# Two-edge connected spanning subgraphs and polyhedra 

Ali Ridha Mahjoub<br>Department of Statistics and Operations Research, College of Sciences, King Saud University, Riyadh, Saudi Arabia

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#### Abstract

This paper studies the problem of finding a two-edge connected spanning subgraph of minimum weight. This problem is closely related to the widely studied traveling salesman problem and has applications to the design of reliable communication and transportation networks. We discuss the polytope associated with the solutions to this problem. We show that when the graph is series-parallel, the polytope is completely described by the trivial constraints and the so-called cut constraints. We also give some classes of facet defining inequalities of this polytope when the graph is general.


Keywords: Two-edge connected graphs; Polyhedra; Facets; Series-parallel graphs

## 1. Introduction and notation

The graphs we consider are finite, undirected, loopless and may have multiple edges. We denote a graph by $G=(V, E)$ where $V$ is the node set and $E$ the edge set of $G$. Given $S \subseteq V$, we denote by $\delta(S)$ the set of edges having exactly one end in $S$. The edge set $\delta(S)$ is called a cut.

A graph $G$ is called $k$-edge connected if $G$ contains no cut having less than $k$ edges. Given a graph $G=(V, E)$ and a function $w: E \rightarrow \mathbb{R}$ which associates the weight $w(e)$ to each edge $e \in E$, the two-edge connected spanning subgraph problem (TECSP for short) is to find a two-edge connected subgraph $H=(V, F)$ of $G$ spanning all nodes in $V$, such that $\sum_{e \in F} w(e)$

[^0]is minimum. This problem has applications to the design of reliable communication and transportation networks [4,24].

In this paper we study the polytope associated with the solutions to this problem. Our aim is, in fact, to propose a polyhedral approach to the study of the TECSP. We show that when the graph is series-parallel, this polytope is completely described by the trivial constraints and the so-called cut constraints. We also discuss various classes of facets of this polytope.

The widely-studied traveling salesman problem $[3,20,23]$ is closely related to the TECSP in that the objective is to find a minimum-weight (Hamiltonian) cycle spanning all vertices in $V$. In fact, as it is pointed out in [11], the problem of determining if a graph $G=(V, E)$ contains a Hamiltonian cycle can be reduced to the TECSP. Thus the TECSP is NP-hard. This relationship between the traveling salesman problem and the TECSP has been widely investigated in the past few years [7,14,22]. Recently, Monma, Munson and Pulleyblank [22] have studied the TECSP in the metric case, that is when the underlying graph is complete and the weight function $w(\cdot)$ is nonnegative and satisfies the triangle inequality (i.e. $w\left(e_{1}\right) \leqslant w\left(e_{2}\right)+w\left(e_{3}\right)$ for every three edges $e_{1}, e_{2}, e_{3}$ defining a triangle in $G$ ). Even in this case the traveling salesman problem and, thus the TECSP are NP-hard. In particular, they showed that the weight of an optimal traveling salesman tour in this case is no greater than $\frac{4}{3} Q$, where $Q$ is the weight of an optimal two-edge connected spanning subgraph of $G$. This has been conjectured and largely proved by Frederickson and Ja'Ja' [14]. From this, it follows that the value of an optimal solution of the linear relaxation of the traveling salesman problem is no greater than $\frac{4}{3} Q$. Recently Cunningham [7] strengthened this by showing that this is never greater than $Q$.

If $G=(V, E)$ is a graph and $F \subseteq E$ an edge set, then the $0-1$ vector $x^{F} \in \mathbb{R}$ with $x^{F}(e)=1$ if $e \in F$ and $x^{F}(e)=0$ if $e \notin F$ is called the incidence vector of $F$. The convex hull TECP $(G)$ of the incidence vectors of all edge sets of two-edge connected spanning subgraphs of $G$ is called the two-edge connected spanning subgraph polytope of $G$, i.e.,
$\operatorname{TECP}(G):=\operatorname{conv}\left\{x^{F} \in \mathbb{R}^{E} \mid(V, F)\right.$
is a two-edge connected spanning subgraph of $G\}$.
Thus the TECSP is equivalent to the following linear program:

$$
\begin{equation*}
\operatorname{Min}\{w x \mid x \in \operatorname{TECP}(G)\} . \tag{1.1}
\end{equation*}
$$

Hence whenever the problem (1.1) can be solved in polynomial time, the problem TECSP can also be solved in polynomial time.

