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# TWO EDGE-DISJOINT HOP-CONSTRAINED PATHS AND POLYHEDRA*

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**Abstract.** Given a graph $G$ with distinguished nodes $s$ and $t$, a cost on each edge of $G$, and a fixed integer $L \geq 2$, the two edge-disjoint hop-constrained paths problem is to find a minimum cost subgraph such that between $s$ and $t$ there exist at least two edge-disjoint paths of length at most $L$. In this paper, we consider that problem from a polyhedral point of view. We give an integer programming formulation for the problem when $L = 2, 3$. An extension of this result to the more general case where the number of required paths is arbitrary and $L = 2, 3$ is also given. We discuss the associated polytope, $P(G, L)$, for $L = 2, 3$. In particular, we show in this case that the linear relaxation of $P(G, L)$, $Q(G, L)$, given by the trivial, the *st*-cut, and the so-called $L$-path-cut inequalities, is integral. As a consequence, we obtain a polynomial time cutting plane algorithm for the problem when $L = 2, 3$. We also give necessary and sufficient conditions for these inequalities to define facets of $P(G, L)$ for $L \geq 2$ when $G$ is complete. We finally investigate the dominant of $P(G, L)$ and give a complete description of this polyhedron for $L \geq 2$ when $P(G, L) = Q(G, L)$.

**Key words.** survivable network, edge-disjoint paths, hop-constraints, polyhedron, facet

**AMS subject classifications.** 90B10, 90C27, 90C57

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## 1. Introduction.

Given a graph $G = (N, E)$, with distinguished nodes $s$ and $t$, and a fixed integer $L \geq 2$, an *$L$-st-path* in $G$ is a path between $s$ and $t$ of length at most $L$, where the length of a path is the number of its edges. Given a function $c : E \rightarrow \mathbb{R}$ which associates a cost $c(e)$ with each edge $e \in E$, the *two edge-disjoint hop-constrained paths problem* (THPP) is to find a minimum cost subgraph such that between $s$ and $t$ there exist at least two edge-disjoint $L$-*st*-paths.

The THPP arises in the design of reliable communication networks. In fact, with the introduction of fiber optic technology in telecommunications, designing a minimum cost survivable network has become a major objective in the telecommunications industry. Survivable networks have to satisfy some connectivity requirements. As pointed out in [28], 2-edge connected networks have been shown to be cost effective and to provide an adequate level of survivability. In such networks, there are at least two edge-disjoint paths between each pair of nodes. So, if a link fails, it is always possible to reroute the traffic between two terminals along the second path.

However, this requirement is often insufficient regarding the reliability of a telecommunications network. In fact, the alternative paths could be too long to guarantee an effective routing. In data networks, such as the Internet, the elongation of the route of the information could cause a strong loss in the transfer speed. For other networks, the signal itself could be degraded by a longer routing. In such cases, the $L$-path requirement guarantees exactly the needed quality for the alternative routes.

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The THPP can also be seen as a special case of the more general problem when more than one pair of terminals is considered. This is the case, for instance, when several commodities have to be routed in the network. Thus an efficient algorithm for solving the THPP would be useful to solve (or produce upper bounds for) this more general problem.

It is clear that an optimal solution of the THPP can be computed in polynomial time by enumerating all the $L$-$st$-paths. However, in a complete graph $G = (N, E)$ with $|N| = n$, there are $\mathcal{O}(n^{L-1})$ $L$-$st$-paths, which can also be enumeratively generated in $\mathcal{O}(n^{L-1})$ time. For every pair of such paths, one has to verify their edge-disjunction, which requires $\mathcal{O}(L^2)$ comparisons. Consequently, the whole enumerative algorithm for the THPP runs in $\mathcal{O}(L^2 n^{2(L-1)})$ time. Clearly, such a method is far from being applicable in practice. One of the principal aims of this paper is to devise a more efficient algorithm for the THPP. This algorithm, which will be a cutting plane method, will be based on a complete description of the associated polytope by a system of linear inequalities.

Given a graph $G = (N, E)$ and an edge subset $F \subseteq E$, the 0-1 vector $x^F \in \mathbb{R}^E$, such that $x^F(e) = 1$ if $e \in F$ and $x^F(e) = 0$ otherwise, is called the *incidence vector* of $F$. For $L \geq 2$, the *convex hull* of the incidence vectors of the solutions of the THPP on $G$, denoted by $P(G, L)$, will be called the THPP *polytope*. Given a vector $w \in \mathbb{R}^E$ and an edge subset $F \subseteq E$, we let $w(F) = \sum_{e \in F} w(e)$. If $W \subset N$ is a node subset of $G$, then the set of edges that have only one node in $W$ is called a *cut* and is denoted by $\delta(W)$. We will write $\delta(v)$ for $\delta(\{v\})$. A cut $\delta(W)$ such that $s \in W$ and $t \in V \backslash W$ will be called an *st-cut*.

If $x^F$ is the incidence vector of the edge set $F$ of a solution of the THPP, then clearly $x^F$ satisfies the inequalities

$$x(\delta(W)) \geq 2 \quad \text{for all } st\text{-cut } \delta(W), \tag{1.1}$$

$$1 \geq x(e) \geq 0 \quad \text{for all } e \in E. \tag{1.2}$$

Inequalities (1.1) will be called *st-cut inequalities* and inequalities (1.2) *trivial inequalities.*

In [12], Dahl considers the problem of finding a minimum cost path between two given terminal nodes $s$ and $t$ of length at most $L$. He describes a class of valid inequalities for the problem and gives a complete description of the associated $L$-path polyhedron when $L \leq 3$. In particular, he introduces a class of valid inequalities as follows.

Let $V_0, V_1, \ldots, V_{L+1}$ be a partition of $N$ such that $s \in V_0$, $t \in V_{L+1}$, and $V_i \neq \emptyset$ for all $i = 1, \ldots, L$. Let $T$ be the set of edges $e = uv$, where $u \in V_i$, $v \in V_j$, and $|i - j| > 1$. Then the inequality

$$x(T) \geq 1$$

is valid for the $L$-path polyhedron.

Using the same partition, this inequality can be generalized in a straightforward way to the THPP polytope as

$$x(T) \geq 2. \tag{1.3}$$

The set $T$ is called an *L-path-cut* (or *L-star*), and a constraint of type (1.3) is called an *L-path-cut (or L-star) inequality.*

Let $Q(G, L)$ be the solution set of the system given by inequalities (1.1)–(1.3). In this paper, we show that inequalities (1.1)–(1.3), together with the integrality constraints, give an integer programming formulation of the THPP and of its generalization when more than two edge-disjoint $L$-$st$-paths are required for $L = 2, 3$. We then discuss the THPP polytope, $P(G, L)$, and show that $P(G, L) = Q(G, L)$ when $L = 2, 3$ for any graph. This yields a polynomial time cutting plane algorithm for the THPP in this case. We also give necessary and sufficient conditions for inequalities (1.1)–(1.3) to define facets for any $L \geq 2$ when the graph is complete. We finally investigate the dominant of $P(G, L)$, for which we give a complete description for any $L \geq 2$ when $P(G, L) = Q(G, L)$. As a consequence, we obtain the dominant of $P(G, L)$ when $L = 2, 3$.

Despite its interesting applications, the THPP has, to the best of our knowledge, never been studied before. There has been, however, a considerable amount of research on many related problems. In [14], Dahl and Johannessen consider the 2-path network design problem, which consists of finding a minimum cost subgraph connecting each pair of terminal nodes by at least one path of length at most 2. This problem is NP-hard. Dahl and Johannessen give an integer programming formulation for the problem and describe some classes of valid inequalities. Using these, they devise a cutting plane algorithm and present some computational results.

The closely related problem of finding a minimum cost spanning tree with hop-constraints is considered in [19], [20], [23]. Here, the hop-constraints limit the number of links between the root and any terminal in the network to a positive integer $H$. This problem is NP-complete even for $H = 2$. Gouveia [19] gives a multicommodity flow formulation for that problem and discusses a Lagrangian relaxation improving the LP bound. Gouveia [20] and Gouveia and Requejo [23] propose more efficient Lagrangian-based schemes for the problem and its Steiner version. Dahl [11] studies the problem for $H = 2$ from a polyhedral point of view and gives a complete description of the associated polytope when the graph is a wheel. Gouveia and Janssen [21] discuss a generalized problem where connectivity requirements are considered. They formulate the problem as a directed multicommodity flow model and use Lagrangian relaxation together with subgradient optimization to derive lower bounds. Gouveia and Magnanti [22] consider the problem that consists in finding a minimum spanning tree such that the number of edges in the tree between any pair of nodes is limited to a given bound (diameter). They present directed and undirected multicommodity formulations along with some computational experiments. Further hop-constrained survivable network design problems are studied in [1], [4], [5], [33], [34], [37].

In the framework of the minimum cost spanning tree problem with hop-constraints, Dahl and Gouveia [13] consider the hop-constrained path problem, that is, the problem of finding between two distinguished nodes $s$ and $t$ a minimum cost path with no more than $K$ edges when $K$ is fixed. They describe various classes of valid inequalities and show that some of these inequalities are sufficient to completely describe the associated polytope when $K \leq 3$. Then they discuss some applications to the hop-constrained minimum spanning tree problem. In [10], Coullard, Gamble, and Liu investigate the structure of the polyhedron associated with the $st$-walks of length $K$ of a graph, where a walk is a path that may go through the same node more than once. They present an extended formulation of the problem, and, using projection, they give a linear description of the associated polyhedron. They also discuss classes of facets of that polyhedron.

Itai, Perl, and Shiloach [30] study the complexity of several variants of the maximum disjoint hop-constrained paths problem. This consists in finding the maximum number of disjoint paths between two nodes $s$ and $t$ of length equal to (or bounded by) $K$, where $K$ is a positive integer. They show that the problem is NP-complete for $K \geq 5$ and polynomially solvable for some of the variants for $K \leq 4$. In particular, they devise a polynomial time algorithm for the problem when the paths must be node- (resp., edge-) disjoint and of length bounded by $K$, with $K \leq 4$ (resp., $K \leq 3$). Bley [7] addresses approximation and computational issues for the edge- (node-) disjoint hop-constrained paths problem. In particular, he shows that the problem of computing the maximum number of edge-disjoint paths between two given nodes of length equal to 3 is polynomial. This answers an open question in [30]. In [35], Li, McCormick, and Simchi-Levi study the problem of finding $K$ disjoint paths of minimum total cost between two distinguished nodes $s$ and $t$, where each edge of the graph has $K$ different costs and the $j$th edge-cost is associated with the $j$th path. They show that all the variants of the problem, when the graph is directed or undirected and the paths are edge- or node-disjoint, are NP-complete, even when $K = 2$.

Besides hop-constraints, another reliability condition, which is used in order to limit the length of the routing, requires that each link of the network belongs to a ring (cycle) of bounded length. In [16], Fortz, Labbé and Maffioli consider the 2-node connected subgraph problem with bounded rings. This problem consists in finding a minimum cost 2-node connected subgraph $(N, F)$ such that each edge of $F$ belongs to a cycle of length at most $L$. They describe several classes of facet defining inequalities for the associated polytope and devise a branch-and-cut algorithm for the problem. In [17], Fortz et al. study the edge version of that problem. They give an integer programming formulation of the problem in the space of the natural design variables and describe different classes of valid inequalities. They study the separation problem for these inequalities and discuss a branch-and-cut algorithm.

The related 2-edge connected subgraph problem and its associated polytope have also been the subject of extensive research in the past years. Grötschel and Monma [25] and Grötschel, Monma, and Stoer [26], [27] study the 2-edge connected subgraph problem within the framework of a general survivable model. They discuss the polyhedral aspects and devise cutting plane algorithms. In [36], Mahjoub shows that if $G$ is series-parallel, then the 2-edge connected subgraph polytope is completely described by the trivial and the cut inequalities. This has been generalized by Baïou and Mahjoub [2] for the Steiner 2-edge connected subgraph polytope and by Didi Biha and Mahjoub [6] for the Steiner $k$-edge connected subgraph polytope for $k$ even. In [3], Barahona and Mahjoub characterize this polytope for the class of Halin graphs. In [15], Fonlupt and Mahjoub study the fractional extreme points of the linear relaxation of the 2-edge connected subgraph polytope. They introduce an ordering on these extreme points and characterize the minimal extreme points with respect to that ordering. As a consequence, they obtain a characterization of the graph for which the linear relaxation of that polytope is integral. Kerivin, Mahjoub, and Nocq [32] describe a general class of valid inequalities for the 2-edge connected subgraph polytope, which generalizes the so-called $F$-partition inequalities [36], and introduce a branch-and-cut algorithm for the problem based on these inequalities, the trivial and the cut inequalities. Further work on the 2-edge and 2-node connected subgraph problems can be found in [9], [18], [28], [31].

The paper is organized as follows. In the next section, we give an integer programming formulation of the THPP and its generalization when the number of paths is arbitrary for $L \leq 3$. In section 3, we study the THPP polytope when $L = 2, 3$ and give our main result. In section 4, we study some structural properties of the facet defining inequalities of $P(G, L)$, which are used in section 5 for proving our main result. In section 6, we describe necessary and sufficient conditions for the inequalities (1.1)–(1.3) to be facet defining. In section 7, we discuss the dominant of $P(G, L)$, and, in section 8, we give some concluding remarks.

The rest of this section is devoted to more definitions and notation. We assume the reader has familiarity with graphs and polyhedra. For specific details, the reader is referred to [8] and [38]. The graphs that we consider are finite, undirected, loopless, and may have multiple edges. A graph is denoted by $G = (N, E)$, where $N$ is the *node set* and $E$ is the *edge set*. Given $W$, $W'$ two disjoint subsets of $N$, $[W, W']$ will denote the set of edges of $G$ having one endnode in $W$ and the other one in $W'$. If $W = \{v\}$, we will write $[v, W']$ instead of $[\{v\}, W']$. If $G$ is a graph and $e$ is an edge of $E$, then $G - e$ will denote the graph obtained from $G$ by removing $e$. A *path* $P$ of $G$ is an alternate sequence of nodes and edges $(u_1, e_1, u_2, e_2, \ldots, u_{q-1}, e_{q-1}, u_q)$, where $e_i \in [u_i, u_{i+1}]$ for $i = 1, \ldots, q-1$. We will denote a path $P$ by either its node sequence $(u_1, \ldots, u_q)$ or its edge sequence $(e_1, \ldots, e_{q-1})$.

