# On perfectly two-edge connected graphs 

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#### Abstract

This paper studies the graphs for which the 2 -edge connected spanning subgraph polytope is completely described by the trivial inequalities and the so-called cut inequalities. These graphs are called perfectly 2 -edge connected. The class of perfectly 2 -edge connected graphs contains for instance the class of series-parallel graphs. We introduce a new class of perfectly 2 -edge connected graphs. We discuss some structural properties of graphs which are (minimally with respect to some reduction operations) nonperfectly 2 -edge connected. Using this we give sufficient conditions for a graph to be perfectly 2 -edge connected.


Keywords: 2-edge connected graphs; Polytopes; Separation problem

## 1. Introduction and notation

We consider finite, undirected and loopless graphs, which may have multiple edges. We denote a graph by $G=(V, E)$ where $V$ is the node set and $E$ is the edge set. Given $S \subseteq V, S \neq \emptyset$, we denote by $\delta(S)$ the set of edges with exactly one endnode is $S$. The edge set $\delta(S)$ is called a cut.

A graph $G$ is called $k$-edge connected if it contains no cut having less than $k$ edges. Given a graph $G=(V, E)$ and a function $w: E \rightarrow R$ which associates the weight $w(e)$ to each edge $e \in E$, the 2-edge connected spanning subgraph problem (TECSP) consists of finding a two edge connected subgraph $H=(V, F)$ of $G$, spanning all the nodes of $G$, and such that $\sum_{e \in F} w(e)$ is minimum. This problem has applications to the design of reliable communication and transportation networks [5,27]. Our objective in this paper is to study the TECSP from a polyhedral point of view. We introduce and discuss a class of graphs, called perfectly 2 -edge connected graphs, in which the TECSP can be solved in polynomial time using a cutting plane algorithm. This class consists of the graphs for which the polytope associated with the solutions to the problem can be completely described by the trivial inequalities and the so-called cut inequalities. This class contains for instance the class of series-parallel graphs. We
introduce a new class of perfectly 2-edge connected graphs. We discuss some structural properties for graphs which are (minimally, with respect to some reduction operations) nonperfectly 2 -edge connected. Using this we describe sufficient conditions for a graph to be perfectly 2 -edge connected.

If $G=(V, E)$ is a graph and $F \subseteq E$ an edge set, then the $0-1$ vector $x^{F} \in R^{|E|}$ with $x_{e}^{F}=1$ if $e \in F$, and $x_{e}^{F}=0$ if $e \notin F$ is called the incidence vector of $F$. The convex hull of the incidence vectors of all edge sets of 2-edge connected spanning subgraphs of $G$, denoted by $\operatorname{TECP}(G)$, is called the 2-edge connected spanning subgraph polytope of $G$, i.e.
$\operatorname{TECP}(G)=\operatorname{conv}\left\{x^{F} \in R^{|E\rangle} \mid(V, E)\right.$ is a 2-edge connected

$$
\text { spanning subgraph of } G\}
$$

Thus, the TECSP is equivalent to the following linear program:

$$
\begin{equation*}
\operatorname{Min}\{w x, x \in \operatorname{TECP}(G)\} \tag{1.1}
\end{equation*}
$$

Hence, whenever problem (1.1) can be solved in polynomial time, the TECSP can be solved in polynomial time.

To solve problem (1.1) using linear programming methods, we need a complete description of the polytope $\operatorname{TECP}(G)$ in terms of linear inequalities. Since the TECSP is NP-hard, such a description is unlikely to be found for all graphs. However, it may be that for certain classes of graphs $G$, the polytope $\operatorname{TECP}(G)$ can be described by means of a few classes of linear inequalities and that for these classes of inequalities, polynomial-time algorithms can be designed so that the TECSP for these graphs can be solved in polynomial time.