Since the TECSP is NP-hard, we cannot expect to find a complete explicit characterization in terms of linear inequalities of $\operatorname{TECP}(G)$ for all graphs $G$. It may however be that for certain classes of graphs $G$, the polytope $\operatorname{TECP}(G)$ can be described by means of a few classes of linear inequalities and that for these classes of inequalities polynomial time algorithms can be designed, so that the TECSP for these graphs can be solved in polynomial time.

Recent work in the $\operatorname{TECP}(G)$ can be found in Grötschel and Monma [17], Grötschel et al. [18,19] and Barahona and Mahjoub [2]. In [17] Grötschel and Monma consider a more general polytope, that is, the polytope associated with the $k$-edge connected spanning subgraphs of a graph $G$, where $k$ is a fixed positive integer. They discuss basic facets of this polytope. In $[18,19]$ the authors describe further classes of facets of this polytope and devise a cutting plane algorithm for the associated optimization problem. In [2] Barahona and Mahjoub give a complete characterization of the $\operatorname{TECP}(G)$ for the class of Halin graphs.

Given $b: E \rightarrow \mathbb{R}$ and $F \subseteq E, b(f)$ will denote $\sum_{e \in F} b(e)$. If ( $\left.V, F\right)$ is a two-edge connected spanning subgraph of $G=(V, E)$, then $x^{F}$ must satisfy the following inequalities:

$$
\begin{align*}
& x(e) \geqslant 0 \quad \text { for all } e \in E,  \tag{1.2}\\
& x(e) \leqslant 1 \quad \text { for all } e \in E,  \tag{1.3}\\
& x(\delta(S) \geqslant 2 \quad \text { for all } S \subseteq V, \emptyset \neq S \neq V . \tag{1.4}
\end{align*}
$$

We will call the inequalities (1.2)-(1.3) trivial constraints and the inequalities (1.4) cut constraints.

Using the famous maximum flow-minimum cut theorem (see Ford and Fulkerson [13]), one can determine a minimum cutset in a weighted undirected graph by solving $|V|-1$ maximum flows. In fact, this can be obtained by calculating the maximum flows between the $|V|-1$ pairs of nodes $(s, t), t \in V\{s\}$, where $s$ is a fixed node in $V$. Because the maximum flow calculation can be carried out in polynomial time (see Dinits [8] and Edmonds and Karp [10]), it follows that the minimum cutset problem and hence the separation problem for constraints (1.4) (i.e. the problem that consists to decide whether a given vector $y \in \mathbb{R}^{|E|}$ satisfies (1.4) and if not to find a violated inequality) can be solved in polynomial time. From [15], this implies that there is a polynomial time algorithm for the solution of (1.1) whenever $\operatorname{TECP}(G)$ is completely described by the inequalities (1.2)-(1.4).

In the next section we study series-parallel graphs and we show that, for this class of graphs, the polytope $\operatorname{TECP}(G)$ is defined by inequalities (1.2)-(1.4). In Section 3 we study the conditions under which inequalities (1.2)-(1.4) define facets for $\operatorname{TECP}(G)$. We also introduce a large class of facet defining inequalities for $\operatorname{TECP}(G)$ called odd-wheel inequalities.

The remainder of this section is devoted to more definitions and notations.
If $G=(V, E)$ is a graph and $e \in V$ is an edge with endnodes $i$ and $j$, we also write $i j$ to denote the edge $e$. If $S \subseteq V$, then $G(S)$ denotes the induced subgraph of $G$ on $S$.

For $e \in E, G-e$ denotes the subgraph of $G$ obtained from $G$ by deleting the edge $e$. For $S \subseteq V$, the set of edges having their endnodes in $S$ is denoted by $\gamma(S)$. If $S_{1}, S_{2}$ are disjoint subsets of $V$, then $\left[S_{1}, S_{2}\right.$ ] denotes the set of edges of $G$ which have one endnode in $S_{1}$ and the other in $S_{2}$. An edge cutset $F \subseteq V$ of $G$ is a set of edges such that $F=\delta(S)=\delta(V-S)$ for some nonempty $S \subseteq V$. The sets $S$ and $V-S$ are called the shores of the edge cutset $F$. We write $k$-edge cutset for an edge cutset having $k$ edges. A cutset having one edge is called a bridge.