**2. Formulation for $L = 2, 3$.** In this section, we show that the $st$-cut, $L$-path-cut, and trivial inequalities, together with integrality constraints, suffice to formulate the THPP as a 0-1 linear program when $L = 2, 3$. To this end, we first give a lemma.

LEMMA 2.1. *Let $G = (N, E)$ be a graph, $s$, $t$ be two nodes of $N$, and $L \in \{2, 3\}$. Suppose that there do not exist $k$ edge-disjoint $L$-$st$-paths in $G$, with $k \geq 2$. Then there exists a set of at most $k - 1$ edges that intersects every $L$-$st$-path.*

*Proof.* We first show the statement for $L = 3$. The proof uses ideas from [30] and [17]. Consider the capacitated directed graph $D = (N', A)$ obtained from $G$ in the following way. The set $N'$ consists of a copy $s', t'$ of $s, t$ and two copies $N_1, N_2$ of $N \backslash \{s, t\}$. For $u \in N \backslash \{s, t\}$, let $u_1$ and $u_2$ be the corresponding nodes in $N_1$ and $N_2$, respectively. To each edge $e \in [s, u]$, with $u \in N \backslash \{s, t\}$, we associate an arc $e'$ from $s'$ to $u_1$ of capacity 1. To each edge $e \in [v, t]$, with $v \in N \backslash \{s, t\}$, we associate an arc $e'$ from $v_2$ to $t'$ of capacity 1. For an edge $e \in [u, v]$, with $u, v \in N \backslash \{s, t\}$, we consider two arcs, one from $u_1$ to $v_2$ and the other from $v_1$ to $u_2$, both of capacity 1. Finally, we consider in $D$ an arc from $s'$ to $t'$ of capacity 1 for every edge in $[s, t]$ and an arc from each node of $N_1$ to its peer in $N_2$ with infinite capacity (see Figure 1 for an illustration). Note that multiple edges in $G$ yield multiple arcs in $D$. Observe that there is a one-to-one correspondence between the 3-$st$-paths in $G$ and the directed $s't'$-paths in $D$.

Now consider a maximum flow $\phi \in \mathbb{R}^A_+$ from $s'$ to $t'$ in $D$. As the capacities of $D$ are integer, $\phi$ can be supposed to be integer. Hence the flow value of each arc of capacity 1 is either 0 or 1. We claim that $\phi$ can be chosen so that no two arcs $(u_1, v_2)$ and $(v_1, u_2)$, corresponding to the same edge $uv$ in $G$, have a positive value. Indeed, suppose that $\phi(u_1, v_2) = 1$ and $\phi(v_1, u_2) = 1$. Let $\phi' \in \mathbb{R}^A_+$ be the flow given by

$$\phi'(e) = \begin{cases} \phi(e) + 1 & \text{if } e \in \{(u_1, u_2), (v_1, v_2)\}, \\ 0 & \text{if } e \in \{(u_1, v_2), (v_1, u_2)\}, \\ \phi(e) & \text{otherwise.} \end{cases}$$

As $(u_1, u_2)$ and $(v_1, v_2)$ have infinite capacity and the flow going into $u_2$ and $v_2$ has not changed, $\phi'$ is still feasible. Moreover, $\phi'$ has the same value as $\phi$.

As a consequence, an $s't'$-flow of value $q$ in $D$ corresponds to $q$ edge-disjoint 3-$st$-paths in $G$. Since there do not exist, in $G$, $k$ edge-disjoint 3-$st$-paths, the maximum flow in $D$ is of value at most $k - 1$. Hence a minimum $st$-cut in $D$ is of value at most $k - 1$ as well. Observe that such a cut does not contain arcs with infinite capacity. Hence, a minimum cut corresponds to a set of at most $k - 1$ edges that intersects all the 3-$st$-paths of $G$, and the proof for $L = 3$ is complete.

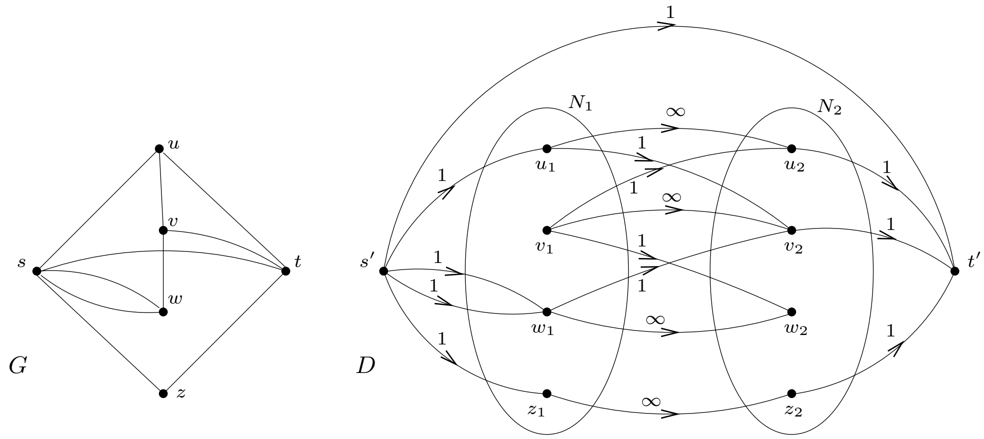


FIG. 1.

If $L = 2$, then we can similarly show the statement by considering the digraph $D = (N', A)$, where $N'$ is a copy of $N$ and to every edge $e \in [s, u]$ (resp., $[u, t]$), where $u \in N\backslash\{s, t\}$, corresponds an arc $e'$ from $s'$ to $u'$ (resp., $u'$ to $t'$) of capacity 1 in $D$. Here $u'$ is the copy of $u$ in $N'$ for every $u \in N$. □

THEOREM 2.2. *Let $G = (N, E)$ be a graph and $L \in \{2, 3\}$. Then the THPP is equivalent to the integer program*

$$\text{Min}\{cx;\ x \in Q(G, L),\ x \in \{0, 1\}^E\}.$$

*Proof.* To prove the theorem, it is sufficient to show that every 0-1 solution $x$ of $Q(G, L)$ induces a solution of the THPP. Let us assume the contrary. Suppose that $x$ does not induce a solution of the THPP but satisfies the $st$-cut and trivial constraints. We will show that $x$ necessarily violates at least one of the $L$-path-cut constraints $x(T) \geq 2$. Let $G_x$ be the subgraph induced by $x$. As $x$ is not a solution of the problem, $G_x$ does not contain two edge-disjoint $L$-$st$-paths. As $L \in \{2, 3\}$, it follows, by Lemma 2.1, that there exists at most one edge in $G_x$ that intersects every $L$-$st$-path. Consider the graph $\tilde{G}_x$ obtained from $G_x$ by deleting this edge. Obviously, $\tilde{G}_x$ does not contain any $L$-$st$-path.

We claim that $\tilde{G}_x$ contains at least one $st$-path of length at least $L+1$. In fact, as $x$ is a 0-1 solution and satisfies the $st$-cut inequalities, $G_x$ contains at least two edge-disjoint $st$-paths. Since at most one edge was removed from $G_x$, at least one path remains between $s$ and $t$ in $\tilde{G}_x$. However, since $\tilde{G}_x$ does not contain an $L$-$st$-path, that path must be of length at least $L + 1$.

Now consider the partition $V_0, \ldots, V_{L+1}$ of $N$, with $V_0 = \{s\}$, $V_i$ the set of nodes at distance $i$ from $s$ in $\tilde{G}_x$ for $i = 1, \ldots, L$, and $V_{L+1} = N \setminus (\bigcup_{i=0}^{L} V_i)$, where the distance between two nodes is the length of a shortest path between these nodes. Since there does not exist an $L$-$st$-path in $\tilde{G}_x$, it is clear that $t \in V_{L+1}$. Moreover, as, by the claim above, $\tilde{G}_x$ contains an $st$-path of length at least $L+1$, the sets $V_1, \ldots, V_L$ are nonempty. Furthermore, no edge of $\tilde{G}_x$ is a chord of the partition (that is, an edge between two sets $V_i$ and $V_j$, where $|i-j| > 1$). In fact, suppose that there exists an edge $e = v_iv_j \in [V_i, V_j]$ with $|i-j| > 1$ and $i < j$. Therefore $v_j$ is at distance $i+1$ from $s$, a contradiction.

Thus, the edge deleted from $G_x$ is the only edge that may be a chord of the partition in $G_x$. In consequence, if $T$ is the set of chords of the partition in $G$, then

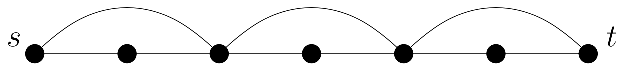


FIG. 2.

$x(T) \leq 1$. But this implies that the corresponding $L$-path-cut inequality is violated by $x$. $\square$

If $L \geq 4$, inequalities (1.1)–(1.3), together with the integrality constraints $x(e) \in \{0,1\}$ for all $e \in E$, do not suffice to formulate the THPP as an integer program. Indeed, suppose that $L = 4$ and consider the graph shown in Figure 2. It is not hard to see that the solution induced by this graph satisfies inequalities (1.1)–(1.3), whereas the graph itself is not a feasible solution of the THPP.

However, using Lemma 2.1, Theorem 2.2 can be easily extended to the case where $L \in \{2,3\}$ and the number $k$ of required $L$-$st$-paths is arbitrary. In other words, the problem in this case is equivalent to the integer program

$$\text{Min } \{cx;\ x \in Q_k(G,L),\ x \in \{0,1\}^E\}, \tag{2.1}$$

where $Q_k(G,L)$ is obtained from $Q(G,L)$ $(= Q_2(G,L))$ by replacing the right-hand side of inequalities (1.1) and (1.3) by $k$.

The separation problem for a system of inequalities consists in verifying whether a given solution $x^* \in \mathbb{R}^E$ satisfies the system and, if not, in finding an inequality of the system that is violated by $x^*$. The separation problem for inequalities (1.1) can be solved in polynomial time using any polynomial max-flow algorithm (see, e.g., [29]). Inequalities (1.3) can also be separated in polynomial time when $L \leq 3$. In fact, in this case, it is not hard to see that the separation problem reduces to finding a minimum weight edge subset that intersects all $L$-$st$-paths. Recently, Fortz et al. [17] have shown that this problem reduces to a max-flow problem (as described in the proof of Lemma 2.1) and hence can be solved in polynomial time.

Thus, by the ellipsoid method [24], problem (2.1) can be solved in polynomial time. It would then be interesting to characterize the graphs for which $Q_k(G,L)$ is integral. In what follows, we will show that for $k = 2$, that is, when (2.1) corresponds to the THPP, $Q_k(G,L)$ is integral for any graph for $L = 2,3$.

## 3. THPP polytope for $L = 2,3$.

We first state our main result.

THEOREM 3.1. *$P(G,L) = Q(G,L)$ if $L = 2,3$.*

The proof of this theorem will be given in section 5. In what follows, we shall discuss the dimension of $P(G,L)$ and study some properties of its facial structure. Let $G = (N,E)$ be a graph. An edge $e \in E$ will be called *$L$-$st$-essential* if $e$ belongs to an $st$-cut of cardinality 2 or an $L$-path-cut of cardinality 2. Let $E^*$ denote the set of $L$-$st$-essential edges. Thus, $P(G-e,L) = \emptyset$ for all $e \in E^*$. The following theorem, which is easily seen to be true, characterizes the dimension of the polytope $P(G,L)$.

THEOREM 3.2. *If $L = 2,3$, $\dim(P(G,L)) = |E| - |E^*|$.*

COROLLARY 3.3. *If $G = (N,E)$ is complete with $|N| \geq 4$ and $L = 2,3$, then $P(G,L)$ is full dimensional.*

The following theorem gives a procedure for obtaining a linear description of the THPP polytope for a subgraph of $G$ from that corresponding to $G$.

THEOREM 3.4. *Let $G = (N,E)$ be a graph, $s,t$ be two nodes of $N$, and $L \geq 2$ be an integer. Let $e$ be an edge of $E$. Let $G' = (N,E')$ be the graph obtained from $G$ by deleting $e$. Then a linear system describing $P(G',L)$ can be obtained from a system describing $P(G,L)$ by removing the variables corresponding to $e$.*

*Proof.* The proof is easy. □

In the following, we will suppose that $G = (N, E)$ is complete with $|N| \geq 4$ and $L = 2, 3$. Hence, by Theorem 3.2, $P(G, L)$ is full dimensional. If $G = (N, E)$ is not complete, then a description of $P(G, L)$ can be obtained from that of $P(\overline{G}, L)$, by repeatedly using Theorem 3.4. Here $\overline{G}$ is the complete graph obtained from $G$ by adding the missing edges. Moreover, it is clear that the problem can be reduced to that case by associating a big cost with the missing edges in the graph.

Let

$$T(G) = \{F \subseteq E \,|\, (N, F) \text{ is a solution of the THPP}\}.$$

Given an inequality $ax \geq \alpha$ that defines a facet of $P(G, L)$, we let

$$\tau_a = \{F \in T(G) \,|\, ax^F = \alpha\}.$$

In what follows, we will consider $a(e)$ as a weight on $e$. Hence, any solution $S$ of $\tau_a$ will have a weight $a(S)$ equal to $\alpha$ and any solution of $T(G)$ a weight $\geq \alpha$.

LEMMA 3.5. (i) *Let $ax \geq \alpha$ be a facet defining inequality of $P(G, L)$ different from the trivial inequalities. Then for every edge $e \in E$, there exists an edge subset in $\tau_a$ that contains $e$ and another one that does not.*

(ii) *Let $ax \geq \alpha$ be a facet defining inequality of $P(G, L)$ different from the st-cut inequalities. Then, for every st-cut $\delta(W)$, there exists an edge subset in $\tau_a$ containing at least three edges of $\delta(W)$.*

*Proof.* The proof is easy. □

Lemma 3.5 will be frequently used in what follows. At times we will use it without referring to it explicitly.

