in polynomial time, the MCP can also be solved in polynomial time.

Since the MCP is NP-complete [28], we cannot expect to find a complete explicit characterization of $P_B(G)$ for all graphs G. This characterization is known only for certain classes of graphs such as planar graphs [2] and weakly bipartite graphs [22]. It may however be that for certain classes of graphs G, the polytope $P_B(G)$ can be described by means of a few classes of linear inequalities and that for these classes of inequalities, polynomial time separation algorithms can be designed so that the MCP for these graphs can be solved in polynomial time. In [5], Barahona, Grotschel and Mahjoub gave various classes of facet defining inequalities of this polytope in the general case. Some of these classes have polynomial time separation algorithms as shown by Gerards [21].

We use the standard notation of polyhedral theory. If $a \in \mathbb{R}^m - \{0\}$, $a_0 \in \mathbb{R}$, then the inequality $a^T x \leqslant a_0$ is said to be *valid* with respect to a polyhedron $P \subseteq \mathbb{R}^m$ if $P \subseteq \{x \in \mathbb{R}^m \mid a^T x \leqslant a_0\}$. We say that a valid inequality $a^T x \leqslant a_0$ *supports* P or defines a *face* of P if $\emptyset \neq p \cap \{x \mid a^T x = a_0\} \neq P$. In this case, the polyhedron $P \cap \{x \mid a^T x = a_0\}$ is called the face associated with $a^T x \leqslant a_0$. A valid inequality $a^T x \leqslant a_0$ defines a *facet* of P if it defines a face of P and if the dimension of $P \cap \{x \mid a^T x = a_0\}$ is one less than the dimension of P.

Let $G = (V, E)$ be a graph. Given $b : E \to \mathbb{R}$ and $F \subseteq E$, $b(F)$ will denote $\sum_{e \in F} b(e)$. The support of b will be $E_b = \{e \in E \mid b(e) \neq 0\}$.

It is well known that a graph is bipartite if and only if it does not contain odd cycles. Thus if (V, F) is a bipartite subgraph of $G = (V, E)$, then x^F, the incidence vector of F, must satisfy the inequalities

$$0 \leqslant x(e) \leqslant 1 \quad \text{for all } e \in E, \tag{1.2}$$

$$x(C) \leqslant |C| - 1 \quad \text{for all odd cycles } C \text{ in } G. \tag{1.3}$$

Barahona [2] showed that constraints (1.2) and (1.3) define facets for $P_B(G)$. (We call the inequalities (1.2) *trivial* and the inequalities (1.3) *odd cycles constraints*). Using matching theory, Grotschel and Pulleyblank [21] devised a polynomial time algorithm for the separation problem associated with the constraints (1.2), (1.3) (i.e., an algorithm which decides in polynomial time whether a given vector x satisfies (1.2) and (1.3), and if not, finds a violated inequality). Thus, the ellipsoid method [22] implies that there is a polynomial time algorithm for the solution of (1.1) whenever $P_B(G)$ is completely determined by the trivial inequalities (1.2) and the odd cycle inequalities (1.3). Grotschel and Pulleyblank called such graphs *weakly bipartite*. A complete characterization of this class of graphs is not known. Two classes of graphs have been shown to be weakly bipartite, cf. Barahona [2, 4], planar graphs and graphs G that contain two nodes which cover all the odd cycles of G. In this paper we extend this to the class of graphs noncontractible to K_5.

Given a graph $G = (V, E)$, the *cut polytope* $P_c(G)$ of G is the convex hull of the incidence vectors of all edge sets of cuts of G, i.e.,

$$P_c(G) = \text{conv}\{x^F \in \mathbb{R}^{|E|} \mid F \text{ is a cut of } G\}.$$

Thus, the MCP is also equivalent to the linear program

$$\max\{cx, x \in P_c(G)\}. \tag{1.4}$$

The polytope $P_c(G)$ has been the subject of intensive investigations in the last few years [2, 3, 6, 14–18]. It is easy to see that the following inequalities are valid for $P_c(G)$.

$$0 \leqslant x(e) \leqslant 1 \quad \text{for all } e \in E, \tag{1.5}$$

$$x(F) - x(C - F) \leqslant |F| - 1 \quad \text{for each cycle } C,\ F \subseteq C,\ |F| \text{ odd}. \tag{1.6}$$

In [6], Barahona and Mahjoub showed that the constraints (1.5) and (1.6) completely describe the polytope $P_c(G)$ if and only if G is noncontractible to K_5. Furthermore, they devised a polynomial time algorithm for solving the separation problem associated with the constraints (1.6). By the ellipsoid method, this yields a polynomial time cutting plane algorithm for the MCP in the class of graphs noncontractible to K_5. As will turn out, this class of graphs is strictly contained in the class of weakly bipartite graphs. Since the MCP is also polynomial in weakly bipartite graphs, it would then be interesting to characterize this class of graphs, or to identify further classes of weakly bipartite graphs. This was, in fact, our motivation for studying the bipartite subgraph polytope $P_B(G)$.

In Section 2, we study the bipartite subgraph polytope in graphs decomposable by means of k-sums, $1 \leq k \leq 3$. In Section 3 we discuss some applications to weakly bipartite graphs. Results on the algorithmic aspect of the composition/decomposition are summarized in the final section. The remainder of this section is devoted to more definitions and notations.

Throughout, if $G = G_1(3)G_2$ (resp $G = G_1(2)G_2$) where $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we let $T_0 = E_1 \cap E_2 = (e_1, e_2, e_3\}$ (resp $E_1 \cap E_2 = \{e_1\}$). We denote by $\bar{G}_1 = (\bar{V}_1, \bar{E}_1)$ and $\bar{G}_2 = (\bar{V}_2, \bar{E}_2)$ respectively the graphs obtained from G_1 and G_2 by adding between the nodes of each edge e_i, $i = 1, 2, 3$ (resp. e_1) a path consisting of two edges e_i', e_i'' (resp. e_2, e_3).

If $G = (V, E)$ is a graph and e an edge then $G - e$ denotes the graph obtained by removing e. If $U \subseteq V$ is a node set, then $G - U$ denotes the graph obtained from G by removing the nodes of U and all the edges adjacent to them.

