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Separation of partition inequalities for the $(1,2)$-survivable network design problem

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Abstract

Given a graph $G=(V,E)$ with edge costs and an integer vector $r\in\mathbb{Z}_+^V$ associated with the nodes of V, the survivable network design problem is to find a minimum cost subgraph of G such that between every pair of nodes s,t of V, there are at least $\min\{r(s),r(t)\}$ edge-disjoint paths. In this paper we consider that problem when $r\in\{1,2\}^V$. This case is of particular interest to the telecommunication industry. We show that the separation problem for the so-called partition inequalities reduces to minimizing a submodular function. This yields a polynomial time separation algorithm for these inequalities in that case.

Keywords: Survivable network; Partition inequalities; Submodular function; Separation algorithm

1. Introduction

Given a graph $G=(V,E)$ with edge costs and an integer vector $r\in\mathbb{Z}_+^V$ associated with the nodes of V, the *survivable network design problem* (SNDP) is to find a minimum cost subgraph of G such that between every pair of nodes s,t of V, there are at least

$$\min\{r(s),r(t)\} \tag{1}$$

edge-disjoint paths. The integers $r(v), v\in V$ are called the *node connectivity types* and conditions (1.1) are called the *survivable conditions*.

The SNDP has applications to the design of survivable telecommunication networks [11,14,16].

In fiber optic networks, all nodes are of connectivity type one or two, and are called ordinary and special offices, respectively. This topology has shown to be cost effective and provides an adequate level of survivability [11,16]. In this paper we consider the SNDP in that case, that is when $r(v)\in\{1,2\}$ for all $v\in V$. We show that the separation problem for the so-called partition inequalities reduces to minimizing a submodular function. This yields a polynomial time seperation algorithm for these inequalities in this case.

The SNDP is NP-hard when $r\in\{1,2\}^V$. It includes as special case the 2-edge connected subgraph problem ($r(v)=2$ for all $v\in V$) which is known to be NP-hard. It has been extensively investigated in the past. Monma and Shallcross [16] devised heuristics for designing survivable networks with node connectivity types $r\in\{1,2\}^V$. Ko and Monma [14] extended these heuristics to the design of k-edge and k-node connected networks. Grötschel et al. [9,10] studied a polyhedral approach for the SNDP. Goemans and

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Bertsimas [7] devised a heuristic with worst case guarantee for the SNDP when the use of multiple copies of an edge is allowed. For a complete survey of the SNDP, see [11,19].

In the next section we present the class of partition inequalities. In Section 3 we discuss the submodular function minimization problem. In Section 4 we study the separation problem for the partition inequalities.

2. Partition inequalities

Let $G = (V,E)$ be a graph and $r \in \{0,1,2\}^V$ a requirement vector. For $W \subseteq V$ let $r(W) = \max\{r(v), v \in W\}$ and $con(W) = \min\{r(W), r(V \setminus W)\}$. If $W \subseteq V$, the set of edges having exactly one node in W is called a *cut* and denoted by $\delta(W)$. Clearly, the following inequalities, called *cut inequalities*, are valid for the SNDP(G,r):

$$x(\delta(W)) \geqslant con(W) \quad \text{for all } W \subseteq V, W \neq \emptyset \neq V. \tag{2.1}$$

In [9] Grötschel et al. introduced a class of valid inequalities for the SNDP (G,r) called partition inequalities, that generalize inequalities (2.1). Let $(V_1,\dots,V_p)$ be a partition of V such that $r(V_i) \geqslant 1$ for $i=1,\dots,p$. Let $\delta(V_1,\dots,V_p)$ be the set of edges between the elements of the partition. The *partition inequality* induced by $(V_1,\dots,V_p)$ is given by

$$x(\delta(V_1,\dots,V_p)) \geqslant \begin{cases} p-1 & \text{if } I_2 = \emptyset, \\ p & \text{otherwise,} \end{cases} \tag{2.2}$$

where $I_2 = \{i \mid con(V_i)=2; i=1,\dots,p\}$. Grötschel et al. [9] gave sufficient conditions and necessary conditions for inequalities (2.2) to define facets.