LEMMA 3.6. *Let $ax \geq \alpha$ be a facet defining inequality of $P(G, L)$ different from a trivial inequality. Then $a(e) \geq 0$ for all $e \in E$ and $\alpha > 0$.*

*Proof.* Assume, on the contrary, that there is an edge $e \in E$ such that $a(e) < 0$. Since $ax \geq \alpha$ is different from $-x(e) \geq -1$, by Lemma 3.5(i), there must exist a solution $S$ of $\tau_a$ that does not contain $e$. As $S' = S \cup \{e\}$ still belongs to $T(G)$, this yields $\alpha \leq ax^{S'} = ax^S + a(e) < ax^S = \alpha$, a contradiction. Thus, $a(e) \geq 0$ for all $e \in E$. Since $ax \geq \alpha$ defines a facet of $P(G, L)$, there must exist at least one edge, say $f$, with $a(f) > 0$. Now, as $ax \geq \alpha$ is different from the inequality $x_f \geq 0$, there is an edge set of $\tau_a$ containing $f$. This implies that $\alpha > 0$. □

The following lemma shows that parallel edges in $G$ have the same coefficient in every nontrivial facet defining inequality of $P(G, L)$ for $L = 2, 3$.

LEMMA 3.7. *Let $ax \geq \alpha$ be a facet defining inequality of $P(G, L)$ different from the trivial inequalities. Let $[u, v] = \{e_1, e_2, \dots, e_p\}$ be the set of the parallel edges between two nodes $u$ and $v$ in $G$. Then $a(e_i) = a(e_j)$ for $i, j = 1, \dots, p$.*

*Proof.* We will show the result for $L = 3$. The proof for $L = 2$ is similar. First we show that all edges in $[u, v]$ have the same coefficient, except possibly one, that may have a smaller coefficient. Indeed, if there are three edges $e_1, e_2, e_3 \in [u, v]$ such that $a(e_1) > a(e_2) \geq a(e_3)$, then there cannot exist an edge subset of $\tau_a$ containing $e_1$. Otherwise, one could replace $e_1$ by either $e_2$ or $e_3$ and get a solution which violates $ax \geq \alpha$, a contradiction. Now, suppose that there are two edges $e_1, e_2 \in [u, v]$ such that $a(e_1) > a(e_2)$. By the remark above, it follows that $a(e) = a(e_1)$ for all $e \in [u, v] \backslash \{e_1, e_2\}$.

*Claim* 1. Let $S$ be a solution of $\tau_a$.

(i) If $S$ contains $e_1$, then it must contain $e_2$.

(ii) If $S$ does not contain $e_2$, then it does not intersect $[u, v]$.

*Proof.* (i) If $e_1 \in S$ and $e_2 \notin S$, then $S' = (S \backslash \{e_1\}) \cup \{e_2\}$ is in $T(G)$. As $ax^{S'} < \alpha$, we have a contradiction.

(ii) Assume the contrary. Then we may suppose that $S$ contains an edge $e_i$, $i \in \{1, \ldots, p\} \backslash \{2\}$, and $e_2 \notin S$. Since $a(e_i) > a(e_2)$, this is impossible by the argument given above. □

Now, since $ax \geq \alpha$ is different from a trivial inequality, by Lemma 3.5(i), there is an edge set of $\tau_a$, say $S_1$, containing $e_1$. Let $L_1$ be a 3-$st$-path of $S_1$ that contains $e_1$. By Claim 1(i), it follows that $e_2$ belongs to the second 3-$st$-path of $S_1$, say $L_2$. Note that $L_1 \cap L_2 = \emptyset$. It is not hard to see that $L_1$ and $L_2$ go through $e_1$ and $e_2$, respectively, in the same direction starting from $s$. If not, one would have one path of the form $(s, u, v, t)$ and the other one of the form $(s, v, u, t)$. But then the edges $e_1, e_2$ might be deleted and one would obtain a feasible solution of weight smaller than $\alpha$, a contradiction. So, let us assume, without loss of generality (w.l.o.g.), that $u$ is the first node of $e_1, e_2$ used by $L_1, L_2$ going in this direction.

Let $L_1^s, L_1^t$ (resp., $L_2^s, L_2^t$) be the subpaths of $L_1$ (resp., $L_2$) between $s$ and $u$ and between $v$ and $t$. Obviously, $|L_i^s \cup L_i^t| \leq 2$ for $i = 1, 2$. Note that we have either $L_1^s = \emptyset = L_2^s$ or $L_1^s \neq \emptyset \neq L_2^s$. Moreover, if the latter case holds, we have that $|L_1^t| \leq 1$ and $|L_2^t| \leq 1$. Note also that, by symmetry, these properties remain true if we exchange $s$ and $t$. Thus every $st$-path consisting of a combination of subpaths $L_i^s \cup \{e_j\} \cup L_k^t$ is of length at most 3 for $i, j, k = 1, 2$. In other words, we have that

$$|L_i^s \cup L_k^t| \leq 2 \qquad \text{for all } i, k \in \{1, 2\}.$$

By Lemma 3.5(i), there must also exist an edge set of $\tau_a$, say $S_2$, that does not contain $e_2$. By Claim 1(ii), we have that $[u, v] \cap S_2 = \emptyset$. Let $P_1$ and $P_2$ be two edge-disjoint 3-$st$-paths in $S_2$. We have the following claim.

*Claim* 2. At least one of the sets $P_1 \cap L_1$ and $P_2 \cap L_2$ ($P_2 \cap L_1$ and $P_1 \cap L_2$) is nonempty.

*Proof.* Assume, on the contrary, that, for instance, $P_1 \cap L_1 = \emptyset = P_2 \cap L_2$. Then, since $P_2 \cup L_2 \in T(G)$, it follows that $a(P_2) \geq a(L_1)$. Now, let $L_1' = (L_1 \backslash \{e_1\}) \cup \{e_2\}$. As $e_2 \notin S_2$ and hence $e_2 \notin P_1$, we have that $P_1 \cap L_1' = \emptyset$. Thus $P_1 \cup L_1' \in T(G)$, and therefore $a(L_1') \geq a(P_2)$. As a consequence, $a(L_1') \geq a(L_1)$, and hence $a(e_2) \geq a(e_1)$, a contradiction. □

By Claim 2, we may assume, w.l.o.g., that $P_1 \cap L_2 \neq \emptyset$. Also by the same claim, at least one of the sets $P_1 \cap L_1$ and $P_2 \cap L_2$ is nonempty. In what follows, we suppose that $P_2 \cap L_2 \neq \emptyset$. The case where $P_1 \cap L_1 \neq \emptyset$ can be treated along the same lines. As $e_2 \notin S_2$, it follows that $|L_2| = 3$. If $|L_2^s| = 2$, then $v = t$, and $L_2$ is of the form $(s, w, u, t)$ with $w \neq s, t, u$. Let $e_0$ be the edge of $L_2 \cap [u, w]$. Note that one of the 3-$st$-paths of $S_2$, say $P_1$, uses $e_0$. Then $P_1$ is of the form $(s, u, w, t)$. Let $\{f\} = P_1 \cap [w, t]$. As $(S_1 \backslash \{e_0, e_1\}) \cup \{f\}$ and $(S_2 \backslash \{e_0, f\}) \cup \{e_2\}$ are edge sets of $T(G)$, we obtain that $a(f) \geq a(e_0) + a(e_1)$ and $a(e_2) \geq a(e_0) + a(f)$, respectively. But this implies that $a(e_2) \geq a(e_1)$, a contradiction.

Consequently, $|L_2^s| \leq 1$, and, by symmetry, we also have that $|L_2^t| \leq 1$. Since $|L_2| = 3$, it follows that $|L_2^s| = |L_2^t| = 1$. So $L_1$ and $L_2$ are both of the form $(s, u, v, t)$. As $P_1 \cap L_2 \neq \emptyset \neq P_2 \cap L_2$ and $S_2 \cap [u, v] = \emptyset$, we may assume, w.l.o.g., that $P_1 \cap [s, u] \neq \emptyset$ and $P_2 \cap [v, t] \neq \emptyset$. Moreover, this implies that $P_1 \cap L_1 = \emptyset = P_2 \cap L_1$. Now, by replacing $e_1$ and $L_1^t$ by the subpath $P_1^{ut}$ of $P_1$ between $u$ and $t$, we get a solution, yielding $a(P_1^{ut}) \geq a(e_1) + a(L_1^t)$. Similarly, if we replace $P_1^{ut}$ by $e_2$ and $L_1^t$ in $S_2$, we obtain that $a(e_2) + a(L_1^t) \geq a(P_1^{ut})$. But this again yields $a(e_2) \geq a(e_1)$, which is impossible. □

By Lemma 3.7, the multiple edges have the same coefficient in any nontrivial facet of $P(G, L)$. For the rest of the paper, if $u, v \in N$, we will denote by $uv$ a fixed edge of $[u, v]$. If $P$ is a path of the form $(u_1, u_2, \ldots, u_q)$, then we will suppose that $P$ uses the edges $u_1u_2, \ldots, u_{q-1}u_q$. If for a solution $S \in T(G)$ and two nodes $u, v \in N$ we have that $S$ intersects $[u, v]$, then we will suppose that $S$ uses edge $uv$ and eventually further edges parallel to $uv$.

**4. Structural properties.** In this section we give some structural properties of the facet defining inequalities of $P(G, L)$ different from the trivial and the $st$-cut inequalities. These will be useful for the proof of our main result in section 5.

Let $L = 2, 3$ and $ax \geq \alpha$ be a facet defining inequality of $P(G, L)$ different from the trivial and the $st$-cut inequalities. First, we give the following technical lemma, which will be frequently used in the subsequent proofs.

LEMMA 4.1. *Let $S_1$ and $S_2$ be two edge sets of $\tau_a$. Let $P_1$ and $P'_1$ be two edge-disjoint $L$-$st$-paths of $S_1$. Suppose that there is an $L$-$st$-path $P_2$ in $S_2$ such that $P_2 \cap P'_1 = \emptyset$. Then, for every $L$-$st$-path $P$ not intersecting $S_2$, we have $a(P) \geq a(P_1)$.*

*Proof.* Let $S'_1$ (resp., $S'_2$) be the edge set obtained from $S_1$ (resp., $S_2$) by replacing $P_1$ by $P_2$ (resp., $P_2$ by $P$). As $S'_1, S'_2 \in T(G)$, it follows that $a(P_2) \geq a(P_1)$ and $a(P) \geq a(P_2)$. Hence, $a(P) \geq a(P_1)$. ∎

LEMMA 4.2. *There cannot exist an $L$-$st$-path containing only edges with zero weight.*

*Proof.* We will show the result for $L = 3$. The proof for $L = 2$ can be done in a similar way.

Let us assume the contrary. Let $P_0$ be a shortest $st$-path such that $a(e) = 0$ for all $e \in P_0$. In what follows, we consider the case where $|P_0| = 3$. The cases where $|P_0| = 2$ or 1 can be treated similarly.

Let $P_0 = (s, u_1, u_2, t)$. Then $a(e) > 0$ for every chord of $P_0$. By Lemma 3.7, we have $a(e) = 0$ for all $e \in [s, u_1] \cup [u_1, u_2] \cup [u_2, t]$. As $ax \geq \alpha$ is different from a trivial inequality, by Lemma 3.5(i), there must exist an edge set $S$ of $\tau_a$ not containing the edge $u_2t$ of $P_0$. Let $P_1, P_2$ be two edge-disjoint 3-$st$-paths of $S$.

*Claim* 1. Let $T$ be a solution of $\tau_a$ and $T_1, T_2$ be two edge-disjoint 3-$st$-paths of $T$. Then at least one of the paths $T_1, T_2$ has only edges with zero value if one of the following statements holds:

(i) $u_2t \notin T$,

(ii) $su_1 \notin T$,

(iii) $u_1u_2 \notin T$ and $|[u_2, t]| \geq 2$.

*Proof.* Suppose that both $T_1$ and $T_2$ use edges with positive weight. We first claim that both $T_1$ and $T_2$ intersect $P_0$. Otherwise, if, for instance, $T_1 \cap P_0 = \emptyset$, then $T_1 \cup P_0 \in T(G)$, yielding $a(P_0) \geq a(T_2)$. As $a(P_0) = 0$, we then have $a(T_2) = 0$, a contradiction.

Now suppose that $u_2t \notin T$. As $T_1 \cap T_2 = \emptyset$, one of the paths, say $T_1$, uses edge $u_1u_2$. Since $T_1$ uses at least one edge of positive weight and $a(e) = 0$ for all $e \in [s, u_1] \cup [u_2, t]$, $T_1$ must be of the form $(s, u_2, u_1, t)$. By the remark above, we have indeed that $a(u_1t) > 0$. Now if we replace in $T$ the edges $u_1u_2$ and $u_1t$ by $u_2t$, we get a solution of $T(G)$. Moreover, as $a(u_2t) = a(u_1u_2) = 0$, it follows that $a(u_1t) = 0$, a contradiction.

If $su_1 \notin T$, then the statement follows by symmetry.

Suppose now that $u_1u_2 \notin T$ and $|[u_2, t]| \geq 2$. Denote by $f$ an edge of $[u_2, t] \backslash \{u_2t\}$. Since $u_1u_2 \notin T$ and $T_1 \cap P_0 \neq \emptyset \neq T_2 \cap P_0$, we may suppose, w.l.o.g., that $su_1 \in T_1$ and $u_2t \in T_2$. Let $T_1^{u_1t}$ be the subpath of $T_1$ between $u_1$ and $t$. Observe that $a(T_1^{u_1t}) > 0$. Consider the solution obtained from $T$ by replacing $T_1^{u_1t}$ by the edges $u_1u_2$ and $f$.