Given $b: E \rightarrow R$ and $F \subseteq E, b(F)$ will denote $\sum_{e \in F} b(e)$. If $(V, E)$ is a 2-edge connected spanning subgraph of $G=(V, E)$, then $x^{F}$ satisfies the following inequalities:

$$
\begin{array}{ll}
x(e) \geqslant 0 \quad \text { for all } e \in E, \\
x(e) \leqslant 1 \quad \text { for all } e \in E, \\
x(\delta(S)) \geqslant 2 & \text { for all } S \subset V, S \neq \emptyset \tag{1.4}
\end{array}
$$

Inequalities (1.2), (1.3) are called trivial inequalities and the inequalities (1.4) are called cut inequalities.

The TECSP is closely related to the widely studied traveling salesman problem [4, $13,23,26]$. In fact, as it is pointed out in [13], the problem of determining whether a graph contains a Hamiltonian cycle, can be reduced to the TECSP. Thus, the TECSP is NP-hard. It has been shown to be polynomially solvable in series-parallel graphs [30] and Halin graphs [29]. The relation between the TECSP and the traveling salesman problem has been widely investigated in the past few years [13, 16, 25]. In [25] Monma, Munson and Pulleyblank studied the TECSP in the metric case,
that is when the underlying graph is complete and the weight function $w(\cdot)$ is nonnegative and satisfies the triangle inequality (i.e. $w\left(e_{1}\right) \leqslant w\left(e_{2}\right)+w\left(e_{3}\right)$ for every three edges $e_{1}, e_{2}, e_{3}$ defining a triangle). Even in this case the traveling salesman problem and thus the TECSP are NP-hard. In particular, they showed that in this case $\tau \leqslant 4 Q_{2} / 3$. Here $\tau$ denotes the weight of an optimal traveling salesman tour and $Q_{k}$ denotes the weight of an optimal $k$-edge connected spanning subgraph of $G$ where $k$ is fixed.

The subtour polytope of the traveling salesman problem is the set of all the solutions of the system given by inequalities (1.2)-(1.4) together with the equalities $x(\delta(v))=2$ for all $v \in V$. Clearly, a $0-1$ solution of this system of constraints corresponds to a Hamiltonian cycle in the graph. Let $\omega$ be the value of a solution of the subtour polytope for which $w x$ is minimized. Obviously, $\omega \leqslant \tau$ and thus $\omega \leqslant 4 Q_{2} / 3$. Cunningham [25] strengthened this by showing that $\omega \leqslant Q_{2}$. Recently, Goemans and Bertsimas [17] extended this result to $k$-edge connected subgraphs by showing that $\omega \leqslant 2 Q_{k} / k$ for every $k$.

The polytope $\operatorname{TECP}(G)$ has been extensively investigated in the past few years. In [24] it is shown that if $G$ is series-parallel [10], then $\operatorname{TECP}(G)$ is completely described by inequalities (1.2)-(1.4). It is also characterized when the inequalities (1.2)-(1.4) define facets for the polytope $\operatorname{TECP}(G)$ and a large class of facet defining inequalities for $\operatorname{TECP}(G)$, called odd-wheel inequalities, is introduced. In [2] Barahona and Mahjoub show that the odd-wheel inequalities together with the inequalities (1.2)-(1.4) completely describe the polytope $\operatorname{TECP}(G)$ when $G$ is a Halin graph. In [1] Baiou and Mahjoub characterized the Steiner 2-edge connected subgraph polytope for series-parallel graphs. In [19] Grötschel and Monma consider a more general model related to the design of minimum-cost survivable networks. They discuss polyhedral aspects of this model. In particular, they study a more general polytope, the extreme points of which are the incidence vectors of the edge sets of the $k$-edge connected spanning subgraphs of a graph $G$, where $k$ is a fixed integer. They describe basic facets of this polytope. In [20-22] Grötschel, Monma and Stoer describe further classes of facets of that polytope and devise a cutting plane algorithm for the associated optimization problem along with a computational study is presented. A complete survey of that model can be found in Stoer [28].