The separation problem for a class of inequalities consists of deciding whether a given vector $\bar{x} \in \mathbb{R}^E$ satisfies the inequalities, and if not to find an inequality that is violated by $\bar{x}$. Grötschel et al. [10] showed that the separation problem for inequalities (2.2) is NP-hard. Actually they showed that the separation problem is NP-hard for the subclass of partition inequalities.

$$x(\delta(V_1,\dots,V_p)) \geqslant p-1, \tag{2.3}$$

for every partition $(V_1,\dots,V_p)$ of V with $r(V_i)=1$ for $i=1,\dots,p$. Inequalities (2.3) are valid for the Steiner tree polytope [3].

If $r(v) = 1$ for all $v \in V$, as shown by Cunningham [4] and Barahona [2], the separation problem for inequalities (2.3) can be solved in polynomial time. Cunningham showed that this can be reduced to $|E|$ minimum cut problems. Barahona's algorithm permits to reduce the problem to $|V|$ minimum cut problems.

In what follows we show that the separation problem for inequalities (2.2) when $r \in \{1,2\}^V$ can be reduced to the minimization of a submodular function, and therefore, can be solved in polynomial time. But first we discuss some properties of submodular functions.

3. Submodular functions

Given a finite set S, a function $f : 2^S \to \mathbb{R}$ is called *submodular* if

$$f(A \cup B) + f(A \cap B) \leqslant f(A) + f(B)$$

$$\text{for all } A,\ B \subseteq S.$$

Minimizing a submodular function has always been an important problem in combinatorial optimization [6,8,15]. Grötschel et al. [8] showed that this problem can be solved in polynomial time using the ellipsoid method. In [17] Queyranne presented a combinatorial and polynomial time algorithm for solving the problem for the class of symmetric submodular functions (a function f is *symmetric* if $f(A) = f(S \setminus A)$ for all $A \subseteq S$). Purely combinatorial and polynomial algorithms for minimizing a submodular function have been recently developed by Schrijver [18], and Iwata et al. [12]. The two algorithms use different approaches.

In [1] Baïou et al. studied the partition inequalities of the form

$$x(\delta(V_1,\dots,V_p)) \geqslant ap + b,$$

where a and b are scalar. They showed that the separation problem for these inequalities reduces to minimizing a symmetric submodular function, and then be solved in polynomial time using Queyranne's algorithm [17].

Given a submodular function f on the subsets of a set V, the *Dilworth truncation* of f, f^* is defined on the subsets of V as follows

$$f^*(\emptyset) = 0,$$
$$f^*(S) = \min\{f(S_1) + \cdots + f(S_k) \mid (S_1,\ldots,S_k) \text{ is a partition of S}\}$$

In [15] Lovàsz proved the following.

Theorem 3.1 (Lovász [15]). *If f is a submodular function, then its Dilworth truncation f^* is also submodular.*

In what follows we shall show that the separation problem for inequalities (2.2) when $r \in \{1,2\}^V$ reduces to $O(|V|)$ minimizations of a function that is the Dilworth truncation of a cut function.

4. Separation of partition inequalities

Let $G = (V,E)$ be a graph and $r \in \{1,2\}^V$ a connectivity type vector. Let $\bar{x} \in \mathbb{R}^E$ and $(V_1,\ldots,V_p)$ be a partition of V. We shall consider two cases.

Case 1: $I_2 = \emptyset$. That is all the nodes of type 2 are in the same set V_i. In this case, inequality (2.2) is of type (2.3). Let $G' = (V',E')$ be the graph obtained by shrinking the set of nodes of type 2. Let us associate with the resulting node, say w, the connectivity type $r(w) = 1$. Therefore all the nodes of V' have connectivity type one. It is clear that finding a violated partition inequality on G (where $I_2 = \emptyset$) with respect to $\bar{x}$ is equivalent to finding a violated inequality of type (2.3) on G' with respect to $\bar{x}'$, the restriction of $\bar{x}$ on E'. Now the separation problem for inequalities (2.3) on G' can be solved in polynomial time using either Cunningham's algorithm [4] or Barahona's algorithm [2].