As $a(f) = a(u_1u_2) = 0$, this yields $a(T_1^{u_1t}) = 0$, a contradiction, which ends the proof of the claim. ☐

As $u_2t \notin S$, by Claim 1(i), it follows that at least one of the paths $P_1$ and $P_2$, say $P_1$, contains only edges with zero coefficient. Moreover, we have that $P_1 \cap P_0 \neq \emptyset$. Otherwise, there would exist a solution formed by $P_1$ and $P_0$ of weight zero, contradicting the fact that $\alpha > 0$.

*Claim* 2. (i) $|[u_2, t]| \geq 2$.

(ii) $|[s, u_1]| \geq 2$.

*Proof.* We will prove (i); the proof of (ii) follows by symmetry. Suppose that $|[u_2, t]| = 1$. We claim that the edge $su_1$ of $P_0$ belongs to $P_1$. In fact, if this is not the case, as $u_2t \notin S$ and $P_1 \cap P_0 \neq \emptyset$, $P_1$ must contain the edge $u_1u_2$. As $|[u_2, t]| = 1$ and $u_2t \notin S$, $P_1$ must use an edge of $[u_1, t]$ which is of positive weight, a contradiction. Thus $P_1$ is of the form $(s, u_1, v, t)$ with $v \neq u_2$. We thus have $|[s, u_1]| = 1$. Otherwise, we would have two edge-disjoint 3-$st$-paths of zero weight, yielding $\alpha = 0$, a contradiction. By considering a solution of $\tau_a$ not containing $su_1$ and using Claim 1(ii) together with similar arguments as above, we can show that there exists a path $P_1'$ of the form $(s, w, u_2, t)$, with $w \neq u_1$, constituted of edges with zero coefficient. As $P_1$ and $P_1'$ are edge-disjoint and hence form a solution of $T(G)$, this yields $\alpha = 0$, a contradiction. ☐

Since there are no two edge-disjoint 3-$st$-paths of weight zero, at least one of the sets $[s, u_1]$, $[u_1, u_2]$, $[u_2, t]$ must be reduced to a single edge. Consequently, by Claim 2, it follows that $|[u_1, u_2]| = 1$. Consider now a solution $S'$ of $\tau_a$ not containing $u_1u_2$. Let $P_1'$ and $P_2'$ be two edge-disjoint 3-$st$-paths of $S'$. As, by Claim 2(ii), $|[u_2, t]| \geq 2$, we may, w.l.o.g., suppose by Claim 1(iii) that $a(P_1') = 0$. Also, since $\alpha > 0$, one should have $P_1' \cap P_0 \neq \emptyset$. Since $u_1u_2 \notin S'$, we may, w.l.o.g., suppose that $su_1 \in P_1'$. Therefore $P_1' = (s, u_1, v', t)$ with $v' \neq u_2$. As, by Claim 2, $|[s, u_1]| \geq 2$, the solution given by $P_0 \cup \tilde{P}_1$, where $\tilde{P}_1 = (f, u_1v', v't)$ with $f \in [s, u_1] \backslash \{su_1\}$, would be in $T(G)$ and of zero weight. But this is a contradiction, and the proof of the lemma is complete. ☐

Let us denote by $U$ (resp., $V$) the subset of nodes $u$ such that $a(e) = 0$ for all $e \in [s, u]$ (resp., $e \in [u, t]$). Note that, by Lemma 3.7, if for an edge $f \in [s, u]$ (resp., $f \in [u, t]$) for some $u \in N \backslash \{s, t\}$ we have $a(f) = 0$, then $u \in U$ (resp., $u \in V$). By Lemma 4.2, we have that $U \cap V = \emptyset$. Moreover, $a(e) > 0$ for all $e \in [s, t] \cup [s, V] \cup [U, t]$. If $L = 3$, we also have that $a(e) > 0$ for all $e \in [U, V]$. Let $W = N \backslash (\{s, t\} \cup U \cup V)$. Note that if $W \neq \emptyset$, $a(e) > 0$ for all $e \in [s, W] \cup [W, t]$.

Lemma 4.3. $U \neq \emptyset \neq V$.

*Proof.* We will prove the lemma for $U$. The proof for $V$ is similar. Since $ax \geq \alpha$ is different from the $st$-cut constraint corresponding to the node $s$, by Lemma 3.5(ii), there is an edge set $F$ of $\tau_a$ that contains at least three edges of $\delta(s)$. As only two of these edges can be used by two edge-disjoint $L$-$st$-paths of $F$, there is an edge of $F \cap \delta(s)$, say $e_0 \in [s, u]$ with $u \in N \backslash \{s, t\}$, such that $F \backslash \{e_0\} \in T(G)$. This implies that $a(e_0) = 0$, and therefore $u \in U$. ☐

Lemma 4.4. *Let $S \in \tau_a$ and $P_1$ be a 3-$st$-path of $S$ going through a node $u$ of $N \backslash \{s, t\}$. Let $\tilde{P}_1$ be the subpath of $P_1$ between $s$ (resp., $t$) and $u$. Let $P$ be a path between $s$ (resp., $t$) and $u$ such that $a(P) = 0$ and $|P| \leq |\tilde{P}_1|$. If $a(\tilde{P}_1) > 0$, then $P \cap P_2 \neq \emptyset$ for any 3-$st$-path $P_2$ of $S$, where $P_2 \cap P_1 = \emptyset$.*

*Proof.* If $P \cap P_2 = \emptyset$, as $|P| \leq |\tilde{P}_1|$, the edge set $(S \backslash \tilde{P}_1) \cup P$ belongs to $T(G)$, and hence $a(\tilde{P}_1) \leq a(P)$. As $a(P) = 0$ and $a(\tilde{P}_1) > 0$, this is impossible. ☐

The following lemma shows that the edges having both endnodes in $U$ ($V$) all have zero coefficient. Moreover, if $L = 2$, the same holds for the edges between $U$ and $V$.

LEMMA 4.5. (i) *If* $L = 2$*, then* $a(e) = 0$ *for all* $e \in [U, V]$.
(ii) $a(e) = 0$ *for all* $e \in E(U) \cup E(V)$.

*Proof.* (i) Let $e \in [U, V]$, and let $S$ be a solution of $\tau_a$ containing $e$. As $e$ cannot belong to a 2-$st$-path of $S$, $S \backslash \{e\}$ is also a solution of $T(G)$, and therefore $a(e) = 0$.

(ii) If $L = 2$ and $e \in E(U) \cup E(V)$, we can show as in (i) that $a(e) = 0$. Now let us consider the case where $L = 3$. Let us assume, on the contrary, that there exists an edge $u_1u_2$ with $u_1, u_2 \in U$ (the case where $u_1, u_2 \in V$ is similar) such that $a(u_1u_2) > 0$. Note that by Lemma 3.7 it follows that $a(e) > 0$ for all $e \in [u_1, u_2]$. Let us consider an edge set of $\tau_a$, say $S_1$, that contains $u_1u_2$, and let $P_1, P_1'$ be two edge-disjoint 3-$st$-paths in $S_1$. As $a(u_1u_2) > 0$, $u_1u_2$ must be in one of the 3-$st$-paths, say $P_1$. We can suppose, w.l.o.g., that $P_1$ is $(s, u_1, u_2, t)$. Moreover, as $a(e) = 0$ for all $e \in [s, u_2]$, by Lemma 4.4, $P_1'$ must contain every edge of $[s, u_2]$. However, this is possible only if $|[s, u_2]| = 1$. Consequently, we will assume in the rest of the proof that $[s, u_2] = \{su_2\}$ and $su_2 \in P_1'$. Let us assume that $P_1'$ is of the form $(s, u_2, z, t)$ with $z \neq s, t, u_2$. If $P_1'$ consists of only two edges, then the proof is similar. Furthermore, $z \notin U$. Otherwise, one can consider the edge set $S_1' = (S_1 \backslash \{su_1, u_1u_2, u_2z\}) \cup \{sz\}$, which is a solution of $T(G)$. As $a(sz) = 0$, we get $a(su_1) + a(u_1u_2) + a(u_2z) \leq 0$, and hence $a(u_1u_2) = 0$, a contradiction. Therefore $z \in V \cup W$.

Moreover, we have that $a(e) > 0$ for all $e \in [U \backslash \{u_1, u_2\}, u_2]$. Indeed, if $a(e) = 0$, then the edge set $(S_1 \backslash \{su_1, u_1u_2\}) \cup \{su, e\}$, where $u$ is the endnode of $e$ different from $u_2$, would be a solution of $T(G)$ with a weight smaller than $\alpha$, a contradiction.

Now, let us consider an edge set of $\tau_a$, say $S_2$, that does not contain the edge $su_2$. Let $P_2, P_2'$ be two edge-disjoint 3-$st$-paths in $S_2$. We claim that $[u_2, t] \cap S_2 = \emptyset$. In fact, if one of the 3-$st$-paths of $S_2$, say $P_2$, uses an edge of $[u_2, t]$, say $u_2t$, as $|[s, u_2]| = 1$ and $su_2 \notin S_2$, one should have $P_2 = (sw, wu_2, u_2t)$, where $w \in N \backslash \{s, u_2, t\}$. Moreover, we have $a(sw) + a(wu_2) > 0$. In fact, this is clear if $w \notin U$. If $w \in U$, then, as shown above, $a(wu_2) > 0$ and the statement follows. Now, by replacing in $S_2$ the subpath $(sw, wu_2)$ by $su_2$, we get a solution of smaller weight, which is impossible.

Thus $[u_2, t] \cap S_2 = \emptyset$, and hence, as $su_2 \notin S_2$, no 3-$st$-path in $S_2$ goes through the node $u_2$. Let $P$ be the path $(su_2, u_2t)$. Thus, $P \cap S_2 = \emptyset$. Moreover, as neither $su_2$ nor $u_2z$ belongs to $S_2$, at most one of the paths $P_2, P_2'$ intersects $P_1'$. W.l.o.g., we may suppose that $P_2 \cap P_1' = \emptyset$. From Lemma 4.1, it then follows that $a(P) \geq a(P_1)$. But this implies that $a(u_1u_2) = 0$, a contradiction. ◻

LEMMA 4.6. (i) *If* $L = 2$*, then* $W = \emptyset$.
(ii) *If* $L = 3$*, then* $W \neq \emptyset$.

*Proof.* (i) Assume the contrary, and let $w \in W$. Then $a(e) > 0$ for all $e \in [s, w] \cup [w, t]$. We will show that $|[s, w] \cap F| = |[w, t] \cap F|$ for every $F \in \tau_a$. In fact, suppose, by contradiction, that there exists $F \in \tau_a$ such that, for instance, $|[s, w] \cap F| > |[w, t] \cap F|$. Since at most $|[w, t] \cap F|$ edge-disjoint 2-$st$-paths can go through $w$, there must exist an edge, say $\bar{e}$, of $[s, w] \cap F$ such that $F \backslash \{\bar{e}\} \in T(G)$. This implies that $a(\bar{e}) = 0$, a contradiction. Thus, the incidence vector of any solution of $\tau_a$ verifies the equation $x([s, w]) = x([w, t])$. As, by Lemma 3.6, this equation cannot be a positive multiple of $ax \geq \alpha$, we get a contradiction.

(ii) Assume that, on the contrary, $W = \emptyset$. Let $U' = U \cup \{s\}$. Since $ax \geq \alpha$ is different from the $st$-cut inequality associated with $\delta(U')$, there exists an edge set of $\tau_a$, say $F_1$, that uses at least three edges of $\delta(U')$. Let $P_1, P_1'$ be two edge-disjoint 3-$st$-paths of $F_1$. Since $W = \emptyset$, $a(e) > 0$ for all $e \in \delta(U')$, and hence every edge of $F_1 \cap \delta(U')$ must belong to one of the paths $P_1$ and $P_1'$. So, one of these paths, say $P_1$, must use at least two edges of $\delta(U')$. As any $st$-path intersects any $st$-cut an odd number of times, we have that $P_1$ contains exactly three edges of $\delta(U')$. Therefore, $P_1$ is of the form

$(s, v, u, t)$, where $u \in U$ and $v \in V$. Let $F_2 = (F_1 \backslash (P'_1 \cup \{vu\})) \cup \{su, vt\}$. Obviously, $F_2 \in T(G)$. As $a(su) = a(vt) = 0$, it follows that $a(vu) = 0$, a contradiction. ∎

For the rest of this section, we assume that $L = 3$.

LEMMA 4.7. (i) *If there are a node $w \in W$ and a node $u_1 \in U$ such that $a(u_1 w) = 0$, then $a(e) = 0$ for all $e \in [U, w]$.*

(ii) *If there are a node $w \in W$ and a node $v_1 \in V$ such that $a(wv_1) = 0$, then $a(e) = 0$ for all $e \in [w, V]$.*

*Proof.* We show the result for $U$; the proof for $V$ is similar. If $|U| = 1$, then the statement follows from Lemma 3.7. So, let us suppose that $|U| \geq 2$ and assume, on the contrary, that there is a node $u_2 \in U$ such that $a(u_2 w) > 0$. Let $S_1$ be a solution of $\tau_a$ such that $u_2 w \in S_1$. As $a(u_2 w) > 0$, $u_2 w$ must belong to a 3-$st$-path $P_1$ in $S_1$. Let $P'_1$ be a further 3-$st$-path of $S_1$ with $P_1 \cap P'_1 = \emptyset$.

*Claim* 1. $P_1 = (s, u_2, w, t)$.

