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ON THE STABLE SET POLYTOPE OF A SERIES-PARALLEL GRAPH

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We give a short proof of Chvátal's conjecture that the nontrivial facets of the stable set polytope of a series-parallel graph all come from edges and odd holes.

Key words: Series-parallel graphs, facets of polyhedra, stable set polytope.

1. Introduction

The graphs we consider are finite, undirected, loopless and without multiple edges. We denote a graph by $G=(V, E)$, where V is the *node set* and E the *edge set* of G. If $e \in E$ is an edge with endnodes u and v we also write uv to denote the edge e.

A homeomorph of K_4 is a graph obtained from K_4 when its edges are subdivided into paths by inserting new nodes of degree two. Graphs which contain no homeomorph of K_4 as a subgraph are called *series-parallel* graphs [4].

If $G=(V, E)$ is a graph, a *stable* (*independent*) *set* in G is a set of nodes, no two of which are adjacent. A path P in G is a sequence of edges $e_1, e_2, \ldots, e_k$ such that $e_1 = v_0v_1$, $e_2 = v_1v_2, \ldots, e_k = v_{k-1}v_k$ and such that $v_i \neq v_j$ for $i \neq j$. The nodes v_0 and v_k are the endnodes of P and we say that P joins v_0 and v_k. (The number k of edges of P is called the length of P.) If $P = e_1, e_2, \ldots, e_k$ is a path joining v_0 and v_k and $e_{k+1} = v_0v_{k+1} \in E$, then the sequence $e_1, e_2, \ldots, e_{k+1}$ is called a *cycle* of length $k+1$. A cycle is called odd if its length is odd, otherwise it is called even. An induced cycle of G is called a *hole*.

Given $S \subseteq V$, the incidence vector of S, x^S is defined by

$$x_u^S = \begin{cases} 1 & \text{if } u \in S, \\ 0 & \text{if } u \in V \setminus S. \end{cases}$$

The *stable set polytope* of a graph $G=(V, E)$, denoted by $P(G)$, is the convex hull of the incidence vectors of all stable sets of G.

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If $G=(V,E)$ is a graph, it is clear that each incidence vector of a stable set of G satisfies the following system of inequalities.

$$\begin{cases} 0 \leqslant x_u \leqslant 1 & \text{for all } u \in V, \\ x_u + x_v \leqslant 1 & \text{for all } uv \in E, \\ \sum_{u \in C} x_u \leqslant \dfrac{|C|-1}{2} & \text{for all odd holes } C \text{ in } G. \end{cases} \tag{1}$$

Chvátal [2] was the first who conjectured that (1) defines the polytope $P(G)$ when G is series-parallel. (A graph G for which $P(G)$ is given by (1) is generally called *h-perfect.*) In [1] Boulala and Uhry described a polynomial algorithm for finding a maximum weight stable set in a series-parallel graph, and using LP-duality they gave a proof for Chvátal's Conjecture. Their proof also implies that, when the graph $G=(V,E)$ is series-parallel, the system (1) is totally dual integral (i.e. given an integer vector $w \in \mathbb{R}^{|V|}$ such that $\max\{w^{\mathrm{T}}x \colon x \in (1)\}$ exists, the corresponding dual linear program has an integer optimum solution). Chvátal's Conjecture has also been proved independently by Clamcy [3]. Recently further results on h-perfect graphs, related to series-parallel graphs, have been obtained by Sbihi and Uhry [6] and Gerards and Schrijver [5].

The purpose of this note is to give a short proof of Chvátal's Conjecture. We shall prove that if G is a series-parallel graph then every subgraph of G, induced by a nontrivial facet of $P(G)$, is an edge or an odd hole.

2. On the structure of facet inducing graphs for $P(G)$

Given a graph $G=(V,E)$, a linear inequality defines a facet of $P(G)$, if and only if (i) it is satisfied by the incident vector x^S of every stable set S of G; and (ii) there are $|V|$ stable sets of G whose incidence vectors are affinely independent and satisfy it with equality.

It is trivial to see that $P(G)$ is full dimensional ($\dim(P(G))=|V|$), which implies that $P(G)$ has a unique (up to multiplication by a positive constant) system of facet inducing inequalities. Further, since $x \in P(G)$ and $0 \leqslant x' \leqslant x$ imply $x' \in P(G)$ it follows that if $ax \leqslant \alpha$ is facet inducing with $\alpha > 0$, then $a \geqslant 0$.

Let $G=(V,E)$ be an arbitrary graph and $ax \leqslant \alpha$ be a facet inducing inequality for $P(G)$ such that $\alpha > 0$ and hence $a \geqslant 0$. Suppose $ax \leqslant \alpha$ is not of the forms described in (1) and denote by G_a the subgraph of G induced by it (i.e. induced by the node set $\{u \in V | a_u \neq 0\}$). (Then G_a is not an edge or odd hole.) Let $\boldsymbol{S}$ be the set of all stable sets S for which $ax^S = \alpha$. Then the only equations satisfied by all members of $\boldsymbol{S}$ are positive multiples of $ax = \alpha$. We then have the following lemmas.

Lemma 1. *If G_a contains a path (pu, uv, vq) such that u and v are of degree two (see Fig. 1), then $a_u = a_v$.*

Proof. Under the hypothesis, we claim that there exists a stable set $S_0 \in \boldsymbol{S}$ such that

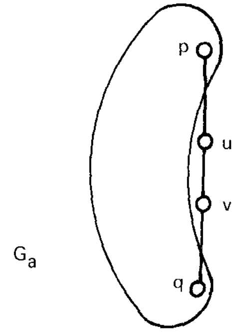

Fig. 1.