A graph G is said to be *contractible* to a graph H, if H may be obtained from G by a sequence of elementary removal and contractions of edges. A contraction consists of identifying a pair of adjacent vertices and of preserving all other vertices and of preserving all other adjacencies between vertices (multiple edges arising from the identification are replaced by single edges).

2. k-Sums and the polytope $P_B(G)$

In this section we shall study the bipartite subgraph polytope in graphs decomposable by means of k-sums, $1 \geq k \geq 3$.

As mentioned above, a complete description of the bipartite subgraph polytope $P_B(G)$ by a finite system of linear inequalities is not known in the general case. However for a graph G which is the k-sum, $1 \leq k \leq 3$, of two graphs G_1 and G_2, it will turn out that such a description can be obtained whenever the polytopes $P_B(\bar{G}_1)$ and $P_B(\bar{G}_2)$ are known. If G is a 1-sum of G_1 and G_2, then $P_B(G)$ is given by the juxtaposition of $P_B(G_1)$ and $P_B(G_2)$. In what follows we study the case when G is a 3-sum of G_1 and G_2. The case when G is a 2-sum of G_1 and G_2 may be obtained as a special case of the second one. For this, we first state some structural properties concerning the facet inducing graphs of the polytope $P_B(G)$.

2.1. *On the facet inducing graphs for $P_B(G)$*

Throughout this section and the remainder of the paper, given a graph $G=(V, E)$, we let

$$\beta(G)=\{B \subseteq E \mid (V, B) \text{ is bipartite}\},$$

denote the set of bipartite edge sets of G. If $a^{\mathrm{T}}x \leqslant \alpha$ is a facet defining inequality of $P_B(G)$, we denote by β_a the set

$$\beta_a=\{B \in \beta(G) \mid a^{\mathrm{T}}x^B=\alpha\}.$$

Remark 2.1. Since $\beta(G)$ defines an independence system, if $a^{\mathrm{T}}x \leqslant \alpha$ is a facet defining inequality of $P_B(G)$ with $\alpha>0$, then $a \geqslant 0$.

Let $G=(V, E)$ be an arbitrary graph and $a^{\mathrm{T}}x \leqslant \alpha$ be a nontrivial facet defining inequality for $P_B(G)$. Then $\alpha>0$ and hence $a \geqslant 0$. Let us denote by G_a the graph induced by this inequality (i.e., induced by the support of a, E_a). Since $P_B(G)$ is full dimensional [5], which implies that $P_B(G)$ has a unique (up to positive scaling) nonredundant defining linear inequality system, then the only equations satisfied by all members of β_a are positive multiples of $a^{\mathrm{T}}x=\alpha$. We then have the following lemma.

Lemma 2.2. (i) *If G_a contains a path (uv, vw) such that v is of degree two, then $a(uv)=a(vw)$.*

(ii) *G_a does not contain a node of degree one.*

(iii) *Let p, q be two nodes of G_a. If G_a is not an odd cycle then at most one path in G_a which joins p and q can have all internal nodes of degree two.*

Proof. (i) and (ii) are easily seen to be true.

(iii) Assume the contrary. Let P_1 and P_2 be the two paths between p and q. Let C be the cycle defined by P_1 and P_2.

Consider a bipartite edge set $B \in \beta_a$. Since B induces a maximum edge set in G_a, for P_i, $i=1, 2$, we have either

$$P_i \subseteq B \quad \text{or} \quad |B \cap P_i|=|P_i|-1. \tag{2.1}$$

Case a: *C is even*

Then P_1 and P_2 have the same parity. And if $B \in \beta_a$, we have

$$P_1 \subseteq B \Leftrightarrow P_2 \subseteq B. \tag{2.2}$$

Suppose $|P_1| \leqslant |P_2|$ and let $\gamma=|P_2|-|P_1|$. From (2.1) and (2.2), for all $B \in \beta_a$ the following holds:

$$|B \cap P_1|=|B \cap P_2|+\gamma \quad \Leftrightarrow \quad \sum_{e \in P_1} x^B(e)-\sum_{e \in P_2} x^B(e)=\gamma.$$

Since the above equation has both positive and negative coefficients, this inequality cannot be a positive multiple of $a^{\mathrm{T}}x \leqslant \alpha$, a contradiction.

Case b: *C is odd.*

It is clear in this case that if $B \in \beta_a$ then

$$\sum_{e \in C} x^B(e) = |C| - 1.$$

Since $G_a \neq C$, we then have a contradiction and our lemma is proved. □

2.2. 3-Sums

Consider a graph $G = (V, E)$ which is a 3-sum of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ (see Fig. 1). Lemma 2.2 implies that the polytopes $P_B(\bar{G}_1)$ and $P_B(\bar{G}_2)$ can be assumed to be described by two minimal linear inequality systems of the following forms, having respectively I and J nontrivial constraints.

$$P_B(\bar{G}_1) = \begin{cases} 0 \leqslant x(e) \leqslant 1 \quad \text{for all } e \in \bar{E}_1, & (2.3) \\ \sum_{e \in E_1} a^i(e)x(e) + \sum_{k=1,\ldots,3} a^k(k)(x(e'_k) + x(e''_k) \leqslant \alpha^i, \quad i = 1, \ldots, I, & (2.4) \end{cases}$$

and

$$P_B(\bar{G}_2) = \begin{cases} 0 \leqslant x(e) \leqslant 1 \quad \text{for all } e \in \bar{E}_2, & (2.5) \\ \sum_{e \in E_2} b^j(e)x(e) + \sum_{k=1,\ldots,3} b^j(k)(x(e'_k) + x(e''_k) \leqslant \beta^j, \quad j = 1, \ldots, J. & (2.6) \end{cases}$$

Note that by Remark 2.1, all the coefficients in (2.4) and (2.6) are nonnegative.