*Proof.* As $u_2 w \in P_1$, $P_1$ is either of the form $(s, w, u_2, t)$ or $(s, u_2, w, t)$. Suppose that the first case holds. As $a(e) > 0$ for all $e \in [s, w]$ and $a(e) = 0$ for all $e \in [s, u_2]$, it follows from Lemma 4.4 that $P'_1$ uses all the edges between $s$ and $u_2$. Therefore $[s, u_2] \subseteq P'_1$. Moreover, since, by Lemma 4.5(ii), all the 2-$su_2$-paths going through $u_1$ have weight zero, again by Lemma 4.4, $P'_1$ must also intersect all these paths. As $P'_1$ cannot use more than one edge incident to $s$, one should have $[u_1, u_2] \subseteq P'_1$. As a consequence, $|[s, u_2]| = |[u_1, u_2]| = 1$, and $P'_1$ is of the form $(s, u_2, u_1, t)$. But, by adding edge $su_1$ and removing the edges $sw, wu_2$, we obtain a solution of lower weight, which is impossible. ∎

Consequently, $P_1 = (s, u_2, w, t)$. As $a(u_2 w) > 0$ and therefore the weight of the subpath of $P_1$ between $s$ and $w$ is positive, it follows by Lemma 4.4 that $P'_1$ must intersect every 2-$sw$-path of weight zero going through $u_1$. Since $a(u_1 w) = 0$, by Lemma 3.7 $a(e) = 0$ for all $e \in [u_1, w]$. Thus, as $a(e) = 0$ for all $e \in [s, u_1]$, we obtain that at least one of the sets $[s, u_1]$ and $[u_1, w]$ is reduced to a single edge. If there is a node $u \in U \backslash \{u_1, u_2\}$ such that $a(e) = 0$ for some edge $e \in [u, w]$, then by Lemma 4.4, $P'_1$ must also intersect the 2-$sw$-paths going through $u$. But as $|P'_1| \leq 3$, this is not possible. Therefore $a(e) > 0$ for all $e \in [U \backslash \{u_1\}, w]$.

*Claim* 2. $P'_1 \cap [u_1, w] = \emptyset$.

*Proof.* Suppose, on the contrary, that $P'_1$ uses, for instance, $u_1 w$. If $P'_1 = (s, w, u_1, t)$, then, as the weight of the subpath of $P'_1$ between $s$ and $u_1$ is positive and $a(e) = 0$ for all $e \in [s, u_1]$, by Lemma 4.4 it follows that $P_1$ uses all the edges between $s$ and $u_1$. But this contradicts Claim 1. Hence $P'_1$ is of the form $(su_1, u_1 w, h)$, where $h \in [w, t] \backslash \{wt\}$. We consider two cases.

*Case* 1. $|[s, u_1]| = 1$. Consider an edge set $S_2$ of $\tau_a$ such that $su_1 \notin S_2$. We may suppose that $S_2$ is minimal. Let $P_2$ and $P'_2$ be the two edge-disjoint 3-$st$-paths of $S_2$. If $S_2$ uses an edge $u_1 z$ with $z \in V \cup W$, then $u_1 z$ belongs to one of the 3-$st$-paths of $S_2$, say $P_2$. As $su_1 \notin S_2$, $P_2 = (s, z, u_1, t)$. Observe that $a(e) > 0$ for all $e \in [s, z]$. Now by replacing the edges $sz, zu_1$ by $su_1$, we get a solution of $T(G)$ of weight less than $\alpha$, a contradiction. As a consequence, we have $[u_1, V \cup W] \cap S_2 = \emptyset$, and therefore $[u_1, w] \cap S_2 = \emptyset$. Suppose now that $S_2 \cap [w, t] \neq \emptyset$ and, for instance, that $P_2 \cap [w, t] \neq \emptyset$. Since $a(e) > 0$ for all $e \in [U \backslash \{u_1\}, w]$, the subpath of $P_2$, say $P_2^{sw}$, between $s$ and $w$ has a positive weight. As $\{su_1, u_1 w\}$ is a 2-$sw$-path of weight zero which does not intersect $S_2$, if we replace $P_2^{sw}$ by $su_1, u_1 w$, we get a solution of lower weight, which is impossible. Thus $S_2 \cap [w, t] = \emptyset$, and, in consequence, $P'_1 \cap S_2 = \emptyset$. Let $P = P'_1$. By Lemma 4.1, it follows that $a(P) = a(P'_1) \geq a(P_1)$. As $a(h) = a(wt)$ and $a(su_1) = a(u_1 w) = 0$, this yields $a(u_2 w) = 0$, a contradiction.

*Case* 2. $|[s,u_1]| \geq 2$. Since one of the sets $[s,u_1],[u_1w]$ contains exactly one edge, we have that $[u_1,w]=\{u_1w\}$. Let $\bar{S}_2$ be a solution of $\tau_a$ not containing $u_1w$. Suppose that $\bar{S}_2$ is minimal, and let $\bar{P}_2$ and $\bar{P}'_2$ be the two edge-disjoint 3-$st$-paths of $\bar{S}_2$. We can show, in a similar way as in Case 1, that $[w,t]\cap\bar{S}_2=\emptyset$. As $u_1w\notin\bar{S}_2$, it follows that $|\bar{S}_2\cap P'_1|\leq 1$. Hence, there is a 3-$st$-path of $\bar{S}_2$, say $\bar{P}_2$, that does not intersect $P'_1$. Therefore $\bar{P}_2\cup P'_1$ is a solution of $T(G)$, yielding $a(\bar{P}_2)\geq a(P_1)$. On the other hand, since $|[s,u_1]|\geq 2$, we may suppose that $\bar{P}'_2\cap P'_1=\emptyset$. So, if we replace, in $\bar{S}_2$, $\bar{P}_2$ by $P'_1$, we get a solution of $T(G)$, implying that $a(P'_1)\geq a(\bar{P}_2)$. Therefore $a(P'_1)\geq a(P_1)$, and hence $a(u_2w)=0$, a contradiction. ☐

By Claim 2, we then have $P'_1\cap[u_1,w]=\emptyset$. As $P'_1$ intersects all the 2-$sw$-paths going through $u_1$, it follows that $[s,u_1]=\{su_1\}$ and $su_1\in P'_1$.

If $P'_1$ uses an edge of $[u_1,t]$, then, by removing the edge $u_2w$ and adding edges $u_1w$ and $u_1u_2$, we get a solution of $T(G)$. But this implies that $a(u_2w)=0$, which is impossible. Along the same lines, we can also show that $P'_1$ does not go through any node of $U$. Hence $P'_1$ must use a node of $V\cup W$, say $v$.

Consider now a solution $S_3$ of $\tau_a$ not containing $su_1$. Let $P_3$ and $P'_3$ be two edge-disjoint 3-$st$-paths of $S_3$. Suppose that there is an edge, say $u_1z$, of $[u_1,V\cup W]$ that belongs to $S_3$. Since $su_1\notin S_3$, the 3-$st$-path containing $u_1z$, say $P_3$, must be of the form $(s,z,u_1,t)$. Note that the subpath between $s$ and $u_1$ has a positive weight. As $a(su_1)=0$, by Lemma 4.4, it follows that $su_1\in P'_3$, and hence $su_1\in S_3$, contradicting our hypothesis. Thus $[u_1,V\cup W]\cap S_3=\emptyset$, and hence $([u_1,v]\cup[u_1,w])\cap S_3=\emptyset$. Thus $|P'_1\cap S_3|\leq 1$. Consequently, there must exist a 3-$st$-path of $S_3$, say $P_3$, such that $P'_1\cap P_3=\emptyset$. Also we may show in a similar way that $[w,t]\cap S_3=\emptyset$. Consider now the path $P=(s,u_1,w,t)$. Observe that $P\cap S_3=\emptyset$. By Lemma 4.1, with respect to $S_1$ and $S_3$, it follows that $a(P)\geq a(P_1)$. But this implies that $a(u_2w)=0$, a contradiction, and the proof of the lemma is complete. ☐

LEMMA 4.8. *For all* $e,e'\in[U,t]$ *(resp.,* $e,e'\in[s,V]$*),* $a(e)=a(e')$.

*Proof.* We will prove the lemma for $U$; the proof for $V$ is similar. If $|U|=1$, the statement follows from Lemma 3.7. So suppose $|U|\geq 2$. Let $u_1,u_2\in U$ such that $a(u_1t)=\min\{a(e),e\in[U,t]\}$ and $a(u_2t)=\max\{a(e),e\in[U,t]\}$. Assume that $a(u_2t)>a(u_1t)$.

*Claim.* (i) Let $S\in\tau_a$. If $S\cap[u_2,t]\neq\emptyset$, then $[u_1,t]\subseteq S$.

(ii) $|[u_1,t]|=1$.

*Proof.* (i) Suppose that $u_2t\in S$, and let $T_1$ and $T_2$ be two edge-disjoint 3-$st$-paths of $S$. As $a(u_2t)>0$, we may suppose, for instance, that $u_2t\in T_2$. Assume that there is an edge $e_1$ of $[u_1,t]$ that is not in $S$. If there is an edge $e\in[s,u_1]$ that is not in $T_1$, then we can replace $u_2t$ by $e$ and $e_1$ and get a solution of $T(G)$ of lower weight, a contradiction. Hence $[s,u_1]\subseteq T_1$, and therefore $[s,u_1]=\{su_1\}$, $su_1\in T_1$, and $[s,u_2]\cap T_1=\emptyset$. Furthermore, if $T_1$ contains an edge $e'\in[u_1,u_2]$, then, as $su_1\in T_1$, $T_1$ must use an edge $f$ of $[u_2,t]\backslash\{u_2t\}$. Now it is easy to see that $(S\backslash\{f\})\cup\{e_1\}\in T(G)$. Since by Lemma 3.7, $a(e_1)=a(u_1t)$ and $a(f)=a(u_2t)$, it follows that $a(u_1t)\geq a(u_2t)$. But this contradicts our hypothesis. Therefore $[u_1,u_2]\cap T_1=\emptyset$. Consider now the solution $S'=(S\backslash\{u_2t\})\cup\{su_2,u_1u_2,e_1\}$. As $a(su_2)=a(u_1u_2)=0$, we have that $a(u_1t)=a(e_1)\geq a(u_2t)$, a contradiction.

(ii) Let $\bar{S}\in\tau_a$ such that $u_2t\in\bar{S}$. We may suppose that $\bar{S}$ is minimal. Let $\overline{T}_1,\overline{T}_2$ be the edge-disjoint 3-$st$-paths of $\bar{S}$, and suppose, w.l.o.g., that $u_2t\in\overline{T}_2$. From (i), it follows that $[u_1,t]\subseteq\bar{S}$. Moreover, as $u_2t\in\overline{T}_2$, $\overline{T}_2\cap[u_1,t]=\emptyset$, and hence $[u_1,t]\subseteq T_1$. This implies that $|[u_1,t]|=1$. ☐

Let $S_1$ be a solution of $\tau_a$ containing $u_2t$. By the claim above, $S_1$ also contains $u_1t$. As $a(su_1)=a(su_2)=0$ and $\{su_1,su_2,u_1t,u_2t\}$ is a solution of $T(G)$, we may assume that $S_1=\{su_1,su_2,u_1t,u_2t\}$.

Consider now a solution $S_2 \in \tau_a$ that does not contain $u_1t$, which may be supposed minimal. Since $u_1t \notin S_2$, by the claim it follows that $[u_2, t] \cap S_2 = \emptyset$; and, as a consequence, $[u_1, u_2] \cap S_2 = \emptyset$. Suppose that $S_2$ contains an edge $su_1$. Since $S_2$ is minimal, one of the two 3-$st$-paths of $S_2$, say $T$, contains $su_1$, and hence $T$ is of the form $(s, u_1, z, t)$, where $z \in N\backslash\{s, t, u_1, u_2\}$. Let $T^{u_1t}$ be the subpath of $T$ between $u_1$ and $t$. As the sets $(S_2\backslash T^{u_1t}) \cup \{u_1t\}$ and $(S_1\backslash\{u_2t\}) \cup (\{u_1u_2\} \cup T^{u_1t})$ are both solutions of $T(G)$, and, as by Lemma 4.5(ii) $a(u_1u_2) = 0$, we have that $a(u_1t) \geq a(T^{u_1t}) \geq a(u_2t)$, a contradiction. Consequently, $[s, u_1] \cap S_2 = \emptyset$.

Let $P_1 = (su_2, u_2t)$ and $P_1' = (su_1, u_1t)$ be the two 3-$st$-paths of $S_1$. Let $P = P_1'$ and $P_2$ be any 3-$st$-path of $S_2$. Note that $P \cap S_2 = P_1' \cap S_2 = \emptyset$, and hence $P_2 \cap P_1' = \emptyset$. By Lemma 4.1, it follows that $a(P) \geq a(P_1)$. However, as $a(su_1) = a(su_2) = 0$, this implies again that $a(u_1t) \geq a(u_2t)$, which is impossible. ☐

Lemma 4.9. *Let $S$ be a minimal solution of $\tau_a$.*

(i) *If $U = \{u\}$ and $S \cap [s, u] = \emptyset$, then $\delta(u) \cap S = \emptyset$.*

(ii) *If $V = \{v\}$ and $S \cap [v, t] = \emptyset$, then $\delta(v) \cap S = \emptyset$.*

*Proof.* We will show (i); the proof of (ii) is similar. We first show that $[u, t] \cap S = \emptyset$. Assume, on the contrary, that $ut \in S$. Then, as $a(ut) > 0$, one of the 3-$st$-paths of $S$, say $P$, must contain $ut$. As $[s, u] \cap S = \emptyset$, $P$ must be of the form $(s, w, u, t)$, where $w \in N\backslash\{s, t, u\}$. Note that $w \notin U$, and hence $a(sw) > 0$. Thus, one can replace $sw$ and $wu$ by $su$ in $S$ and get a solution of $T(G)$ of weight less than $\alpha$, a contradiction. Thus $[u, t] \cap S = \emptyset$. Now, by the minimality of $S$, no other edge of $\delta(u)$ may be used by $S$. ☐

Lemma 4.10. *$a(e) = a(e')$ for all $e \in [U, t]$ and $e' \in [s, V]$.*

*Proof.* Assume the contrary. Thus, by Lemma 4.8, we may assume, w.l.o.g., that

$$a(e) > a(e') \qquad \text{for all } e \in [U, t] \text{ and } e' \in [s, V]. \tag{4.1}$$

Let $u_1 \in U$. Consider a solution $S_1$ of $\tau_a$ that contains $u_1t$, and suppose that $S_1$ is minimal. Let $P_1$ and $P_1'$ be the two edge-disjoint 3-$st$-paths of $S_1$, and suppose that $u_1t \in P_1$.

*Claim.* $|V| = 1$.