Related work can also be found in [3,6-8, 14]. In particular, Fonlupt and Naddef [14] characterized the class of graphs $G$ for which the polyhedron described by the inequalities (1.2) and (1.4) is the convex hull of the incidence vectors of the tours of $G$ (a tour is a cycle going at least once through each node). This yields a polynomial-time algorithm for the graphical traveling salesman problem [6] in that class of graphs. In [6] Cornuéjols, Fonlupt and Naddef studied the polyhedron defined by the inequalities (1.2) and (1.4). They showed that when the graph is series-parallel, this polyhedron has integral extreme points. In [7, 8] Coullard et al. study the polytope, the extreme points of which are the edge sets of the 2-node connected Steiner subgraphs of $G$. They characterized that polytope for series-parallel graphs and its dominant for graphs which do not have $W_{4}$ (the wheel on 5 nodes) as a minor. In [3] Chopra
discussed the polyhedron associated with the $k$-edge connected spanning subgraphs of a graph $G$ where multiple copies of an edge may be considered. In particular, he characterized that polyhedron for the class of the outerplanar graphs when $k$ is odd.

Let us denote by $P(G)$ the polytope defined by the inequalities (1.2)-(1.4). The polytope $P(G)$ is a relaxation of both the polytope $\operatorname{TECP}(G)$ and the subtour polytope. Thus, minimizing $w x$ over the polytope $P(G)$ provides a lower bound for the optimal solutions of both the traveling salesman problem and the TECSP. In general, the polytope $P(G)$ may have fractional extreme points.

Using the famous maximum flow-minimum cut theorem (see [15]), one can determine a minimum cutset in a weighted undirected graph by solving $|V|-1$ maximum flows. In fact, this can be obtained by calculating the maximum flows between the $|V|-1$ pairs of nodes $(s, t), t \in V \backslash\{s\}$, where $s$ is a fixed node in $V$. Because the maximum flow problem can be solved in polynomial time (see [9, 12]), it follows that the minimum cutset problem and hence the separation problem over the polytope $P(G)$ (i.e the problem that consists to decide whether a given vector $y \in R^{|E|}$ satisfies the inequalities (1.2)-(1.4) and if not to find a violated inequality) can be solved in polynomial time. From [18], this implies that there is a polynomial-time algorithm for the solution of (1.1) whenever $\operatorname{TECP}(G)=P(G)$.

Thus, it seems to be interesting to characterize the class of graphs $G$ for which $\operatorname{TECP}(G)=P(G)$. This was our motivation for studying this class of graphs. In this paper we give a partial characterization of this class of graphs.

We will call a graph $G$ perfectly 2-edge connected (perfectly-TEC) if $\operatorname{TECP}(G)=$ $P(G)$. Thus, series-parallel graphs are perfectly-TEC. In Section 2 we introduce a new class of perfectly-TEC graphs. In Section 3 we discuss structural properties for (minimally, with respect to some reduction operations) nonperfectly 2 -edge connected graphs. In Section 4 we use these properties to give sufficient conditions for a graph to be perfectly-TEC.

The remainder of this section is devoted to more definitions and notations.
If $G=(V, E)$ is a graph and $e \in E$ is an edge with endnodes $i$ and $j$, we also write $i j$ to denote $e$. If $e \in E$ is an edge, then $G-e$ is the graph obtained from $G$ by deleting $e$. If $W \subseteq V$ is a subset of nodes, then $G \backslash W$ is the graph obtained by deleting $W$ and the edges adjacent to the nodes of $W$, and $G / W$ is the graph obtained by contracting the nodes in $W$ to a new node (retaining multiple edges). For $W, W^{\prime} \subseteq V, \delta\left(W, W^{\prime}\right)$ denotes the set of edges having one endnode in $W$ and the other in $W^{\prime}$. An edge cutset $F \subseteq V$ of $G$ is a set of edges such that $F=\delta(S)=\delta(V \backslash S)$ for some nonempty set $S \subseteq V$. For $W \subseteq V$, we denote by $E(W)$ the set of edges having both nodes in $W$ and by $G(W)$ the subgraph induced by $W$.