Given a nonnegative I-row vector $\pi = (\pi_1, \ldots, \pi_I)$ (resp. J-row vector $\mu = (\mu_1, \ldots, \mu_J)$), let $[\pi]$ (resp. $[\mu]$) denote the constraint, linear combination of the constraints (2.4) (resp. (2.6)) with respect to π (resp. μ), given by

$$[\pi] := \sum_{e \in E_1} a(\pi, e)x(e) + \sum_{k=1,\ldots,3} a(\pi, k)(x(e'_k) + x(e''_k)) \leqslant \alpha(\pi), \tag{2.7}$$

resp.

$$[\mu] := \sum_{e \in E_2} b(\mu, e)x(e) + \sum_{k=1,\ldots,3} b(\mu, k)(x(e'_k) + x(e''_k)) \leqslant \beta(\mu), \tag{2.8}$$

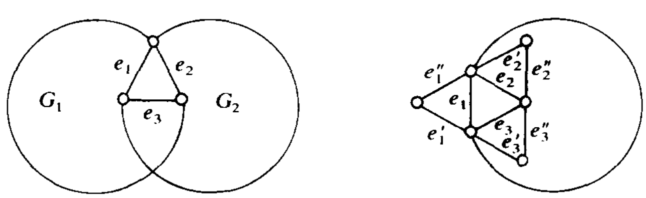

Fig. 1. The graphs G (left) and $\bar{G}_2$ (right).

where

$$a(\pi, e) = \sum_{i=1,\dots,I} \pi_i a^i(e) \quad \text{for all } e \in E_1,$$

$$a(\pi, k) = \sum_{i=1,\dots,I} \pi_i a^i(k) \quad \text{for } k = 1, 2, 3,$$

$$\alpha(\pi) = \sum_{i=1,\dots,I} \pi_i \alpha^i,$$

resp.,

$$b(\mu, e) = \sum_{j=1,\dots,J} \mu_j b^j(e) \quad \text{for all } e \in E_2,$$

$$b(\mu, k) = \sum_{j=1,\dots,J} \mu_j b^j(k) \quad \text{for } k = 1, 2, 3,$$

$$\beta(\mu) = \sum_{j=1,\dots,J} \mu_j \beta^j.$$

Definition 2.3. Given a constraint $[\pi]$ of type (2.7) and a constraint $[\mu]$ of type (2.8), defined respectively with respect to two row vectors $\pi = (\pi_1, \dots, \pi_I)$ and $\mu = (\mu_1, \dots, \mu_J)$ which satisfy the following system,

$$(R)\begin{cases} a(\pi, e_k) \geq b(\mu, k) & \text{for } k = 1, 2, 3, & (2.9)\\ b(\mu, e_k) \geq a(\pi, k) & \text{for } k = 1, 2, 3, & (2.10)\\ \sum_i \pi_i + \sum_j \mu_j = 1, & & (2.11)\\ \pi_i \geq 0 & \text{for } i = 1, \dots, I, & (2.12)\\ \mu_j \geq 0 & \text{for } j = 1, \dots, J, & (2.13) \end{cases}$$

we call *mixed constraint* of $[\pi]$ and $[\mu]$, denoted by $[\pi, \mu]$, the constraint given by

$$\begin{aligned} [\pi, \mu] := &\sum_{e \in E_1'} a(\pi, e)x(e) + \sum_{e \in E_2'} b(\mu, e)x(e) \\ &+ \sum_{k=1,\dots,3} (a(\pi, e_k) + b(\mu, e_k) - a(\pi, k) \\ &- b(\mu, k))x(e_k) \leq \alpha(\pi) + \beta(\mu) \\ &- 2 \sum_{k=1,\dots,3} (a(\pi, k) + b(\mu, k)), \end{aligned} \tag{2.14}$$

where $E_i' = E_i - \{e_1, e_2, e_3\}$, $i = 1, 2$.

Remark 2.4. The constraint (2.11) in (R) is just to ensure that at least one of the vectors π and μ is nonzero. Moreover, if $(\pi, \mu) \neq 0$ and (π, μ) satisfies (2.9), (2.10), (2.12), (2.13) (but (π, μ) does not satisfy (2.11)), then $[\pi, \mu]$ is a positive multiple of the constant $[\pi', \mu']$ where

$$(\pi', \mu') = \Big(\sum_{i=1,\ldots,I} \pi_i + \sum_{j=1,\ldots,J} \mu_j \Big)^{-1} (\pi, \mu) \in (R).$$

Hence $[\pi, \mu]$ can be considered as a mixed constraint and for convenience, we will also say that (π, μ) is feasible for (R).

Note that any mixed constraint has a nonnegative coefficients. Moreover we have the following.

Lemma 2.5. *Any mixed constraint $[\pi, \mu]$ is valid for $P_B(G)$.*

Proof. Easy. □

Clearly, the system (R) is a polyhedron in $\mathbb{R}^{I+J}$ which, in consequence, has a finite number of extreme points. Let (P) be the polytope defined by all the mixed constraints $[\pi, \mu]$, where (π, μ) is an extreme point of (R), together with the constraints

$$0 \leqslant x(e) \leqslant 1 \quad \text{for all } e \in E.$$

The following theorem shows that the bipartite subgraph of G, $P_B(G)$ is precisely the polytope (P).

Theorem 2.6. $P_B(G) = (P)$.

The proof of Theorem 2.6 is lengthy and appears in [19]. Although Theorem 2.6 does not provide a simple linear description of the polytope $P_B(G)$, as it will turn out, it has very interesting applications to graphs defined by means of k-sums, $1 \leqslant k \leqslant 2$ and weakly bipartite graphs. Before showing this, we need to state some properties concerning the structure of the mixed constraints of $P_B(G)$.

2.3. *On the mixed constraints of $P_B(G)$*

Having proved that $P_B(G) = (P)$, in this subsection we shall discuss the structure of the mixed constraints that may define facets for $P_B(G)$. We will show that any mixed constraint defined from more than four constraints from both $P_B(\bar{G}_1)$ and $P_B(\bar{G}_2)$ is redundant in the description of (P), and hence cannot define a facet for $P_B(G)$. This will be used in subsequent proofs.