*Proof.* Assume that $|V| \geq 2$. First observe that $P_1$ cannot go through a node $v \in V$. Otherwise, $P_1$ would be of the form $(s, v, u_1, t)$. Since the subpaths of $P_1$ between $s$ and $u_1$ and between $v$ and $t$ both have positive weight, by Lemma 4.4, $P_1'$ must use edges $su_1$ and $vt$. Now, if we remove the edges of $S_1$ between $u_1$ and $v$, we still have a solution of $T(G)$. This implies that $a([u_1, v]) = 0$. But this contradicts the fact that $a(u_1v) > 0$. In consequence, since $S_1$ is minimal, $S_1$ may contain at most one edge from $[s, V]$. Suppose that $S_1$ contains edge $sv_1$, where $v_1 \in V$. Note that $sv_1 \in P_1'$. As $|V| \geq 2$, there is an edge $sv_2$, with $v_2 \in V$, that does not belong to $S_1$. If there is an edge $e \in [v_2, t]$ such that $e \notin S_1$, then, by replacing $u_1t$ by $sv_2$ and $e$, we get a solution of $T(G)$. As $a(e) = 0$, this yields $a(sv_2) \geq a(u_1t)$, which contradicts (4.1). Thus $[v_2, t] \subseteq S_1$ and therefore $[v_2, t] \subseteq P_1'$. This implies that $[v_2, t] = \{v_2t\}$ and $P_1' = (s, v_1, v_2, t)$. By considering the solution obtained by replacing $u_1t$ by $sv_2$ and $v_1t$, we obtain that $a(sv_2) \geq a(u_1t)$, which once again contradicts (4.1).

Consequently, $S_1 \cap [s, V] = \emptyset$. Now we remark that, since $S_1$ is minimal and $u_1t \in S_1$, $S_1$ cannot use two edges of $[V, t]$. Thus there is a node $z \in V$ such that $([s, z] \cup [z, t]) \cap S_1 = \emptyset$. By replacing $u_1t$ by $sz$ and $zt$ in $S_1$, we get a solution of $T(G)$, yielding $a(sz) \geq a(u_1t)$. This contradicts (4.1), and the claim is proved. ☐

Let $V = \{v\}$. Let $P = (s, v, t)$ be an $st$-path of length 2 going through $v$. We claim that $P_1' \cap P \neq \emptyset$. In fact, if this is not the case, then, as the edge set obtained from $S_1$ by replacing $P_1$ by $P$ is in $T(G)$, we would have that $a(sv) \geq a(u_1t)$. But this contradicts (4.1). Therefore, $P_1'$ must contain at least one of the sets $[s,v]$ and $[v,t]$. Thus at least one of the sets $[s,v]$ and $[v,t]$ is reduced to a single edge.

*Case* 1. $[v,t]=\{vt\}$. Consider a solution $S_2 \in \tau_a$ not containing $vt$, which is supposed minimal. Then, by Lemma 4.9, $S_2 \cap \delta(v)=\emptyset$, and hence $P \cap S_2 = \emptyset$. Moreover, as $P_1' \cap P \neq \emptyset$, $P_1'$ does meet $v$, and therefore $|P_1' \cap S_2| \leq 1$. Thus there exists a 3-$st$-path of $S_2$, say $P_2$, that does not intersect $P_1'$. As $P \cap S_2 = \emptyset$, by Lemma 4.1, we have that $a(P) \geq a(P_1)$, and hence $a(sv) \geq a(u_1t)$. But this contradicts (4.1).

*Case* 2. $[s,v]=\{sv\}$. By Case 1, we may suppose that $|[v,t]| \geq 2$. As $P_1'$ contains one of the sets $[s,v]$ and $[v,t]$, it follows that $sv \in P_1'$. Note that $\{su_1, u_1t, sv, vt\} \in T(G)$. As $a(su_1)=a(vt)=0$ and $S_1$ is minimal, we may suppose, w.l.o.g., that $S_1=\{su_1,u_1t,sv,vt\}$. Hence $P_1=(su_1,u_1t)$ and $P_1'=(sv,vt)$. Consider now an edge set $S_3$ of $\tau_a$ not containing $sv$ and suppose that $S_3$ is minimal. Since $|P_1' \cap S_3| \leq 1$, there must exist a 3-$st$-path in $S_3$, say $P_3$, such that $P_3 \cap P_1' = \emptyset$. If we replace, in $S_1$, $P_1$ by $P_3$, the resulting set is still a solution of $T(G)$, and therefore $a(P_3) \geq a(P_1)$. On the other hand, if there is an edge $h \in [v,t]$ such that $h \notin S_3$, then one can replace the path $P_3$ by the one formed by $sv$ and $h$ and get a solution of $T(G)$. But this implies that $a(P_3) \leq a(sv)+a(h)$. As $a(P_3) \geq a(P_1)$ and $a(h)=0$, we obtain that $a(u_1t) \leq a(sv)$, contradicting (4.1). Thus $[v,t] \subseteq S_3$. As $|[v,t]| \geq 2$ and $S_3$ is minimal, it follows that $P_3 \cap [v,t] \neq \emptyset$. Let $P_3^{sv}$ be the subpath of $P_3$ between $s$ and $v$. By replacing, in $S_3$, $P_3^{sv}$ by $sv$, we get a solution of $T(G)$, which yields $a(sv) \geq a(P_3^{sv})$. As $a(P_3) \geq a(P_1)$ and therefore $a(P_3^{sv}) \geq a(u_1t)$, we get $a(sv) \geq a(u_1t)$. But this again contradicts (4.1), which ends the proof of the lemma. □

Lemma 4.7 allows a partition of the set $W$ into four subsets:

$W_1 = \{w \in W \mid a(e)=0 \text{ for all } e \in [U,w], \text{ and } a(e')>0 \text{ for all } e' \in [w,V]\}$,

$W_2 = \{w \in W \mid a(e)=0 \text{ for all } e \in [U,w] \cup [w,V]\}$,

$W_3 = \{w \in W \mid a(e)>0 \text{ for all } e \in [U,w], \text{ and } a(e')=0 \text{ for all } e' \in [w,V]\}$,

$Z = W \backslash (W_1 \cup W_2 \cup W_3)$.

LEMMA 4.11. (i) *If $U=\{u\}$, then $a(e)=a(e')$ for all $e \in [u,t]$ and $e' \in [W_1 \cup W_2, t]$.*

(ii) *If $V=\{v\}$, then $a(e)=a(e')$ for all $e \in [s,v]$ and $e' \in [s, W_2 \cup W_3]$.*

*Proof.* We will prove only (i); the proof of (ii) is similar. Assume by contradiction that $a(ut) \neq a(wt)$ for some $w \in W_1 \cup W_2$. We will first give the following claim.

*Claim.* No solution of $\tau_a$ uses at the same time an edge of $[u,t]$ and an edge of $[w,t]$.

*Proof.* It suffices to show that there is no solution of $\tau_a$ containing at the same time $ut$ and $wt$. Let us suppose, on the contrary, that there exists a solution $S \in \tau_a$ with $ut, wt \in S$. Let $T_1$ and $T_2$ be two edge-disjoint 3-$st$-paths of $S$. As $a(ut)>0$ and $a(wt)>0$, we may suppose that $ut \in T_1$ and $wt \in T_2$.

Suppose that $a(wt)<a(ut)$. The case where $a(wt)>a(ut)$ can be treated along the same lines. If $[s,u] \cap T_1 = \emptyset$, $T_1$ must go through a node $z \in N \backslash \{s,t,u\}$, and hence the subpath $T_1^{su}$ of $T_1$ between $s$ and $u$ is of positive weight. By Lemma 4.4, it follows that $[s,u] \subseteq T_2$, and therefore $[s,u]=\{su\}$ and $T_2=(s,u,w,t)$. If $z \in V$, then, by replacing $wt$ by $zt$ in $S$, we get a solution of $T(G)$. But, as $a(zt)=0$, this implies that $a(wt)=0$, a contradiction. Thus $T_1$ cannot go through $V$. As a consequence, as by Lemma 4.3, $V \neq \emptyset$, there is a node $v \in V$ such that $sv$ and $vt$ belong neither to $T_1$ nor to $T_2$. So, by replacing $T_1$ by $(sv,vt)$, we get a solution of $T(G)$. However, since, from Lemma 4.10, we have $a(ut)=a(sv)$, we get $a(T_1^{su})=0$, a contradiction. Consequently, $[s,u] \cap T_1 \neq \emptyset$ and $T_1=(s,u,t)$. By using similar arguments, we can also show that $T_2$ is of the form $(f,uw,wt)$, where $f$ is an edge parallel to $su$, and hence $|[s,u]| \geq 2$. Furthermore, at least one of the sets $[u,w]$ and $[w,t]$ is reduced to a single edge. If not, one may replace $ut$ by a 2-$ut$-path going through $w$ and get a solution of $T(G)$. But this would imply that $a(wt) \geq a(ut)$, a contradiction.

Suppose that $|[w,t]| = 1$. The case where $|[u,w]| = 1$ is similar. Hence $[w,t] = \{wt\}$. Let $S' \in \tau_a$ such that $wt \notin S'$ and suppose that $S'$ is minimal. If $S'$ contains an edge $e \in [u,w]$, then, as $S'$ is minimal, there must exist in $S'$ a 3-$st$-path $T$ containing $e$. Therefore $T$ is of the form $(s,w,u,t)$. Observe that in this case, the edge set obtained by deleting $ut$ and adding $wt$ is in $T(G)$, and then $a(ut) \leq a(wt)$, a contradiction. Consequently, $[u,w] \cap S' = \emptyset$. Hence, as $|T_2 \cap S'| \leq 1$, there is a 3-$st$-path, say $T_1'$, in $S'$ such that $T_1' \cap T_2 = \emptyset$. By replacing $T_1$ by $T_1'$ in $S$, we get a solution of $T(G)$, and hence $a(T_1') \geq a(T_1)$. Note that only one edge of $[s,u]$ can be used by the second 3-$st$-path of $S'$. Thus one can replace $T_1'$ by $T_2$ in $S'$ and obtain a feasible solution, which yields $a(T_2) \geq a(T_1')$, and therefore $a(T_2) \geq a(T_1)$. But this implies that $a(wt) \geq a(ut)$, which is impossible. □

Suppose that $a(ut) > a(wt)$. The case where $a(ut) < a(wt)$ can be treated similarly. Let $S_1$ be a minimal solution of $\tau_a$ that contains $ut$, and let $P_1$ and $P_1'$ be two edge-disjoint 3-$st$-paths of $S_1$. Suppose, w.l.o.g., that $ut \in P_1$. By the claim, we have $[w,t] \cap S_1 = \emptyset$. If $S_1$ contains an edge of $[u,w]$, then there is a 3-$st$-path of $S_1$ of the form $(s,w,u,t)$. However, by removing $ut$ and adding $wt$, we obtain a solution of $T(G)$, yielding $a(wt) \geq a(ut)$, a contradiction. Thus $[u,w] \cap S_1 = \emptyset$. Moreover, if there is an edge $e$ of $[s,u]$ such that $e \notin P_1'$, one can replace $ut$ by $(e, uw, wt)$ and get a solution of $T(G)$. But this implies that $a(wt) \geq a(ut)$, a contradiction. Consequently, we have that $[s,u] \subseteq P_1'$. Hence $[s,u] = \{su\}$ and $P_1 = (s,z,u,t)$ with $z \in N \backslash \{s,t,u,w\}$. Observe that the subpath $P_1^{su}$ of $P_1$ between $s$ and $u$ is of positive weight. If there are two edges $f \in [s,v]$ and $f' \in [v,t]$ such that $f, f' \notin P_1'$, where $v \in V$, then we can replace $P_1$ by the edges $f$ and $f'$ and still have a feasible solution. As by Lemma 4.10, $a(f) = a(ut)$, we obtain that $a(P_1^{su}) = 0$, a contradiction. Thus, for every node $v \in V$, the path $P_1'$ must use all the edges of at least one of the sets $[s,v]$ and $[v,t]$. This implies that $V = \{v\}$. Moreover, as $su \in P_1'$, we have that $[s,v] \cap P_1' = \emptyset$, $[v,t] = \{vt\}$, and $P_1' = (s,u,v,t)$.

Let $S_2$ be a solution of $\tau_a$ that does not contain $su$. Recall that $[s,u] = \{su\}$. Suppose that $S_2$ is minimal. Thus $S_2$ consists of two edge-disjoint 3-$st$-paths, say $P_2$ and $P_2'$. As $|U| = 1$, by Lemma 4.9, we have that $\delta(u) \cap S_2 = \emptyset$. If $S_2$ contains an edge $e$ of $[w,t]$, as $a(e) > 0$, $e$ must belong to one of the 3-$st$-paths of $S_2$, say $P_2$. Since $(\{su\} \cup [u,w]) \cap S_2 = \emptyset$, $P_2$ must be of the form $(s,z',w,t)$, where $z' \notin \{s,t,u\}$. We remark that the subpath of $P_2$ between $s$ and $w$ is of positive weight. Hence, by Lemma 4.4, $P_2'$ must intersect every 2-$sw$-path going through $u$. But this contradicts the fact that $(\{su\} \cup [u,w]) \cap S_2 = \emptyset$. It then follows that $[w,t] \cap S_2 = \emptyset$. As $|P_1' \cap S_2| \leq 1$, there is a 3-$st$-path in $S_2$, say $P_2$, which does not intersect $P_1'$. Let $P$ be a 3-$st$-path going through the nodes $s,u,w,t$. From Lemma 4.1, it follows that $a(P) \geq a(P_1)$. But then we have that $a(wt) \geq a(ut)$, a contradiction. □

**5. Proof of Theorem 3.1.** In this section, we prove Theorem 3.1; that is, $P(G,L) = Q(G,L)$ for $L = 2,3$. For this, we consider an inequality $ax \geq \alpha$ that defines a facet of $P(G,L)$ different from the trivial and the $st$-cut inequalities. We will show that $ax \geq \alpha$ is necessarily an $L$-path-cut inequality.