A polyhedron $P \subseteq \mathbb{R}^{m}$, is the intersection of finitely many halfspaces in $\mathbb{R}^{m}$. A polytope is a bounded polyhedron or, equivalently, the convex hull of finitely many points. The dimension of a polyhedron $P$, denoted by $\operatorname{dim}(P)$, is the maximum number of affinely independent points in $P$ minus one.

If $a \in \mathbb{R}^{m}-\{0\}, a_{0} \in \mathbb{R}$, then the inequality $a^{\mathrm{T}} x \leqslant a_{0}$ is said to be valid with respect to a polyhedron $P \subseteq \mathbb{R}^{m}$ if $P \subseteq\left\{x \in \mathbb{R}^{m} \mid a^{\mathrm{T}} x \leqslant a_{0}\right\}$. We say that a valid inequality $a^{\mathrm{T}} x \leqslant a_{0}$ supports $P$ or defines a face of $P$ if $\emptyset \neq P \cap\left\{x \mid a^{\mathrm{T}} x=a_{0}\right\} \neq P$. In this case the polyhedron $P \cap\left\{x \mid a^{\mathrm{T}} x=a_{0}\right\}$ is called the face associated with $a^{\mathrm{T}} x \leqslant a_{0}$. A valid inequality $a^{\mathrm{T}} x \leqslant a_{0}$ defines a facet of $P$ if it defines a face of $P$ and if there exist $\operatorname{dim}(P)$ affinely independent points in $P \cap\left\{x \mid a^{\mathrm{T}} x=a_{0}\right\}$. Two face defining inequalities $a^{\mathrm{T}} x \leqslant a_{0}$ and $b^{\mathrm{T}} x \leqslant b_{0}$ are called equivalent if $P \cap\left\{x \mid a^{\mathrm{T}} x=a_{0}\right\}=P \cap\left\{x \mid b^{\mathrm{T}} x=b_{0}\right\}$.

## 2. The polytope $\operatorname{TECP}(G)$ of a series-parallel graph

A homeomorph of $K_{4}$ is a graph obtained from $K_{4}$ when its edges are subdivided into paths by inserting new nodes of degree two. A graph is called series-parallel if it contain no homeomorph of $K_{4}$ as a subgraph.

Given a graph $G=(V, E)$, we let $P(G)$ (resp. $P^{\prime}(G)$ ) denote the polyhedron defined by the inequalities (1.2)-(1.4) (resp. (1.2) and (1.4)). Clearly, TECP $(G) \subseteq P(G)$ $\subseteq P^{\prime}(G)$. In general, both $P(G)$ and $P^{\prime}(G)$ may have fractional extreme points Cornuéjols, Fonlupt and Naddef [5] showed that for a series-parallel graph $G, P^{\prime}(G)$ is integral. In [12] Fonlupt and Naddef characterized the class of graphs $G$ for which $P^{\prime}(G)$ defines the convex hull of the tours of $G$ (a tour is a cycle going at least once through each node of $G$ ). This yields a polynomial time algorithm for the graphical salesman problem [5] in that class of graphs.

A natural question that may arise is whether or not $P(G)$ is integral whenever $P^{\prime}(G)$ so is. The answer to this question is in the negative as shown by the following example:

Consider the graph $K_{4}$ given in Fig. 1. It has been shown in [12] that $P^{\prime}\left(K_{4}\right)$ is integral. Let $x$ be the vector given by $x\left(e_{\mathrm{i}}\right)=\frac{1}{2}$ for $i=1,2,3$ and $x\left(e_{\mathrm{i}}\right)=1$ for $i=4,5,6$. Clearly, $x$


Fig. 1.
satisfies all the constraints (1.2)-(1.4). However, it is not hard to see that $x$ is an extreme point of $P\left(K_{4}\right)$, showing that $P\left(K_{4}\right)$ is not integral.

Thus it seems to be interesting to characterize the class of graphs $G$ for which $P(G)$ is integral, i.e., $P(G)=\operatorname{TECP}(G)$, since an efficient cutting plane algorithm can be used to solve the TECSP for this class of graphs. We do not have a complete characterization of these graphs, but, in what follows, we shall show that series-parallel graphs belong to this class.