A path $P$ in $G$ is a sequence of nodes $v_{0}, v_{1}, \ldots, v_{k}$, such that $v_{i} v_{i+1}$ is an edge for $i=0, \ldots, k-1$, and no node appears more than once in $P$.

Let $G=(V, E)$ be a graph and let $x^{*}$ be an extreme point of $P(G)$. An inequality $a x \leqslant \alpha$ is said to be tight for $x^{*}$ if $a x^{*}=\alpha$. We will denote by $\mathbf{C}\left(x^{*}\right)\left(\mathbf{T}\left(x^{*}\right)\right)$ the set of all the cut (trivial) inequalities tight for $x^{*}$.

## 2. A further class of perfectly-TEC graphs

In this section we shall introduce a further class of perfectly-TEC graphs. Let $\Gamma$ be the class of graphs $G=(V, E)$ for which there is a node subset $S \subseteq V$ such that:
(1) $|S|=3$ and $S$ covers all the edges of $G$,
(2) $|V \backslash S| \geqslant 3$ and if $|V \backslash S|=3$ then $G=K_{3,3}$,
(3) $\delta(S)$ does not contain multiple edges.

Fig. 1 shows some graphs of $\Gamma$. Note that graphs in $\Gamma$ can be recognised in polynomial time and may be non series-parallel. If $|V \backslash S|=3$ and $G \neq K_{\mathbf{3}, \mathbf{3}}, G$ may not be perfectly-TEC. In fact, consider the case where the graph $G$ is $K_{3,3}$ plus one edge in $G(S)$. Then by removing certain edges, one can obtain the graph that consists of $K_{4}$ with two adjacent edges subdivided. This graph is not perfectly-TEC (see Section 3), which implies that the graph $G$ itself is not perfectly-TEC.

Theorem 2.1. The graphs in $\Gamma$ are perfectly-TEC.

Proof. Let $G(V, E)$ be a graph of $\Gamma$. Let $S=\left\{s_{1}, s_{2}, s_{3}\right\}$ and $\bar{S}=V \backslash S=$ $\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}, n \geqslant 3$. In what follows, we are going to consider the case where $n \geqslant 4$. The case where $n=3$ (for which $G=K_{3,3}$ ) can be treated in a similar way.

Without loss of generality, we may assume that each node $v_{i}$ is adjacent to every node in $S$ (if the theorem holds in this case, then it holds for every graph in $\Gamma$ (see Lemma 3.1)). Thus the graph $G$ is 3-edge connected. Let $P^{\prime}(G)$ be the polytope given by the constraints

$$
\begin{array}{ll}
0 \leqslant x(e) \leqslant 1 & \text { for all } e \in E, \\
x(\delta(v)) \geqslant 2 & \text { for all } v \in V . \tag{2.2}
\end{array}
$$

We shall show that $\operatorname{TECP}(G)=P^{\prime}(G)$. First note that a $0-1$ vector is a solution of $P^{\prime}(G)$ if and only if it induces a 2-edge connected spanning subgraph of $G$. Thus, it suffices to show that the polytope $P^{\prime}(G)$ has integral extreme points.


Fig. 1. Graphs in $\Gamma$.