*Case* 1. $L = 2$. Let $U, V, W$ be as defined in the previous section. By Lemma 4.6, it follows that $W = \emptyset$, and thus each 2-$st$-path uses exactly one edge with a nonzero coefficient. Thus, any solution of $\tau_a$ contains exactly two edges with a positive coefficient, which are exactly the edges of the 2-path-cut inequality induced by the

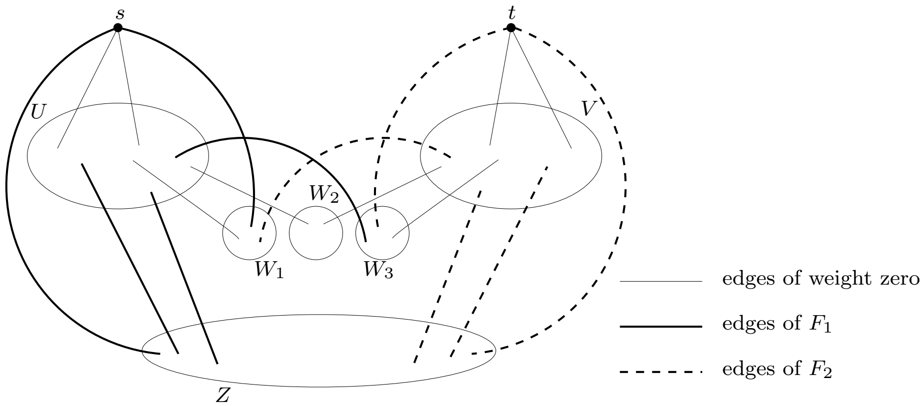


FIG. 3.

partition $\{s\}, U, V, \{t\}$. This implies that $ax \geq \alpha$ is the 2-path-cut inequality induced by this partition.

*Case* 2. $L = 3$. Let $U, V, W_1, W_2, W_3, Z$ be as defined in the previous section. We consider two cases.

*Case* 2.1. $W_1 \cup W_3 \cup Z \neq \emptyset$. Let $F_1 = [\{s\} \cup U, Z] \cup [s, W_1] \cup [U, W_3]$ and $F_2 = [Z, V \cup \{t\}] \cup [W_3, t] \cup [W_1, V]$ (see Figure 3). We remark that $F_1 \cap F_2 = \emptyset$ and that there is no $st$-path of length 3 in $G$ formed by edges only from $F_1$ and $F_2$. We have the following.

LEMMA 5.1. *For every solution $S$ of $\tau_a$, we have that* $|S \cap F_1| = |S \cap F_2|$.

*Proof.* Assume the contrary. Then there exists a solution, say $S_1$, such that, for one of its 3-$st$-path, say $P_1$, we have $|P_1 \cap F_1| \neq |P_1 \cap F_2|$. Let $P_1'$ be the second 3-$st$-path in $S_1$. W.l.o.g., we may suppose that $P_1 \cap F_1 \neq \emptyset$.

*Claim* 1. $P_1 \cap F_2 = \emptyset$.

*Proof.* Since $P_1 \cap F_1 \neq \emptyset$ and $F_1 \cap F_2 = \emptyset$, we have that $|P_1 \cap F_2| \leq 2$. If $|P_1 \cap F_2| = 1$, as $|P_1 \cap F_1| \neq |P_1 \cap F_2|$ and $P_1 \cap F_1 \neq \emptyset$, $|P_1 \cap F_1| = 2$. Then, $P_1$ is of length 3 and contained in $F_1 \cup F_2$, which is impossible by the remark above. If $|P_1 \cap F_2| = 2$, then $|P_1 \cap F_1| = 1$, and again we have that $P_1$ is of length 3 and contained in $F_1 \cup F_2$, a contradiction. Thus, $|P_1 \cap F_2| = 0$ and the claim is proved. $\square$

*Claim* 2. (i) $P_1 \cap [s, U] = \emptyset$.

(ii) $P_1 = (s, z, w, t)$ with $z \in Z \cup W_1$ and $w \in U \cup W_1 \cup W_2$ ($z$ and $w$ may be the same).

(iii) $[s, U] \subset P_1'$.

(iv) $|U| = 1$ and $|[s, U]| = 1$.

*Proof.* First note that (iv) is a consequence of (iii).

(i) If $P_1$ uses an edge of $[s, U]$, say $su$ with $u \in U$, as $P_1 \cap F_1 \neq \emptyset$, $P_1$ would be of the form $(s, u, z, t)$, where $z$ belongs to either $Z$ or $W_3$. But this implies that $P_1 \cap F_2 \neq \emptyset$, which contradicts Claim 1.

(ii) Suppose that $P_1$ contains an edge of $[U, W_3]$, say $uw_3$. Note that $a(uw_3) > 0$. As, by (i), $[s, U] \cap P_1 = \emptyset$, it follows that $P_1 = (s, w_3, u, t)$. By removing $uw_3$ and adding $su$ and edges $w_3v, vt$ for some $v \in V$, we get a solution of $T(G)$. As the added edges all have zero weight, this implies that $a(uw_3) = 0$, a contradiction. Consequently, we have that $P_1 \cap [U, W_3] = \emptyset$. Then, by (i) and the fact that $P_1 \cap F_1 \neq \emptyset$, it follows that $P_1$ uses one of the edges of $[s, Z \cup W_1]$. As, by Claim 1, $P_1 \cap F_2 = \emptyset$, we obtain that $P_1 = (s, z, w, t)$, where $z \in Z \cup W_1$ and $w \in U \cup W_1 \cup W_2$.

(iii) Suppose that there is an edge of $[s, U]$, say $su_0$, that does not belong to $P_1'$. We have that $w \neq u_0$. Otherwise, $P_1$ would be $(s, z, u_0, t)$. As by (ii) $z \in Z \cup W_1$ and hence $a(sz) > 0$, it follows that the subpath of $P_1$ between $s$ and $u_0$ has a positive weight. But this implies by Lemma 4.4 that $su_0 \in P_1'$, a contradiction. We claim that $[u_0, w] \subseteq P_1'$. In fact, if, for instance, $u_0w \notin P_1'$, then consider the solution, say $S_1'$, obtained from $S_1$ by replacing $sz$ and $zw$ by $su_0$ and $u_0w$. Clearly, $S_1' \in T(G)$, which implies that $a(su_0) + a(u_0w) \geq a(sz) + a(zw)$. As $a(u_0w) = a(su_0) = 0$, we obtain that $a(sz) = 0$, a contradiction. Thus $[u_0, w] \subseteq P_1'$, and hence $[u_0, w] = \{u_0w\}$. Suppose now that $P_1' = (f, u_0w, g)$, where $f$ (resp., $g$) is an edge of $[s, u_0]$ (resp., $[w, t]$) different from that used by $P_1$. By removing $sz, zw$, and $g$ and adding the edges $su_0$ and $u_0t$, we get a solution of $T(G)$. As by Lemma 4.11 $a(u_0t) = a(g)$, it follows that $a(sz) = 0$, a contradiction. Consequently, $P_1' = (s, w, u_0, t)$. Now, by considering the solution $\tilde{S}_1 = (S_1 \backslash \{sz, zw\}) \cup \{su_0\}$, one can get a contradiction along the same lines. This ends the proof of the claim. ∎

Now, by Claim 2(iv), we may suppose that $U = \{u\}$ and $[s, u] = \{su\}$. Let $S_2$ be a solution of $\tau_a$ that does not contain $su$. W.l.o.g., we may suppose that $S_2$ is minimal. Then, by Lemma 4.9, it follows that $S_2 \cap \delta(u) = \emptyset$. Let $P = \{s, u, t\}$. Clearly, $P \cap S_2 = \emptyset$. Moreover, as $P_1'$ goes through node $u$, $|P_1' \cap S_2| \leq 1$. As a consequence, there must exist a 3-$st$-path of $S_2$, say $P_2$, such that $P_2 \cap P_1' = \emptyset$. Now, by Lemma 4.1, we obtain that $a(P) \geq a(P_1)$. By Claim 2(ii), together with Lemma 4.11, it follows that $a(sz) \leq 0$. We then have a contradiction, and the lemma is proved. ∎

From Lemma 5.1, it follows that the facet defined by $ax \geq \alpha$ is contained in the face induced by the equation $x(F_1) - x(F_2) = 0$. As, by Lemma 3.6, this equation cannot be a positive multiple of $ax = \alpha$, we have a contradiction.

*Case* 2.2. $W_1 \cup W_3 \cup Z = \emptyset$. Since, by Lemma 4.6, $W \neq \emptyset$, we have necessarily that $W_2 \neq \emptyset$. Thus $\{s\}, U, W_2, V, \{t\}$ is a partition of $N$. Let $T$ be the set of edges of the 3-path-cut induced by this partition (these edges are represented by solid lines in Figure 4). Note that $a(e) > 0$ for all $e \in T$. Moreover, $a(e) = 0$ for all $e \in E \backslash T$. This is clear for the edges of $E \backslash (T \cup E(W_2))$ from Lemma 4.5(ii) and the definition of $U, V, W_2$. If $a(z_1z_2) > 0$ for some $z_1, z_2 \in W_2$, then there must exist a solution $\tilde{S}$ of $\tau_a$ and a 3-$st$-path $\tilde{P}$ of $\tilde{S}$ containing $z_1z_2$. W.l.o.g., we may suppose that $\tilde{P} = (s, z_1, z_2, t)$. Let $\tilde{S}' = (\tilde{S} \backslash \{z_1z_2\}) \cup \{su, uz_2, z_1v, vt\}$ for some nodes $u \in U$ and $v \in V$. As $\tilde{S}' \in T(G)$ and all the added edges have zero weight, it follows that $a(z_1z_2) = 0$, a contradiction.

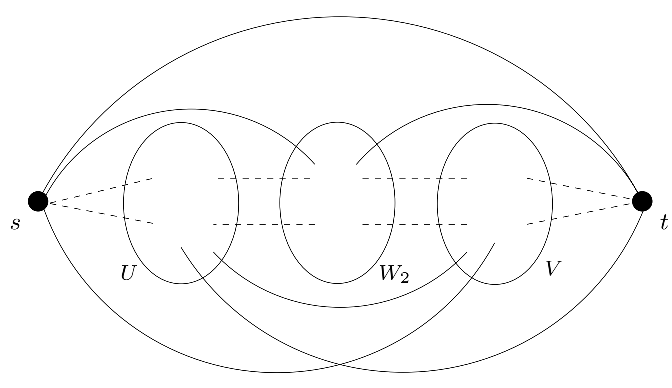


FIG. 4.

Now we claim that each solution of $\tau_a$ contains exactly two edges of $T$. First of all, note that, as the constraint (1.3) associated with $T$ is valid for $P(G,3)$, every solution of $\tau_a$ must contain at least two edges of $T$. Assume that there is a solution $S$ of $\tau_a$ with more than two edges of $T$. So, there must exist in $S$ a 3-$st$-path $P$ that contains at least two edges of $T$. We consider the case where $P = (s, w_2, w_2', t)$ with $w_2, w_2' \in W_2$. The other possible cases for $P$ can be treated similarly ($(s, w_2, t)$, $(s, w_2, u, t)$ with $u \in U$, $(s, v, w_2, t)$ with $v \in V$, $(s, v, u, t)$). Let $P'$ be the second 3-$st$-path of $S$. By replacing $P'$ by the edges $su, uw_2', w_2v, vt$ in $S$, we get a solution of $T(G)$. As all these edges have zero weight, $a(P') = 0$, contradicting Lemma 4.2.

Thus, every solution of $\tau_a$ uses exactly two edges of $T$. This implies that $ax \geq \alpha$ is nothing but the 3-path-cut inequality induced by $T$, which ends the proof of Theorem 3.1.

**6. Facets of $P(G, L)$.** In this section, we give necessary and sufficient conditions for inequalities (1.1)–(1.3) to be facet defining for $P(G,L)$. This yields a minimal description of this polytope when $L \leq 3$. Throughout this section, $G = (N, E)$ is a complete graph with $|N| \geq 4$, which may contain multiple edges. Hence, by Corollary 3.3, $P(G,L)$ is full dimensional. The first two theorems, given without proof, describe when the trivial and the $st$-cut inequalities define facets of $P(G,L)$.

THEOREM 6.1. (i) *For $L \geq 2$, inequality $x(e) \leq 1$ defines a facet of $P(G,L)$.*

(ii) *For $L \geq 2$, inequality $x(e) \geq 0$ defines a facet of $P(G,L)$ if and only if $|N| \geq 5$, or $|N| = 4$ and $e$ does not belong to either an $st$-cut or an $L$-path-cut, with exactly three edges.*

THEOREM 6.2. (i) *If $L = 2$, then the only $st$-cut inequalities that define facets of $P(G,2)$ are those induced by $\{s\}$ and $N \setminus \{t\}$.*

(ii) *For $L \geq 3$, every $st$-cut inequality defines a facet of $P(G,L)$.*

We give now necessary and sufficient conditions for the $L$-path-cut inequalities to be facet defining for $P(G,L)$.

THEOREM 6.3. *For $L \geq 2$, inequality* (1.3) *defines a facet of $P(G,L)$ if and only if $|V_0| = |V_{L+1}| = 1$.*

*Proof. Necessity.* We will show that $x(T) \geq 2$ does not define a facet of $P(G,L)$ if $|V_0| \geq 2$. The case where $|V_{L+1}| \geq 2$ follows by symmetry.

Suppose that $|V_0| \geq 2$, and consider the partition given by

$\overline{V}_0 = \{s\}$,
$\overline{V}_1 = V_1 \cup (V_0 \setminus \{s\})$,
$\overline{V}_i = V_i$, $i = 2, \ldots, L+1$.

This partition induces the $L$-path-cut inequality $x(\overline{T}) \geq 2$, where $\overline{T} = T \setminus [V_0 \setminus \{s\}, V_2]$. As $G$ is complete, we have that $\overline{T}$ is strictly contained in $T$, and hence $x(T) \geq 2$ cannot be facet defining.

*Sufficiency.* Now, suppose that $|V_0| = |V_{L+1}| = 1$, that is, $V_0 = \{s\}$ and $V_{L+1} = \{t\}$. Let us denote inequality (1.3) by $ax \geq \alpha$, and let $bx \geq \beta$ be a facet defining inequality of $P(G,L)$ such that

$$\{x \in P(G,L) \mid ax = \alpha\} \subseteq \{x \in P(G,L) \mid bx = \beta\}.$$

We will show that $a = \rho b$ for some $\rho > 0$.