Definition 2.7. Let $G = (V, E)$ be a graph. Suppose that the polytope $P_B(G)$ is given by the system $\{Ax \leqslant b; x(e) \geqslant 0, e \in E\}$ where A is an (m, n) matrix and b

is an m-column vector. If $a_1^T x \leqslant \alpha_1$ and $a_2^T x \leqslant \alpha_2$ are two valid constraints of $P_B(G)$, then we say that $a_2^T x \leqslant \alpha_2$ *dominates* $a_1^T x \leqslant \alpha_1$ if

(i) there exists an m-row vector $y \geqslant 0$ such that $a_2 = yA$, $\alpha_2 = yb$ (i.e., $a_2^T x \geqslant \alpha_2$ is a linear combination of the constraints of the system $Ax \leqslant b$),

(ii) $a_2 \geqslant a_1$, $\alpha_2 \leqslant \alpha_1$.

We then have the following lemmas.

Lemma 2.8. *Let $a^T x \leqslant \alpha$ be a valid constraint of $P_B(\bar{G}_i)$, $i = 1, 2$, with $a(e'_k) = a(e''_k)$ for $k = 1, 2, 3$. Then there exists a valid constraint $\bar{a}^T x \leqslant \bar{\alpha}$* of *$P_B(\bar{G}_i)$, $i = 1, 2$, that dominates $a^T x \leqslant \alpha$ such that $\bar{a}(e'_k) = \bar{a}(e''_k)$ for $k = 1, 2, 3$.*

Proof. First notice that if $a^T x \leqslant \alpha$ is a linear combination of the constraints of $P_B(G_i)$, then one can take $\bar{a} = a$, $\bar{\alpha} = \alpha$. Now assume that this is not the case and, for instance, that $a^T x \leqslant \alpha$ is valid for $P_B(\bar{G}_1)$. Hence, the linear program: $\max\{a^T x : x \in P_B(\bar{G}_1)\}$ has an optimal solution x_0 such that $a^T x_0 \leqslant \alpha$. By LP-duality, there exists an optimal dual solution $(\pi, Z) \geqslant 0$ where $\pi = (\pi_1, \ldots, \pi_l)$ and $Z = (Z(e), e \in E_1)$ such that

$$a(\pi, e) + Z(e) \geqslant a(e) \quad \text{for all } e \in \bar{E}_1,$$

$$\alpha(\pi) + \sum_{e \in E_1} Z(e) = a^T x_0 \leqslant \alpha,$$

(recall that $a(\pi, e)$, $e \in \bar{E}_1$ and $\alpha(\pi)$ denote respectively the coefficients and the right hand side of the constraint $[\pi]$).

Since the dual of the above linear program is to be minimized and

$$a(\pi, e'_k) = a(\pi, e''_k) \quad \text{for } k = 1, 2, 3,$$

we then should have

$$Z(e'_k) = Z(e''_k) \quad \text{for } k = 1, 2, 3.$$

Let

$$\bar{a}(e) = a(\pi, e) + Z(e) \quad \text{for all } e \in \bar{E}_1,$$

$$\bar{\alpha} = \alpha(\pi) + \sum_{e \in \bar{E}_1} Z(e).$$

Then $\bar{a}^T x \leqslant \bar{\alpha}$ is the required constraint. $\square$

Lemma 2.9. *Let $a^T x \leqslant \alpha$ be a valid constraint of $P_B(\bar{G}_i)$, $i = 1, 2$, with $a(e'_k) = a(e''_k)$ for $k = 1, 2, 3$. Then there exists a valid constraint $\bar{a}^T x \leqslant \bar{\alpha}$ of $P_B(\bar{G}_i)$, $i = 1, 2$, that dominates $a^T x \leqslant \alpha$ such that*

$$\bar{a}(e) = a(e) \quad \textit{for all } e \in \{e'_k, e''_k, k = 1, 2, 3\}. \tag{2.15}$$

Proof. By Lemma 2.8, there exists a valid constraint $\bar{a}^T x \leqslant \bar{\alpha}$ of $P_B(\bar{G}_i)$ that dominates $a^T x \leqslant \alpha$ and such that $\bar{a}(e'_k) = \bar{a}(e''_k)$ for $k = 1, 2, 3$. Let us assume that

$\bar{\alpha}$ is minimum with respect to this property. We then claim that $\bar{a}$ satisfies (2.15). In fact, if this is not the case then by letting

$$\rho_k = \bar{a}(e_k') - a(e_k'), \quad \text{for } k = 1, 2, 3,$$

we obtain that

$$\sum_{k=1,2,3} \rho_k > 0.$$

Set

$$\bar{a}_1(e) = \bar{a}(e) \qquad \text{for all } e \in E_i,$$
$$\bar{a}_1(e) = \bar{a}(e) - \rho_k \quad \text{for all } e \in \{e_k', e_k'', k = 1, 2, 3\},$$
$$\bar{\alpha}_1 = \bar{\alpha} - \sum_{k=1,2,3} \rho_k.$$

The constraint $\bar{a}_1^{\mathrm{T}} x \leqslant \bar{\alpha}_1$ is easily seen to be valid for $P_B(\bar{G}_i)$. Consequently, by Lemma 2.8 there exists a valid constraint $\bar{a}_2^{\mathrm{T}} x \leqslant \bar{\alpha}_2$ of $P_B(\bar{G}_i)$ that dominates $\bar{a}_1^{\mathrm{T}} x \leqslant \bar{\alpha}_1$ and hence dominates $a^{\mathrm{T}} x \leqslant \alpha$ with $\bar{a}_2(e_k') = \bar{a}_2(e_k'')$ for $k = 1, 2, 3$ and $\bar{\alpha}_2 \leqslant \bar{\alpha}_1 < \bar{\alpha}$, a contradiction. □

Lemma 2.10. *Let $[\pi, \mu]$ be a mixed constraint. If $[\pi, \mu]$ is nonredundant in the description of (P), then*

$$\begin{aligned} a(\pi, e_k)a(\pi, k) &= 0, \quad k = 1, 2, 3, \\ b(\mu, e_k)b(\mu, k) &= 0, \quad k = 1, 2, 3. \end{aligned} \tag{2.16}$$