Duffin [9] showed that every connected series-parallel graph can be obtained by a recursive application of the following operations starting from the graph consisting of two nodes joined by an edge:
(a) duplicate an edge (i.e. add an edge joining the same endnodes);
(b) subdivide an edge (i.e. replace an edge $u v$ by two edges $u w$ and $w v$ where $w$ is a new node).

We then have the following property for connected series-parallel graphs.

Remark 2.1. A connected series-parallel graph $G=(V, E)$, with $|E| \geqslant 2$, which does not contains multiple edges, has at least one node of degree 2.

Throughout we consider a 2-edge connected graph $G=(V, E)$ and we let $m=|E|$.
Let $x$ be a feasible solution of $P(G)$. A cut $\delta(W)$ is said to be tight for $x$ if $x(\delta(W))=2$. If $x$ is an extreme point of $P(G)$, then there is a set $\left\{\delta\left(W_{i}\right): i=1, \ldots, r\right\}$ of tight cuts and two edge sets $E_{1}, E_{2} \subseteq E$ such that $x$ is the unique solution of the linear system

$$
\left\{\begin{array}{l}
x(e)=1 \quad \text { for all } e \in E_{1}  \tag{2.1}\\
x(e)=0 \text { for all } e \in E_{2} \\
x\left(\delta\left(W_{i}\right)\right)=2 \text { for } i=1, \ldots, r
\end{array}\right.
$$

Theorem 2.1. If $G$ is series-parallel, then $\operatorname{TECP}(G)=P(G)$.

Proof. Since $G$ is 2-edge connected (otherwise TECP $(G)=P(G)=\emptyset$ ) we have that $m \geqslant 2$. Let us assume the contrary, that there is a (2-edge connected) series-parallel graph $G=(V$, $E)$ such that $\operatorname{TECP}(G) \neq P(G)$. Assume that $|E|$ is minimum i.e., that for any seriesparallel graph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ for which $\left|E^{\prime}\right|<|E|$ we have $\operatorname{TECP}(G)=P(G)$. Let $x$ be a fractional extreme point of $P(G)$.

Claim 1. $x(e)>0$ for every edge $e \in E$.
Proof of Claim 1. Suppose there is an edge $e_{0} \in E$ such that $x\left(e_{0}\right)=0$. Then let $x^{\prime} \in \mathbb{R}^{m-1}$ be defined by $x^{\prime}(e)=x(e)$ for $e \in E-\left\{e_{0}\right\}$. Obviously, $x^{\prime}$ is an extreme point of $P\left(G-e_{0}\right)$. Since $x^{\prime}$ is fractional and $P\left(G-e_{0}\right)=\operatorname{TECP}\left(G-e_{0}\right)$ we have a contradiction.

Claim 2. $G$ is 3-edge connected.
Proof of Claim 2. Assume $G$ is not 3-edge connected. Since $G$ is 2-edge connected, then $G$ contains a 2-edge cutset, say $\left\{e_{1}, e_{2}\right\}$. Hence $x\left(e_{1}\right)=x\left(e_{2}\right)=1$. Let $G^{*}=\left(V^{*}, E^{*}\right)$ be the graph obtained from $G$ by contracting $e_{1}$. Let $x^{*} \in \mathbb{R}^{m-1}$ be the restriction of $x$ on $E^{*}$.

It is clear that $x^{*}$ is feasible for $P\left(G^{*}\right)$. Moreover $x^{*}$ is an extreme point of $P\left(G^{*}\right)$. Indeed, if this is not the case, then there must exist two solutions $y^{\prime}$ and $y^{\prime \prime}$ of $P\left(G^{*}\right)$ such that $x^{*}=\frac{1}{2}\left(y^{\prime}+y^{\prime \prime}\right)$. Thus $y^{\prime}\left(e_{2}\right)=y^{\prime \prime}\left(e_{2}\right)=1$. Now consider the solutions $y^{* \prime}, y^{* \prime \prime} \in \mathbb{R}^{m}$ defined by

$$
y^{* \prime}(e)= \begin{cases}y^{\prime}(e) & \text { for } e \in E^{*} \\ 1 & \text { for } e=e_{1}\end{cases}
$$

and

$$
y^{* \prime \prime}(e)= \begin{cases}y^{\prime \prime}(e) & \text { for } e \in E^{*} \\ 1 & \text { for } e=e_{1}\end{cases}
$$

Clearly, $y^{* \prime}, y^{* \prime \prime}$ are feasible for $P(G)$. Also we have $x=\frac{1}{2}\left(y^{* \prime}+y^{* \prime \prime}\right)$, a contradiction. Consequently, $x^{*}$ is an extreme point of $P\left(G^{*}\right)$. Since $G^{*}$ is series-parallel and $x^{*}$ is fractional, by our minimality assumption, we then have a contradiction.