Let $V_0 = \{s\}, V_1, \ldots, V_L, V_{L+1} = \{t\}$ be the partition inducing $ax \geq \alpha$. Let $\overline{E} = E \setminus T = (\bigcup_{i=1}^{L} E(V_i)) \cup (\bigcup_{i=0}^{L} [V_i, V_{i+1}])$. Let $f \in [s,t]$ and $T_f = T \setminus \{f\}$. As the graph $G$ is complete, it is easy to see that the sets given by

$$F_e = \overline{E} \cup \{f, e\} \qquad \text{for all } e \in T_f$$

induce solutions of the THPP whose incidence vectors satisfy $ax \geq \alpha$ with equality. Thus,

$$0 = bx^{F_e} - bx^{F_{e'}} = b(e) - b(e') \qquad \text{for all } e, e' \in T_f.$$

Hence,

$$b(e) = b(e') \qquad \text{for all } e, e' \in T_f. \tag{6.1}$$

Now let $g \in [V_0, V_L]$, $g' \in [V_1, V_{L+1}]$, and $F^* = \overline{E} \cup \{g, g'\}$. It is obvious that $F^*$ induces a solution whose incidence vector satisfies $ax \geq \alpha$ with equality. Thus $bx^{F^*} - bx^{F_g} = b(g') - b(f) = 0$. This together with (6.1) yields

$$b(e) = \gamma \qquad \text{for all } e \in T \text{ for some } \gamma \in \mathbb{R}.$$

Now, we shall show that $b(e) = 0$ for all $e \in \overline{E}$. Suppose first that $e \in [V_0, V_1]$. Consider an edge $h \in [s, w]$ with $w \in V_2$ and the edge set $F_h \backslash \{e\}$, where $F_h$ is as defined above. It is easy to see that $F_h \backslash \{e\}$ still induces a solution of the THPP whose incidence vector satisfies $ax \geq \alpha$ with equality. Thus,

$$0 = bx^{F_h} - bx^{F_h \backslash \{e\}} = b(e).$$

Similarly, we obtain that $b(e) = 0$ for all $e \in \bigcup_{i=0}^{L} [V_i, V_{i+1}]$. Consider now an edge $e \in E(V_i)$, $i \in \{1, \ldots, L\}$. Let $v \in V_L$ and $h' \in [s, v]$. Clearly, the set $F_{h'} \backslash \{e\}$ induces a solution of the problem. As $ax^{F_{h'}} = ax^{F_{h'} \backslash \{e\}} = \alpha$, we have that $bx^{F_{h'}} = bx^{F_{h'} \backslash \{e\}} = \alpha$, and hence $b(e) = 0$.

Consequently, we have that

$$\begin{aligned} b(e) &= 0 \qquad \text{for all } e \in \overline{E}, \\ b(e) &= \gamma \qquad \text{for all } e \in T. \end{aligned}$$

Since $\alpha > 0$, we have that $\gamma > 0$, and by setting $\rho = 1/\gamma$, we obtain that $a = \rho b$. ☐

Let $E'$ be the set of edges that belong neither to an $st$-cut nor to an $L$-path-cut, consisting of exactly three edges. From the previous theorems, we have the following.

Corollary 6.4. *For $L = 2$, if $G = (N, E)$ is complete and $|N| \geq 4$, then a minimal complete linear description of $P(G, L)$ is given by*

$$\begin{aligned} & x(\delta(s)) \geq 2, \\ & x(\delta(t)) \geq 2, \\ & x(T) \geq 2 \qquad \textit{for all 2-path-cut } T \textit{ induced by } V_0 = \{s\}, V_1, V_2, V_3 = \{t\}, \\ & x(e) \leq 1 \qquad \textit{for all } e \in E, \\ & x(e) \geq 0 \qquad \textit{for all } e \in E'. \end{aligned}$$

Corollary 6.5. *For $L = 3$, if $G = (N, E)$ is complete and $|N| \geq 4$, then a minimal complete linear description of $P(G, L)$ is given by*

$$\begin{aligned} & x(\delta(W)) \geq 2 \quad \textit{for all st-cut } \delta(W), \\ & x(T) \geq 2 \qquad \textit{for all 3-path-cut } T \textit{ induced by } V_0 = \{s\}, V_1, V_2, V_3, V_4 = \{t\}, \\ & x(e) \leq 1 \qquad \textit{for all } e \in E, \\ & x(e) \geq 0 \qquad \textit{for all } e \in E'. \end{aligned}$$

**7. Dominant of $P(G, L)$.** In this section, we consider the dominant of the polytope $P(G, L)$. We give a complete description of that polyhedron for any graph $G$ and integer $L \geq 2$ such that $P(G, L) = Q(G, L)$.

Let $Dom(P(G, L))$ be the dominant of $P(G, L)$, that is,

$$Dom(P(G,L)) = \{y \in \mathbb{R}^E \mid \exists x \in P(G,L), x \leq y\}.$$

Let $D(G, L)$ be the polyhedron given by

$$\begin{array}{lll} & y(\delta(W)) \geq 2 & \text{for all } st\text{-cut } \delta(W), \\ (7.1) & y(\delta(W)\backslash\{e\}) \geq 1 & \text{for all } st\text{-cut } \delta(W),\ e \in \delta(W), \\ & y(T) \geq 2 & \text{for all } L\text{-path-cut } T, \\ (7.2) & y(T\backslash\{e\}) \geq 1 & \text{for all } L\text{-path-cut } T,\ e \in T, \\ (7.3) & y(e) \geq 0 & \text{for all } e \in E. \end{array}$$

THEOREM 7.1. *For every $L \geq 2$, if $P(G, L) = Q(G, L)$, then $Dom(P(G, L)) = D(G, L)$.*

*Proof.* We first prove that $Dom(P(G, L)) \subseteq D(G, L)$. Let $y \in Dom(P(G, L))$. Then there exists $\bar{x} \in P(G, L)$ such that $\bar{x} \leq y$. Hence, $y$ satisfies (1.1), (1.3), and (7.3). We show that $y$ also satisfies constraints (7.1) and (7.2).

Consider a constraint $y(\delta(W)\backslash\{e\}) \geq 1$ of type (7.1). As $\bar{x}(\delta(W)) \geq 2$ and $\bar{x}(e) \leq 1$, we have that

$$\begin{aligned} y(\delta(W)\backslash\{e\}) &\geq \bar{x}(\delta(W)\backslash\{e\}) \\ &= \bar{x}(\delta(W)) - \bar{x}(e) \\ &\geq 2 - \bar{x}(e) \\ &\geq 1. \end{aligned}$$

Now, in a similar way, we obtain that $y(T\backslash\{e\}) \geq 1$ for all $L$-path-cut $T$ and $e \in T$. Therefore $Dom(P(G, L)) \subseteq D(G, L)$.

Next we prove that $D(G, L) \subseteq Dom(P(G, L))$. To this end, first let us note that the dominant of $D(G, L)$, $Dom(D(G, L))$, is $D(G, L)$ itself. Thus, to prove that $D(G, L) \subseteq Dom(P(G, L))$, it is sufficient to show that any extreme point $\bar{y}$ of $D(G, L)$ belongs to $P(G, L)$. Indeed, suppose that this is the case. Then any convex combination of extreme points of $D(G, L)$ is also in $P(G, L)$. On the other hand, since $Dom(D(G, L)) = D(G, L)$, any solution $y \in D(G, L)$ can be seen as $\tilde{y} + z$, where $\tilde{y}$ belongs to the convex hull of the extreme points of $D(G, L)$ and $z \geq 0$. As $\tilde{y} \in P(G, L)$, we have therefore that $y \in Dom(P(G, L))$.

So let $\bar{y}$ be an extreme point of $D(G, L)$. As $P(G, L) = Q(G, L)$ and all inequalities in $Q(G, L)$ are in $D(G, L)$ except $x(e) \leq 1, e \in E$, in order to show that $\bar{y} \in P(G, L)$, it suffices to show that $\bar{y}(e) \leq 1$ for all $e \in E$.

Suppose that $\bar{y}(e_0) > 1$ for some $e_0 \in E$. Since $\bar{y}$ is an extreme point of $D(G, L)$, there exists at least one constraint among (1.1), (7.1), (1.3) (7.2) involving the variable $x(e_0)$ and that is tight for $\bar{y}$.

If $\bar{y}(\delta(W)\backslash\{f\}) = 1$ with $e_0 \in \delta(W)\backslash\{f\}$, then, clearly, $\bar{y}(e_0) \leq \bar{y}(\delta(W)\backslash\{f\}) = 1$, a contradiction.

If $\bar{y}(\delta(W)) = 2$ with $e_0 \in \delta(W)$, then $\bar{y}(e_0) + \bar{y}(\delta(W)\backslash\{e_0\}) = 2$, and hence $\bar{y}(e_0) = 2 - \bar{y}(\delta(W)\backslash\{e_0\})$. As $\bar{y}$ satisfies (7.1), it follows that $\bar{y}(e_0) \leq 1$, which is impossible.

We obtain a similar contradiction if one of the constraints (1.3), (7.2) is tight for $\bar{y}$ . ∎

It would be interesting to investigate the dominant of the THPP polytope when $P(G, L) \neq Q(G, L)$.

An immediate consequence of Theorems 3.1 and 7.1 is the following.

COROLLARY 7.2. *If $L = 2, 3$, then $Dom(P(G, L)) = D(G, L)$.*

**8. Concluding remarks.** In this paper, we have considered the problem of finding a minimum cost edge set containing at least two edge-disjoint paths between two terminals $s$ and $t$ of length no more than $L$, where $L \geq 2$ is a given integer. We have given a formulation for this problem and extended this formulation to the case where more than two paths are required between $s$ and $t$. We have also investigated its polyhedral structure when $L = 2, 3$. In particular, we have shown in that case that the associated polytope $P(G, L)$ is described by the trivial, $st$-cut, and $L$-path-cut inequalities. Moreover, we have given necessary and sufficient conditions for these inequalities to be facet defining for any $L \geq 2$. This yielded a minimal linear description for $P(G, L)$ when $L = 2, 3$. We have finally considered the dominant of $P(G, L)$, for which we have given a complete description for any $L \geq 2$ when $P(G, L)$ is given by those inequalities.

Since the separation problems for inequalities (1.1) and (1.3) can be solved in polynomial time when $L \leq 3$, from Theorem 3.1 it follows that, for $L \leq 3$, the THPP can be solved in polynomial time using a cutting plane algorithm. To the best of our knowledge, this is the first (nonenumerative) polynomial algorithm devised for this problem.

Let $P_k(G, L)$ be the polytope associated with the problem where the number of edge-disjoint paths $k$ is arbitrary. A natural question that may be posed is whether the linear relaxation of this problem is integral. We have made some investigations in this direction. These motivate us to give the following conjecture.

CONJECTURE 8.1. *$P_k(G, L) = Q_k(G, L)$ if $L = 2, 3$, where $Q_k(G, L)$ is as defined in section 2.*

As already mentioned, if $L \geq 4$, the formulation given in section 2 is no longer valid for the THPP. Unfortunately, so far we do not know a formulation for the problem in that case. However, for $L \leq 3$, it is not hard to see that the formulation given in section 2 for the THPP (and also for its generalization when the number $k$ of required edge-disjoint $L$-$st$-paths is more than two) can be easily extended to the case where more than one pair of terminals is considered. Here the formulation is given by the $st$-cut and $L$-path-cut inequalities for every pair $\{s, t\}$ of terminals, together with the trivial inequalities. However, these inequalities do not suffice to completely describe the associated polytope for this general case even for $L \leq 3$. In fact, consider the graph shown in Figure 5 with two pairs of terminals $\{s, t\}$ and $\{s', t'\}$. Suppose that $L = 3$. Here, a feasible solution must contain at least two edge-disjoint 3-$st$-paths and at least two edge-disjoint 3-$s't'$-paths. It is not hard to see that the fractional point $\bar{x} = (1, 1, 1, 1, 0, 0, 0, 1/2, 1/2, 1/2)$ satisfies all trivial, $st$-cut, and $L$-path-cut inequalities (with respect to the two pairs of terminals). Moreover, $\bar{x}$ is an extreme point of the polyhedron given by these inequalities. Actually, one can easily see that the inequality

$$x(e_5) + x(e_6) + x(e_7) + x(e_8) + x(e_9) + x(e_{10}) \geq 2$$

is valid for the problem but violated by $\bar{x}$. Furthermore, this inequality is facet

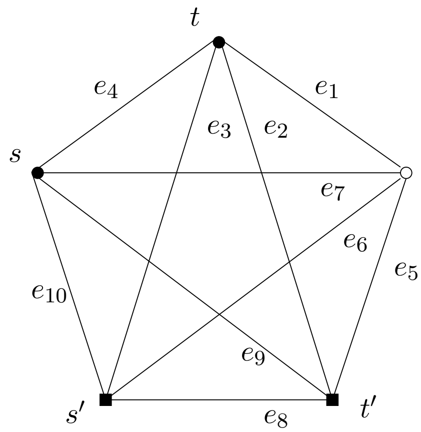


Fig. 5.

defining for the associated polytope.

Finally, let us note that the results given in this paper can be exploited to devise a branch-and-cut algorithm for that general problem when $L = 2, 3$. For this, we should identify further families of facet defining inequalities. These should take into account the interaction between the different pairs of terminals. Our results can also be used to obtain upper bounds for that problem. If $L \leq 3$, one can solve the THPP in the underlying graph $G$ for every pair of terminals using the cutting plane algorithm developed in this paper. Then, by considering the union of the different solutions obtained this way, one get a feasible solution for the problem. This approach can be used to provide upper bounds even when $L \geq 4$. On the other hand, it would be interesting to investigate the extension of the results, related to the formulation of the THPP when $L \leq 3$ as well as the facial structure of its associated polytope, to the more general case when $k$ and $L$ are both arbitrary. This is our aim for future work.

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