Proof. Let us suppose that (2.16) does not hold. Since (π, μ) is feasible for the system (R), we may w.l.o.g., assume that there is $\gamma \in \{1, 2, 3\}$ such that $a(\pi, e_\gamma) \geqslant a(\pi, \gamma) > 0$. We will show that $[\pi, \mu]$ is then redundant in (P). For this, let us assume that $b(\mu, e_\gamma) \geqslant b(\mu, l)$. (The case where $b(\mu, e_\gamma) < b(\mu, \gamma)$ can be treated in a similar way.) It is clear that constraint $[\pi]$ (resp. $[\mu]$) can be obtained by summing the two constraints:

$$\begin{aligned} &\sum_{e \in E_1'} a(\pi, e)x(e) + \sum_{k \neq \gamma} a(\pi, e_k)x(e_k) + (a(\pi, e_\gamma) - a(\pi, \gamma))x(e_\gamma) \\ &\quad + \sum_{k \neq \gamma} a(\pi, k)(x(e_k') + x(e_k'')) \leqslant \alpha(\pi) - 2a(\pi, \gamma), \end{aligned} \tag{2.17}$$

and

$$a(\pi, \gamma)((x(e_\gamma) + x(e_\gamma') + x(e_\gamma'')) \leqslant 2a(\pi, \gamma), \tag{2.18}$$

resp.,

$$\begin{aligned} &\sum_{e \in E_2'} b(\mu, e) + \sum_{k \neq \gamma} b(\mu, e_k)x(e_k) + (b(\mu, e_\gamma) - b(\mu, \gamma))x(e_\gamma) \\ &\quad + \sum_{k \neq \gamma} b(\mu, k)(x(e_k') + x(e_k'')) \leqslant \beta(\mu) - 2b(\mu, \gamma), \end{aligned} \tag{2.19}$$

and

$$b(\mu, \gamma)(x(e_\gamma) + x(e_\gamma') + x(e_\gamma'')) \leqslant 2b(\mu, \gamma) \tag{2.20}$$

Notice that all the coefficients of (2.17) and (2.19) are nonnegative. We claim that (2.17) (resp. (2.19)) is valid for $P_B(\bar{G}_1)$ (resp. $P_B(\bar{G}_2)$). In fact, consider a bipartite edge set B of $\bar{G}_1$ (resp. $\bar{G}_2$) which may be assumed maximal. Then $|B \cap \{e'_k, e'_k, e''_k\}| = 2$ for $k = 1, 2, 3$. Since x^B satisfies $[\pi]$ (resp. $[\mu]$), it also satisfies (2.17) (resp. 2.19).

Now let us denote (2.17) (resp. 2.20) by $a^{\mathrm{T}}x \leq \alpha$ (resp. $b^{\mathrm{T}}x \leq \beta$). In consequence, by Lemma 2.9 together with Definition 2.7, there is a vector

$$(\pi', Z) = (\pi'_1, \ldots, \pi'_I; Z(e), e \in \bar{E}_1) \geq 0$$

resp.,

$$(\mu', t) = (\mu'_1, \ldots, \mu'_J; t(e), e \in \bar{E}_2) \geq 0)$$

such that

$$\begin{cases} a(\pi', e) + Z(e) \geq a(e) \quad \text{for all } e \in E_1, \\ a(\pi', k) + Z(e) = a(e) \\ \quad \text{for all } e \in \{e'_k, e''_k\},\ k = 1, 2, 3, \\ \alpha(\pi') + \sum_{e \in \bar{E}_1} Z(e) \leq \alpha, \end{cases} \tag{2.21}$$

resp.,

$$\begin{cases} b(\mu', e) + t(e) \geq b(e) \quad \text{for all } e \in E_2, \\ b(\mu', k) + t(e) = b(e) \\ \quad \text{for all } e \in \{e'_k, e''_k\},\ k = 1, 2, 3, \\ \beta(\mu') + \sum_{e \in \bar{E}_2} t(e) \leq \beta. \end{cases} \tag{2.22}$$

We assume that $\alpha(\pi')$ (resp. $\beta(\mu')$) is minimum with respect to (2.21) (resp. (2.22)). By Lemma 2.9 together with Definition 2.7, it is not difficult to show that (π', μ') is feasible for (R).

Now let $\bar{a}^{\mathrm{T}}x \leq \bar{\alpha}$ be the constraint obtained by summing the following constraints:

$$\begin{aligned} &[\pi', \mu'], \\ &Z(e)x(e) \leq Z(e) \quad \text{for all } e \in E_1, \\ &t(e)x(e) \leq t(e) \quad\ \ \text{for all } e \in E_2. \end{aligned}$$

It is easily seen that $\bar{a}^{\mathrm{T}}x \leq \bar{\alpha}$ dominates $[\pi, \mu]$. From (2.21) it follows that $a(\pi', \gamma) = 0$, hence $(\pi', \mu') \neq (\pi, \mu)$. Which implies that the constraint $[\pi, \mu]$ is redundant in (P) and we are done. □

Lemma 2.11. *Let $[\pi, \mu]$ be a nonredundant mixed constraint in the description of (P). Then (π, μ) has at most four positive components.*

Proof. Since $[\pi, \mu]$ is nonredundant in (P), then (2.16) holds. Let (R') be the system obtained from (R) as follows: For $k = 1, 2, 3$, if $a(\pi, e_k) = 0$ (resp. $a(\pi, k) = 0$) delete the constraint of (2.9) (resp. 2.10) that corresponds to k. Thus (R') is a polytope having at most four constraints together with the nonnegative constraints. We claim that (π, μ) is an extreme point of (R'). In fact, if not, there are two solutions (π^1, μ^1), (π^2, μ^2) of (R') such that $(\pi^1, \mu^1) \neq (\pi^2, \mu^2)$ and $(\pi, \mu) = \frac{1}{2}(\pi^1, \mu^1) + \frac{1}{2}(\pi^2, \mu^2)$. Therefore, for $k = 1, 2, 3$, if $a(\pi, e_k) = 0$ (resp. $a(\pi, k) = 0$, $b(\mu, e_k) = 0$, $b(\mu, k) = 0$), then $a(\pi^i, e_k) = 0$ (resp. $a(\pi^i, k) = 0$, $b(\mu^i, e_k) = 0$, $b(\mu^i, k) = 0$) for $i = 1, 2$. This implies that (π^1, μ^1) and (π^2, μ^2) are feasible for (R) and thus (π, μ) is not an extreme point of (R). This contradicts the fact that $[\pi, \mu]$ is a constraint of (P). As a consequence, (π, μ) cannot have more than four nonzero components. □

Remark 2.12. Lemma 2.11 implies that every nonredundant mixed constraint $[\pi, \mu]$ of $P_B(G)$, when $\pi \neq 0 \neq \mu$, can be obtained by mixing at most four constraints from (2.4) and (2.6).