Claim 3. Each variable $x(e)$ has a nonzero coefficient in at least two of the equations of (2.1).

Proof of Claim 3. Clearly, each variable $x(e)$ has a nonzero coefficient in at least one of the equations of (2.1). Let us assume that there is an edge $e_{0}$ such that $x\left(e_{0}\right)$ has a nonzero coefficient in exactly one of those equations. Without loss of generality, we may assume that between the nodes of $e_{0}$, there is only one edge $\left(e_{\mathrm{o}}\right)$. Let (2.1)' be the system obtained from (2.1) by deleting the equation containing $x\left(e_{0}\right)$. Let $x^{0} \in \mathbb{R}^{m-1}$ be the solution given by $x^{0}(e)=x(e)$ for $e \in E \backslash\left\{e_{0}\right\}$. Then $x^{0}$ is fractional. In fact, this is clear if $x\left(e_{0}\right)$ is integer. If not, as $x$ satisfies (2.1), then $x$ must have at least two fractional components and thus $x^{0}$ is fractional. Moreover we have that $x^{0}$ is the unique solution of the system (2.1)' and, consequently, $x^{0}$ is an extreme point of $P\left(G^{*}\right)$, where $G^{*}$ is the graph obtained from $G$ by contracting $e_{0}$. Since $G-e_{0}$ is series-parallel, by our assumption, we have that $P\left(G-e_{0}\right)$ is integral, a contradiction.

Since $G$ is series-parallel and $m \geqslant 2$, by Remark 2.1 and Claim 2 it follows that $G$ contains two multiple edges, say $f, g$. Since any cut contains either both edges $f, g$ or neither of them, at most one of the variables $x(f)$ and $x(g)$ must take a fractional value. Let us assume, for instance, that $x(f)=1$. By Claim 3 , there is a cut $\delta\left(W_{i^{*}}\right), i^{*} \in\{1, \ldots, r\}$ which contains $f$ and hence also $g$. Moreover we have that $x(g)$ is fractional. In fact, if $x(g)=1$, then, from Claims 1 and 2 it follows that $2=x\left(\delta\left(W_{i^{*}}\right)>x(f)+x(g)=2\right.$, a contradiction. Consequently, by Claim 3, there must exist a cut $\delta\left(W_{j^{*}}\right), j^{*} \in\{1, \ldots, r\} \backslash\left\{i^{*}\right\}$ containing $f$ and $g$. Thus we have ( $\triangle$ means the "symmetric difference" which, for two arbitrary sets $I$ and $J$, is defined by $I \triangle J=(\Lambda J) \cup(\Omega \backslash I)$, the symmetric difference of two cuts is a cut $)$.

$$
\begin{aligned}
2 & \leqslant x\left(\delta\left(W_{i^{*}}\right) \triangle \delta\left(W_{j^{*}}\right)\right) \\
& =x\left(\delta\left(W_{i^{*}}\right)\right)+x\left(\delta\left(W_{j^{*}}\right)\right)-2\left(x\left(\delta\left(W_{i *}\right) \cap \delta\left(W_{j^{*}}\right)\right)\right) \\
& \leqslant 4-2(x(f)+x(g)) \\
& <2
\end{aligned}
$$

## 3. Facets of $\operatorname{TECP}(\boldsymbol{G})$

The successful applications of the polyhedral approach to some NP-hard combinatorial optimization problems, as the Traveling Salesman Problem [6], the Linear Ordering Problem [16] and the Max-Cut Problem [1], demonstrated that even a partial characterization of the polyhedron associated with an NP-hard problem could be sufficient to prove optimality of a solution to the problem. In this section we shall present a partial nonredundant system of inequalities for the polytope $\operatorname{TECP}(G)$.