Let $A$ denote the matrix of the system $x(\delta(v))=2$ for all $v \in V$. If $E(S)=\emptyset$, then $A$ is a transportation matrix and, therefore, is totally unimodular. Thus, the matrix defining $P^{\prime}(G)$ is also totally unimodular and, consequently, $P^{\prime}(G)$ has integral extreme points. So let us suppose that $E(S) \neq \emptyset$ and let $x^{*}$ be an extreme point of $P^{\prime}(G)$. Let $B$ be the matrix of the cut constraints that define $x^{*}$. If $x^{*}$ has zero values for all the edges of $E(S)$, then the matrix $B$ is a submatrix of $A$ and hence it is totally unimodular, which implies that $x^{*}$ is integral. Thus, let us suppose that $x^{*}(f)>0$ for some edge $f \in E(S)$. Without loss of generality, we may assume that $f=s_{1} s_{2}$. From the structure of the graph $G$ and constraints (2.2), it follows that $x^{*}\left(\delta\left(\left\{s_{1}, s_{2}\right\}, v\right)\right) \geqslant 1$ for all $v \in \bar{S}$. Thus, $x^{*}\left(\delta\left(\left\{s_{1}, s_{2}\right\}, \bar{S}\right)\right) \geqslant 4$, since $|\bar{S}| \geqslant 4$. As $x^{*}(f)>0$, it follows that at most one of the cuts $\delta\left(s_{1}\right)$ and $\delta\left(s_{2}\right)$ is tight for $x^{*}$. Thus, we may suppose, for instance, that either $x^{*}\left(\delta\left(s_{1}\right)\right)>2$ and $x^{*}\left(\delta\left(s_{2}\right)\right)>2$ or $x^{*}\left(\delta\left(s_{1}\right)\right)>2$ and $x^{*}\left(\delta\left(s_{2}\right)\right)=2$ holds. If the latter situation happens, we should also have $x^{*}\left(\delta\left(s_{3}\right)\right)>2$, since $x^{*}\left(\delta\left(\left\{s_{2}, s_{3}\right\}, \bar{S}\right)\right) \geqslant 4$ also holds. Consequently, at most one of the cuts $\delta\left(s_{1}\right), \delta\left(s_{2}\right)$, $\delta\left(s_{3}\right)$ is tight for $x^{*}$. And hence the matrix $B$ defining $x^{*}$ can be obtained from $A$ by deleting at least two of the rows corresponding to $s_{1}, s_{2}, s_{3}$ and adding columns (corresponding to the edges in $E(S)$ ) containing at most one nonzero entry. Obviously, this matrix is totally unimodular and thus $x^{*}$ is integral, which finishes the proof of our theorem.

Remark 2.2. Theorem 2.1 can also be proved in the following way.
A 2-cover of a graph $G$ is a subgraph in which each node is of degree at least two. It is easy to see that for a graph $G$ in $\Gamma$, a subgraph of $G$ is a 2 -cover if and only if it is a 2-edge connected spanning subgraph of $G$. Thus if $G$ is a graph of $\Gamma$, the 2-cover polytope and the polytope $\operatorname{TECP}(G)$ coincide. A complete characterization of the 2-cover polytope is known, it is given by the inequalities (2.1), (2.2) and the so-called 2 -cover inequalities. This characterization is a consequence of the $b$-matching polytope (see $[11,21]$ ). One can easily show that in the case of a graph in $\Gamma$, the 2-cover inequalities do not define facets.

## 3. On the structure of nonperfectly-TEC graphs

In this section we shall discuss some structural properties of the nonperfectly 2-edge connected graphs. These properties will be used in the next section to give sufficient conditions for a graph to be perfectly-TEC. In the rest of the paper we consider 2-edge connected graphs.

### 3.1. Operations preserving perfectly-TEC

First, we describe three operations on graphs which preserve perfectly-TEC, these operations are given by the following lemmas.

Lemma 3.1. Let $G=(V, E)$ be a graph and fbe an edge of $G$. If $G$ is perfectly-TEC and $G-f$ is 2-edge connected, then G-f is perfectly-TEC.

Proof. Suppose not, then let $x$ be an extreme point of $P(G-f)$ which is fractional. Let $\bar{x} \in R^{|E|}$ such that

$$
\bar{x}(e)= \begin{cases}x(e) & \text { if } e \neq f \\ 0 & \text { if } e=f\end{cases}
$$

Thus, $\bar{x}$ is an extreme point of $P(G)$. Since $\bar{x}$ is fractional, this contradicts the fact that $G$ is perfectly-TEC.

Lemma 3.2. Let $G=(V, E)$ be a graph and let $W$ be a node subset of $V$ such that $G(W)$ is 2-edge connected. If $G$ is perfectly-TEC then $G / W$ is perfectly-TEC.