2.3. 2-Sums

Now, consider a graph $G = (V, E)$ which is a 2-sum of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ (see Fig. 2). As mentioned above, the characterization of the polytope $P_B(G)$ in this case may be obtained as a special case of the 3-sum.

Lemma 2.2 shows that $P_B(\bar{G}_\gamma)$ can be described by a minimal linear system having the following form (where I_1^γ, I_2^γ, I_3^γ denote respectively the index sets of all the constraints of $P_B(\bar{G}_\gamma)$ whose support does not intersect T_0, contains e_1 but

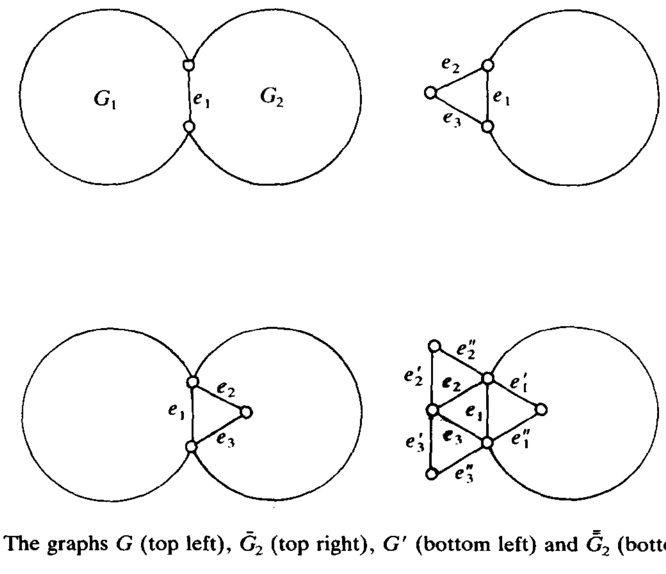

Fig. 2. The graphs G (top left), $\bar{G}_2$ (top right), G' (bottom left) and $\bar{\bar{G}}_2$ (bottom right).

not e_2, e_3, contains e_2, e_3 but not e_1)

$$P_B(\bar{G}_\gamma)=\begin{cases} 0 \leq x(e) \leq 1, \quad \text{for all } e \in \bar{E}_\gamma & (2.23)\\ x(e_1)+x(e_2)+x(e_3) \leq 2, & (2.24)\\ \sum\limits_{e\in E'_\gamma} a^i_\gamma(e)x(e) \leq \alpha^i_\gamma, \quad \text{for } i \in I^\gamma_1, & (2.25)\\ \sum\limits_{e\in E'_\gamma} a^i_\gamma(e)x(e)+x(e_1) \leq \alpha^i_\gamma, \quad \text{for } i \in I^\gamma_2, & (2.26)\\ \sum\limits_{e\in E'_\gamma} a^i_\gamma(e)x(e)+x(e_2)+x(e_3) \leq \alpha^i_\gamma, \quad \text{for } i \in I^\gamma_3. & (2.27) \end{cases}$$

for $\gamma = 1, 2$, where $E'_i = E_i - \{e_1\}$, $i = 1, 2$.

2.3.1. *The polytope $P_B(G)$*

Let $G' = (V', E')$ be the 3-sum of $\bar{G}_1$ and $\bar{G}_2$, obtained by identifying the triangle T_0 (see Fig. 2). (Notice that G' is also the graph obtained from G by adding the path (e_2, e_3) between the nodes of e_1.)

Lemma 2.2 implies that any facet defining inequality of $P_B(\bar{\bar{G}}_\gamma)$ (see Fig. 2 for $\bar{\bar{G}}_2$), $a^{\mathrm{T}}_\gamma x \leq \alpha_\gamma$ whose support intersects both E'_γ and the edge set $\{e_k, e'_k, e''_k;$ $k = 1, 2, 3\}$ has one of the following forms:

$$S_\gamma=\begin{cases} \sum\limits_{e\in E'_\gamma} a_\gamma(e)x(e)+x(e_1) \leq \alpha_\gamma, & (2.28)\\ \sum\limits_{e\in E'_\gamma} a_\gamma(e)x(e)+x(e'_1)+x(e''_1) \leq \alpha_\gamma, & (2.29)\\ \sum\limits_{e\in E'_\gamma} a_\gamma(e)x(e)+x(e_2)+x(e_3) \leq \alpha_\gamma, & (2.30)\\ \sum\limits_{e\in E'_\gamma} a_\gamma(e)x(e)+x(e_2)+x(e'_3)+x(e''_3) \leq \alpha_\gamma, & (2.31)\\ \sum\limits_{e\in E'_\gamma} a_\gamma(e)x(e)+x(e_3)+x(e'_2)+x(e''_2) \leq \alpha_\gamma, & (2.32)\\ \sum\limits_{e\in E'_\gamma} a_\gamma(e)x(e)+x(e'_2)+x(e''_2)+x(e'_3)+x(e''_3) \leq \alpha_\gamma, & (2.33) \end{cases}$$

for $\gamma = 1, 2$.

The following theorem, given by Barahona, Grotschel and Mahjoub [5], shows that if we replace an odd path of a facet defining inequality by an edge, we get another facet defining inequality.

Theorem 2.13. *Let $H = (W, F)$ be a graph and $a^{\mathrm{T}}x \leq \alpha$ be a nontrivial facet defining inequality for $P_B(H)$. Suppose the support of a contains a path $P(vv_1, v_1v_2, \ldots, v_{p-1}v_p, v_pw)$ of odd length $p+1 \geq 3$ with $a(ij) = \gamma$ for all $ij \in P$ and where the degree of all nodes $v_1, \ldots, v_p$ in the support of a is 2. Let*

$H' = (W', F')$ be the graph obtained from H by removing the nodes $v_1, \ldots, v_p$ and adding the edge vw (if vw is not already contained in H). Let $\tilde{a} \in \mathbb{R}^{|E'|}$ be defined as follows:

$$\tilde{a}(ij) = a(ij) \quad \text{for all } ij \in F \cap F',$$

$$\tilde{a}(vw) = \gamma, \qquad \tilde{\alpha} = \alpha - p\gamma.$$

Then $\tilde{a}^{\mathrm{T}}x \leq \tilde{\alpha}$ defines a facet of $P_B(H')$.