### 3.1. Basic facets of TECP(G)

For a graph $G=(V, E)$, let us denote by $E^{*} \subseteq E$ the set of edges that belong to 2-edge cutsets of $G$. The two following theorems provide some basic properties of TECP $(G)$ which are easily seen to be true.

Theorem 3.1. (i) $\operatorname{dim}(\operatorname{TECP}(G))=|E|-\left|E^{*}\right|$. Consequently, $\operatorname{TECP}(G)$ is full dimensional (i.e., $\operatorname{dim}(P)=m$ ) if and only if $G$ is 3 -edge connected.
(ii) The inequality $x(e) \geqslant 0$ defines a facet of TECP $(G)$ if and only if $e$ is not in a 2edge cutset, and $e$ is not in a 3-edge cutset $U$ with $(U-\{e\}) \cap\left(E-E^{*}\right) \neq \emptyset$.
(iii) The inequality $x(e) \leqslant 1$ defines a facet of $\operatorname{TECP}(\mathrm{G})$ if and only if $e$ is not in a 2edge cutset.

Theorem 3.2. Let $S \subseteq V$ be a nonempty node subset of $V$. The valid inequality $x(\delta(S)) \geqslant 2$ defines a facet of $\operatorname{TECP}(G)$, if and only if the following are satisfied:
(a) $|\delta(S)| \geqslant 3$;
(b) $G(S)$ and $G(V-S)$ are both connected;
(c) if $G(S)(G(V-S))$ is not 2-edge connected, then (c.1) every bridge of $G(S)(G(V-S))$ is contained in a 2-edge cutset; (c.2) $1 \leqslant\left|\delta(S) \cap E^{*}\right| \leqslant 2$;
(c.3) for every partition $S_{1}, S_{2}$ of $G(V-S)$ with $\left|\left[S_{1}, S_{2}\right]\right|=1$ we have
(c.3.1) if $\left|\delta(S) \cap E^{*}\right|=1$ then either $\delta\left(S_{1}\right) \subseteq E^{*}$ or $\delta\left(S_{2}\right) \subseteq E^{*}$ and there do not exist two edges $u_{1} v_{1}, u_{2} v_{2} \in \delta(S)-E^{*}$ such that $v_{1}$ and $v_{2}$ are both of degree three, belong to $V-S(S)$ and are adjacent, with $v_{1} v_{2} \in \gamma(V-S)-E^{*}\left(v_{1} v_{2} \in \gamma(S)-E^{*}\right)$;
(c.3.2) if $\left|\delta(S) \cap E^{*}\right|=2$, then $|\delta(S)|=3$ and $\{f\}=\delta(S)-E^{*}$ is not contained in a 3-edge cutset $U$ with $(U-\{f\}) \cap\left(E-E^{*}\right) \neq \emptyset$.

From Theorem 3.2 it is not hard to obtain the following theorem which characterizes when cut constraints are equivalent.

Theorem 3.3 Two distinct cut constraints $x(\delta(S)) \geqslant 2$ and $x\left(\delta\left(S^{\prime}\right)\right) \geqslant 2$, defining facets for $\operatorname{TECP}(G)$, are equivalent if and only if
(i) $|\delta(S)|=\left|\delta\left(S^{\prime}\right)\right|$; and
(ii) $\delta(S)-E^{*}=\delta\left(S^{\prime}\right)-E^{*}$.

Theorem 3.3 is, in fact, important from the point of view of cutting planes, it characterizes the minimal set of cut constraints that should be included in any description of the polytope $\operatorname{TECP}(G)$ by a linear inequality system.

Consider the graph $K_{4}$ (see Fig. 1). As mentioned before, the vector $x$ such that $x\left(e_{i}\right)=\frac{1}{2}$ for $i=1,2,3$ and $x\left(e_{i}\right)=1$ for $i=4,5,6$ is an extreme point of $\operatorname{TECP}\left(K_{4}\right)$. In fact, this vector satisfies the equality $x\left(e_{1}\right)+x\left(e_{2}\right)+x\left(e_{3}\right)=\frac{3}{2}$. Therefore it is easy to see that for any 2-edge connected spanning subgraph of $K_{4}$ whose edge set is, say $F, x^{F}$ should satisfy $x\left(e_{1}\right)+x\left(e_{2}\right)+x\left(e_{3}\right) \geqslant 2$. Furthermore, this inequality defines a facet for the polytope $\operatorname{TECP}\left(K_{4}\right)$. In the following we prove this as a special case of a more general class of facet defining inequalities for the polytope $\operatorname{TECP}(G)$.