$v \in S_0$, $p \notin S_0$. In fact, if this is not the case then for every stable set $S \in \mathbf{S}$ the following holds:

$$v \in S \Rightarrow p \in S,$$

$$v \notin S \Rightarrow |S \cap \{p, u\}| = 1.$$

Thus $x_p^S + x_u^S = 1$ for all $S \in \mathbf{S}$, a contradiction.

Let $S_0' = (S_0 \backslash v) \cup \{u\}$. Since S_0' is also a stable set of G, we have $a_u \leqslant a_v$. Now by considering u and q we can deduce $a_v \leqslant a_u$ and hence $a_u = a_v$. □

Lemma 2. *Let p, q be nodes of G_a. Then at most one path in G_a which joins p and q can have all internal nodes of degree two.*

Proof. Assume the contrary. Let C be the hole defined by two paths between p and q (see Fig. 2) and let L_1 and L_2 be the sets of nodes of these paths, different from p and q. From Lemma 1, we have

$$a_u = a_v \quad \forall u, v \in L_i, \;\; i = 1, 2. \tag{2}$$

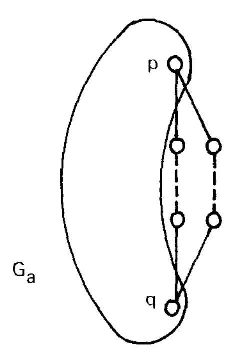

Fig. 2.

Consider a stable set $S \in \mathcal{S}$. Then S must induce a maximum stable set in each of the paths depending on which of p, q belongs to S. Thus from (2) it is easy to verify for L_i, $i=1,2$ that, if $|L_i|$ is even (odd), then

$$\begin{aligned}
&\text{if } \{p, q\} \subset S && \text{then } |S \cap L_i| = \frac{|L_i|-2}{2}\left(\frac{|L_i|-1}{2}\right),\\
&\text{if } |\{p, q\} \cap S| = 1 && \text{then } |S \cap L_i| = \frac{|L_i|}{2}\left(\frac{|L_i|-1}{2}\right),\\
&\text{if } \{p, q\} \cap S = \emptyset && \text{then } |S \cap L_i| = \frac{|L_i|}{2}\left(\frac{|L_i|+1}{2}\right).
\end{aligned} \tag{3}$$

Now let us examine the hole C.

Case a. C is odd: Then sets L_1 and L_2 are such that if one is odd the other is even. From (3) it is clear that if $S \in \mathcal{S}$ then

$$\sum_{u \in C} x_u^S = \frac{|C|-1}{2},$$

a contradiction.

Case b. C is even: Then sets L_1 and L_2 have the same parity. Suppose $|L_1| \leqslant |L_2|$ and let $\gamma = |L_2| - |L_1|$. Note that γ is even. From (3) every stable set $S \in \mathcal{S}$ satisfies the following conditions:

$$|S \cap L_2| = |S \cap L_1| + \frac{\gamma}{2} \quad \Leftrightarrow \quad \sum_{u \in L_2} x_u^S - \sum_{u \in L_1} x_u^S = \frac{\gamma}{2}.$$

Since p and q have zero coefficients in the above equation, but p, q are nodes of G_a, this inequality cannot be a positive multiple of $ax \leqslant \alpha$, a contradiction. This completes the proof of our Lemma. □

Lemma 3. *G_a does not contain a node of degree one.*

Proof. Suppose G_a contains a node, say u, of degree one, and let v be the node of G_a adjacent with u. Then every stable set $S \in \mathcal{S}$, contains either u or v, which implies that $x_u^S + x_v^S = 1$ for all $S \in \mathcal{S}$, a contradiction. □

3. Application to series-parallel graphs

A *cut vertex* in a graph $G = (V, E)$ is a vertex whose removal increases the number of connected components in G. A connected graph $G = (V, E)$ is called *2-connected* if $|E| \geqslant 2$ and G contains no cut vertex.

Duffin [4] showed that every 2-connected series-parallel graph (without multiple edges) can be obtained by a recursive application of the following operations starting from the graph consisting of a triangle.

(a) Subdivide an edge;

(b) if uv is an edge, add the path (uw, wv) where w is a new node.

Theorem 4. *If G is a series-parallel graph, then $P(G)$ is defined by* (1).

Proof. Suppose $P(G)$ has a nontrivial facet inducing inequality $ax \leqslant \alpha$. (We have seen earlier that $a \geqslant 0$). Then G_a is series-parallel and we can assume G_a has more than one edge. We claim that G_a contains an induced 2-connected subgraph. In fact, if G_a is 2-connected, the claim is true. If this is not the case, then G_a contains a cut vertex v_0 that separates it into two (induced) subgraphs G_a^1 and G_a^2 having exactly the node v_0 in common and no edges in common. Moreover, we may assume that one of those two subgraphs, say G_a^1, has no cut vertex (because if not we can separate again G_a^1). But from Lemma 3, G_a^1 must have more than one edge. Thus G_a^1 is 2-connected and our claim is proved. Consequently, by Duffin's results, G_a^1 must contain two nodes that are joined by more than one path in which all internal nodes are of degree two. But then it follows from Lemma 2 that G_a is an odd hole and the proof is complete.

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