Now we may make the following remarks.

Remark 2.14. The polytope $P_B(G)$ can be obtained from $P_B(G')$ by deleting the constraints whose support intersects the edge set $\{e_2, e_3\}$.

Remark 2.15. From Lemma 2.11 and Theorem 2.13, it follows that for the constraints of type (2.31), (2.32) and (2.33), the constraints

$$\sum_{e \in E'_\gamma} a_\gamma(e)x(e) + x(e_1) \leq \alpha_\gamma - 2,$$

and

$$\sum_{e \in E'_\gamma} a_\gamma(e)x(e) + x(e_2) + x(e_3) \leq \alpha_\gamma - 2,$$

also define facets for $P_B(\bar{\bar{G}}_\gamma)$, for $\gamma = 1, 2$.

Remark 2.16. If we mix one (resp. two or three) constraints of S_1 with two or three (resp. one) constraints of S_2 we obtain a constraint whose support intersects $\{e_2, e_3\}$, hence it cannot define a facet of $P_B(G)$.

Remark 2.17. A mixed constraint of $P_B(G')$, defined from two constraints of S_1 and two constraints of S_2 whose support does not intersect $\{e_2, e_3\}$ can be written as a sum of two mixed constraints. Thus it cannot define a facet of $P_B(G)$.

Remark 2.18. A mixed constraint of $P_B(G')$ which is defined from one constraint of S_1 and one constraint of S_2 defines a facet of $P_B(G)$ only if it is obtained by mixing a constraint of type (2.28) with a constraint of type (2.29).

Proof. It suffices to show that a mixed constraint which is either obtained by
(i) mixing (2.31) with (2.32), or
(ii) mixing (2.30) with (2.33))
cannot define a facet of $P_B(G)$. Indeed, let us assume that the constraints of types (2.30) and (2.31) (resp. (2.32) and (2.33)) are in S_1 (resp. S_2).

In both cases (i) and (ii) the corresponding mixed constraint is given by

$$\sum_{e \in E'_1} a_1(e)x(e) + \sum_{e \in E'_2} a_2(e)x(e) \leq \alpha_1 + \alpha_2 - 4. \tag{2.34}$$

Case (i): By Remark 2.15, the constraint

$$\sum_{e\in E'_\gamma} a_\gamma(e)x(e) + x(e_1) \leq \alpha_\gamma - 2,$$

defines a facet for $P_B(\bar{\bar{G}}_\gamma)$, for $\gamma = 1, 2$. Thus (2.34) is redundant in $P_B(G')$, hence it cannot define a facet for $P_B(G)$.

Case (ii): By Remark 2.15, the constraint

$$\sum_{e\in E'_2} a_2(e)x(e) + x(e_2) + x(e_3) \leq \alpha_2 - 2,$$

defines a facet of $P_B(\bar{\bar{G}}_2)$. Now it is easy to see that the constraints

$$\sum_{e\in E'_1} a_1(e)x(e) \leq \alpha_1 - 1,$$

$$\sum_{e\in E'_2} a_2(e)x(e) \leq \alpha_2 - 3,$$

are valid for $P_B(\bar{\bar{G}}_1)$, $P_B(\bar{\bar{G}}_2)$. Thus (2.34) cannot define a facet for $P_B(G')$, and hence for $P_B(G)$. □

Now, from Remarks 2.16–2.18 together with Remarks 2.12 and 2.14, it follows that the mixed constraints of $P_B(G')$ that may define facets for $P_B(G)$ are those obtained by mixing one constraint of type (2.28) with one constraint of type (2.29). Thus from Theorem 2.6 together with Remark 2.14 we can state the following theorem.

Theorem 2.19. *The polytope $P_B(G)$ is defined by all the constraints* (2.25), (2.26), $0 \leq x(e) \leq 1$ *for all* $e \in E$, *and the mixed constraints*

$$\sum_{e\in E'_\gamma} a^i_\gamma(e)x(e) + \sum_{e\in E'_t} b^j_t(e)x(e) \leq \alpha^i_\gamma + \beta^j_t - 2 \tag{2.35}$$

with $\gamma, t \in \{1, 2\}$, $\gamma \neq t$, $(i, j) \in I^\gamma_2 \times I^t_3$.

Theorem 2.19 permits one to obtain, in a very simple way, a linear system defining $P_B(G)$ from those defining $P_B(\bar{G}_1)$ and $P_B(\bar{G}_2)$. The following theorem shows that this system is also minimal. Its proof follows the same way as other proofs given for similar results in [11, 12], hence it is omitted.

Theorem 2.20. *Inequalities* (2.35) *define facets of* $P_B(G)$.

3. Applications to weakly bipartite graphs

In this section we shall discuss some applications of the results presented above to weakly bipartite graphs.

Corollary 3.1. *Let $G=(V, E)$ be a k-sum, $1 \leq k \leq 3$, of two graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$. If $\bar{G}_1$ and $\bar{G}_2$ are weakly bipartite then G is weakly bipartite.*

Proof. This is clear for $k=1, 2$.

Suppose G is a 3-sum of G_1 and G_2. Since $\bar{G}_1$ and $\bar{G}_2$ are weakly bipartite, the nontrivial facets of $P_B(\bar{G}_1)$ and $P_B(\bar{G}_2)$ all come from odd cycles. Let $[\pi, \mu]$, where $\pi \neq 0 \neq \mu$, be a mixed constraint which defines a nontrivial facet of $P_B(G)$. Thus by Remark 2.12, $[\pi, \mu]$ is obtained by mixing at most four constraints of (2.2) and (2.4).

Let $H=(W, F)$ be the graph induced by $[\pi, \mu]$. It suffices to show that H is an odd cycle.

– If $[\pi, \mu]$ is defined from one or two constraints of $P_B(\bar{G}_1)$ and one or two constraints of $P_B(\bar{G}_2)$, then H is planar and thus should be an odd cycle.