### 3.2. Odd-wheel inequalities

Given a 2-edge connected graph $G=(V, E)$, an odd-wheel configuration (see Fig. 2) is defined by an integer $k \geqslant 1$, integers $p_{i}$, for $i=1, \ldots, 2 k+1$, and a partition of the node set $V$ into $V_{i}^{s}, V_{0}$ for $i=1,2, \ldots, 2 k+1$ and $s=0,1, \ldots, p_{i}$ such that
(1) the graphs $G\left(V_{i}^{s}\right)$ and $G\left(V_{0}\right)$ are 3-edge connected for $i=1,2, \ldots, 2 k+1$ and $s=0$, $1, \ldots, p_{i}$;
(2) the edge set $\left[V_{i}^{0}, V_{i+1}^{0}\right]$ is nonempty for $i=1,2, \ldots, 2 k+1$ (modulo $2 k+1$ );
(3) the edge set $\left[V_{i}^{s}, V_{i}^{s+1}\right]$ is nonempty and, if $p_{i}>0,\left|\left[V_{i}^{s}, V_{i}^{s+1}\right]\right|=1$ for $i=1,2$, $\ldots, 2 k+1$ and $s=0,1, \ldots, p_{i}$ (for convenience we let $V_{i}^{p_{i}+1}=V_{0}$ for $i=1,2, \ldots, 2 k+1$ );
(4) the edge set $\left[V_{i}^{s}, V_{i}^{q}\right]$ is empty for $1 \leqslant i \leqslant 2 k+1,0 \leqslant s, q \leqslant p_{i}+1, q \neq s+1$ and ( $s$, q) $=\left(0, p_{i}+1\right)$;
(5) the edge set $\left[V_{i}^{s}, V_{i}^{q}\right]$ is empty for $1 \leqslant i, t \leqslant 2 k+1, i \neq t, 1 \leqslant s \leqslant p_{i}+1$ and $1 \leqslant q \leqslant p_{t}+1$.

Let $r_{i}$, for $1 \leqslant i \leqslant 2 k+1$, denote the largest integer such that $0 \leqslant r_{i} \leqslant p_{i}$ and $\left|\delta\left(V_{i}^{r_{i}}\right)\right| \geqslant 3$. We denote by $e_{i, s}$ a fixed edge in $\left[V_{i}^{s}, V_{i}^{s+1}\right.$ ] for $i=1, \ldots, 2 k+1$ and $s=0,1, \ldots, p_{i}$. Let

$$
F=E^{1}-\left\{e_{i, s} ; 1 \leqslant i \leqslant 2 k+1, r_{i} \leqslant s \leqslant p_{i}\right\},
$$

where $E^{1}$ is the set of edges that are in the edge cutsets $\delta\left(V_{i}^{s}\right)$ for $1 \leqslant i \leqslant 2 k+1$ and $0 \leqslant s \leqslant p_{i}+1$, that is

$$
E^{1}=\bigcup_{\substack{1 \leqslant i \leqslant 2 k+1 \\ 0 \leqslant s \leqslant p_{i}+1}} \delta\left(V_{i}^{S}\right)
$$

The odd-wheel inequality associated with the odd-wheel configuration is


Fig. 2. An odd-wheel configuration.

$$
x(F) \geqslant k+1+\sum_{i=1, \ldots, 2 k+1} r_{i}
$$

It is easy to see that odd-wheel inequalities are valid for TECP $(G)$. Moreover, we have:

Theorem 3.4. Odd-wheel inequalities define facets of $\operatorname{TECP}(G)$.

Proof. See [21].

In [2] Barahona and Mahjoub show that for a Halin graph $G, \operatorname{TECP}(G)$ is described by the trivial, cut and odd-wheel constraints.

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[^0]:    Present address: Laboratoire d'Informatique de Brest (LIBr),Université de Bretagne Occidentale, 6 avenue Le Gorgeu, 29287 Brest cedex, France.
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