Case 2: $|I_2| \geqslant 2$. Thus there are at least two sets V_i and V_j with $r(V_i) = r(V_j) = 2$. In consequence, inequality (2.2) can be written as

$$x(\delta(V_1,\ldots,V_p)) \geqslant p. \tag{4.1}$$

Consider the problem

$$\min\{x(\delta(V_1,\ldots,V_p)) - p\} \tag{4.2}$$

with $p \geqslant 2$, which is called the *multicut problem*. Problem (4.2) is equivalent to minimizing

$$g(S) = x(\delta(S)) - 2 + \min\{x(\delta_S(S_1,\ldots,S_k)) - (k-1)\},$$

where $S \subset V$ and $(S_1,\ldots,S_k)$ is a partition of S (see [1]). Here $\delta_S(S_1,\ldots,S_k)$ is the set of edges between the S_i.

Lemma 4.1. *g is submodular.*

Proof. Let $f(S) = \frac{1}{2}x(\delta(S)) - 1$ for $S \subset V$. The Dilworth truncation of f, f^*, is given by

$$f^*(S) = \min\left\{\sum_{i-1}^{k} (\tfrac{1}{2}x(\delta(S_i)) - 1) \mid (S_1,\ldots,S_k) \text{ is a partition of } S\right\}$$
$$= \min\{x(\delta_s(S_1,\ldots,S_k)) - k\} + \tfrac{1}{2}x(\delta(S)).$$

Thus $g(s) = f^*(S) + \frac{1}{2}x(\delta(S)) - 1$. As f is submodular, by Theorem 3.1, we have that f^* is submodular, and hence the lemma follows. □

Let $U_2 = \{v_1,\ldots,v_s\}$ be the set of nodes of connectivity type 2. Then the separation problem for inequalities of type (4.1) is equivalent to

$$\min\{g(S), S \cap U_2 \neq \emptyset \neq (V \backslash S) \cap U_2\}. \tag{4.3}$$

For every node $v_i \in U_2 \backslash \{v_1\}$, let us consider two weight vectors $w^i, \bar{w}^i \in \mathbb{R}^V$, associated with the nodes of V, given by

$$w^i(v) = \begin{cases} 0 & \text{if } v \in V \backslash \{v_1, v_i\}, \\ M & \text{if } v = v_1, \\ -M & \text{if } v = v_i \end{cases}$$

and

$$\bar{w}^i(v) = \begin{cases} 0 & \text{if } v \in V \backslash \{v_1, v_i\}, \\ -M & \text{if } v = v_1, \\ M & \text{if } v = v_i, \end{cases}$$

where M is a big value. Let g_i and $\bar{g}_i$ be the functions on the subsets of V such that $g_i(S) = g(S) + w^i(S)$ and $\bar{g}_i(S) = g(S) + \bar{w}^i(S)$. Clearly, as g is submodular, g_i and

$\bar{g}_i$ are so. Moreover, problem (4.3) is now equivalent to minimize

$$\{\min\{g_i(S)\}, \min\{\bar{g}_i(S)\}\},$$

for $v_i \in U_2 \backslash \{v_1\}$. So problem (4.3) reduces to solving $O(|U_2|)$ submodular function minimization problems. As this can be done in polynomial time [8,12,18], we then have the following.

Theorem 4.2. *If* $r \in \{1,2\}^V$, *then the separation problem for inequalities* (2.2) *can be solved in polynomial time.*

By the ellipsoid method [8], Theorem 4.2 implies that the $(1,2)$-survivable network design problem can be solved in polynomial time in the graphs for which the partition inequalities together with the trivial and the so-called cut inequalities suffice to describe the (1,2)-survivable network polytope. It would thus be interesting to characterize that class of graphs. A partial characterization of these graphs is given in [13]. Fonlupt and Mahjoub [5] have characterized these graphs when $r(v) = 2$ for all $v \in V$.

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