– If $[\pi, \mu]$ is defined from, say, one constraint of $P_B(\bar{G}_1)$ and three constraints of $P_B(\bar{G}_2)$, then Let $L=F \cap (E_1 - \{e_1, e_2, e_3\})$. Hence L is a path. If $L=\emptyset$, then H is contained in $\bar{G}_2$ and thus is an odd cycle. If $L \neq \emptyset$, then, by Lemma 2.2, the coefficients of $[\pi, \mu]$ which correspond to the edges of L all are equal. If L is even (resp. odd), then let H' be the graph obtained from H by contracting $|L|-2$ (resp. $|L|-1$) edges of L. Clearly H' is a subgraph of $\bar{G}_2$. By Theorem 2.13, H' induces a facet for $P_B(H)$ and hence for $P_B(\bar{G}_2)$. Thus $H'$is an odd cycle and H so is. □

Now consider a graph $G=(V, E)$ which is obtained by means of k-sums, $1 \leq k \leq 3$, from n graphs $G_1=(V_1, E_1), G_2=(V_2, E_2), \ldots, G_n=(V_2, E_2)$. Let $\bar{G}_i$, $i=1, \ldots, n$, be the graph obtained from G_i by adding between the vertices of every edge $e \in E_i \cap E_j$, for $i \neq j$, a path consisting of two edges. A consequence of Corollary 3.1 is the following.

Corollary 3.2. *Let G be a graph obtained by means of k-sums, $1 \leq k \leq 3$, starting from n graphs $G_1, \ldots, G_n$. If $\bar{G}_1, \ldots, \bar{G}_n$ are weakly bipartite, then G is weakly bipartite.*

Corollary 3.2 can be seen as a first step toward the characterization of weakly bipartite graphs. Indeed, that characterization may need decompositions of graphs by means of k-sums, $1 \leq k \leq 3$ and then one has to make use of Corollary 3.2. In what follows we shall use Corollary 3.2 to give an alternative proof that graphs noncontractible to K_5 are weakly bipartite. As pointed out by Barahona [10], this can also be obtained from Seymour [25].

Wagner [26, 27] gave the following characterizations for graphs noncontractible to K_5 and graphs noncontractible to $K_{3,3}$.

Theorem 3.3. *A graph $G=(v, E)$ is maximally noncontractible to K_5 (resp. $K_{3,3}$) (i.e., G is not contractible to K_5 (resp. $K_{3,3}$) and for all $v_i, v_j \in V$ such that*

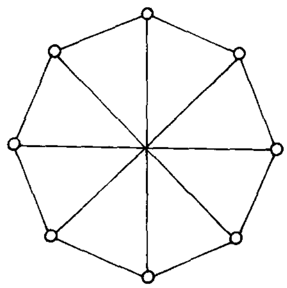

Fig. 3. The graph V_8.

$v_iv_j \notin E$, $G + v_iv_j$ is contractible to K_5 (resp. $K_{3,3}$)) if and only if it can be obtained by means of k-sums, $1 \leqslant k \leqslant 3$, of maximal planar graphs and copies of V_8 (resp. K_5) (V_8 is the graph shown in Fig. 3).

Now let $\tilde{V}_8$ (resp. $\tilde{K}_5$) be the graph obtained by adding a path consisting of two edges between the nodes of each edge in V_8 (resp. K_5). Using the results given in [5], it is not difficult to show that $\tilde{V}_8$ is weakly bipartite. Now since planar graphs are weakly bipartite, we can deduce the following theorem.

Theorem 3.4. *Graphs noncontractible to K_5 are weakly bipartite.*

Theorem 3.4 is equivalent to the following result which has been conjectured by Johnson and Gastou in [20].

Corollary 3.5. *Given a graph $G = (V, E)$ noncontractible to K_5, the polyhedron*

$$\begin{aligned} &\sum_{e \in C} x(e) \geqslant 1 \quad && \text{for all odd cycles } C \text{ in } G, \\ &x(e) \geqslant 0 && \text{for all } e \in E, \end{aligned}$$

has all vertices in 0–1.

As applications of Corollary 3.2 we also have the following corollaries.

Corollary 3.6. *Let G be a 2-sum of two graphs G_1 and G_2 such that G_1 is weakly bipartite and G_2 is bipartite, then G is weakly bipartite.*

Let Ω be the class of graphs G such that there exists a node v_0 where $G - V_0$ is bipartite. From [4], the graphs of Ω are weakly bipartite. Let G be a graph of Ω. If we subdivide an edge or a triangle of G, the resulting graph is also in Ω. Thus from the results above we may state the following corollary.

Corollary 3.7. *The k-sum, $1 \leqslant k \leqslant 3$, of a graph in Ω and a graph noncontractible to K_5 is weakly bipartite.*

4. Final remarks

The algorithmic aspect of the composition/decomposition of graphs by means of k-sums, $1 \leqslant k \leqslant 3$, introduced in Section 2, has also been studied intensively (see [19]).

Using ideas similar to those of Barahona [3], we showed that if a graph G is a k-sum, $1 \leqslant k \leqslant 3$, of two graphs G_1 and G_2, then a max-weight cut in G can be obtained from the max-weight cuts in $\bar{G}_1$ and $\bar{G}_2$ with respect to appropriate weight systems associated with $\bar{G}_1$ and $\bar{G}_2$. This yields a kind of decomposition in the determination of a max-weight cut in a graph G when G is obtained by means of k-sums, $1 \leqslant k \leqslant 3$, of more than two graphs. Moreover, we have the following main result:

Let $G = (V, E)$ be a graph obtained by means of k-sums, $1 \leqslant k \leqslant 3$, of n graphs $G_1, \ldots, G_n$. If the MCP is polynomial for the graphs $\bar{G}_1, \ldots, \bar{G}_n$, then it is polynomial for G.

Since the MCP is polynomial in planar graphs [24] and in $\tilde{V}_8$ (by enumeration of solution), by combining our result above and Theorem 3.3, we obtain that the MCP is polynomial in graphs noncontractible to K_5 as shown by Barahona [3]. Similarly we also obtain that the MCP is polynomial in graphs noncontractible to $K_{3,3}$.

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