Proof. Assume the contrary. Let $x$ be a fractional extreme point of $P(G / W)$ and let $\bar{x}$ be the solution given by

$$
\bar{x}(e)= \begin{cases}x(e) & \text { if } e \in E \backslash E(W) \\ 1 & \text { if } e \in E(W)\end{cases}
$$

It is clear that $\bar{x} \in P(G)$. Moreover, $\bar{x}$ is an extreme point of $P(G)$. In fact, if this is not the case, then there are two solutions $y^{1}, y^{2} \in R^{|E|}$ such that $\bar{x}=\frac{1}{2}\left(y^{1}+y^{2}\right)$. Furthermore, we have that $y^{1}(e)=y^{2}(e)=1$ for all $e \in E(W)$. Now let $\bar{y}^{1}, \bar{y}^{2} \in R^{|E \backslash E(W)|}$ be the restrictions of $y^{1}$ and $y^{2}$ on $E \backslash E(W)$, respectively. It is clear that $\bar{y}^{1}$ and $\bar{y}^{2}$ belong to $P(G / W)$. Moreover, we have $x=\frac{1}{2}\left(\bar{y}^{1}+\bar{y}^{2}\right)$, which contradicts the fact that $x$ is an extreme point of $P(G / W)$. Thus $\bar{x}$ is an extreme point of $P(G)$. Since $\bar{x}$ is fractional, this is a contradiction.

Lemma 3.3. Let $G=(V, E)$ be a graph. Let uv be an edge of $G$ such that $u$ and $v$ are of degree two. If $G$ is perfectly-TEC then $G /\{u, v\}$ is perfectly-TEC.

Proof. Let $e_{1}$ and $e_{2}$ be the edges different from $u v$ adjacent to $u$ and $v$, respectively. If $G /\{u, v\}$ is not perfectly-TEC, then let $x$ be a fractional extreme point of $P(G /\{u, v\})$. Since $\left\{e_{1}, e_{2}\right\}$ is a 2 -edge cutset, we have $x\left(e_{1}\right)=x\left(e_{2}\right)=1$. Consider the solution $\bar{x} \in R^{|E|}$ given by

$$
\bar{x}(e)= \begin{cases}x(e) & \text { if } e \neq u v \\ 1 & \text { if } e=u v\end{cases}
$$

Clearly, this solution is fractional, moreover it defines an extreme point of $P(G)$, a contradiction.

Lemma 3.4. Let $G$ be a graph and $H$ a graph obtained from $G$ by subdividing one edge. If $G$ is perfectly-TEC then $H$ is perfectly-TEC.

## Proof. Easy. $\square$

Given a graph $G$, a subdivision of $G$ is any graph obtained from $G$ by inserting nodes of degree two on some edges of $G$. We then have the following corollaries.

Corollary 3.5. Any subdivision of a perfectly-TEC graph is perfectly-TEC.

Corollary 3.6. Any subdivision of a graph of $\Gamma$ is perfectly-TEC.

Now let $\theta_{1}, \theta_{2}, \theta_{3}$, be the operations described by Lemmas $3.1-3.3$, respectively, that is,
$\theta_{1}$ : delete an edge,
$\theta_{2}$ : contract a node set inducing a 2-edge connected subgraph,
$\theta_{3}$ : contract an edge whose endnodes are of degree two.
We say that a graph $G=(V, E)$ is reducible to a graph $H=(W, F)$ if $H$ can be obtained from $G$ by repeated applications of the operations $\theta_{1}, \theta_{2}, \theta_{3}$. A graph $G=(V, E)$ is called minimally nonperfectly-TEC if $G$ is not perfectly-TEC, but any graph obtained from $G$ by one of the operations $\theta_{1}, \theta_{2}, \theta_{3}$, is perfectly-TEC (see Fig. 2 for some minimally nonperfectly-TEC graphs. Remark that all these graphs are odd wheels with some internal edges subdivided. If $C$ is the (odd) exterior cycle of the wheel, then the polytope $P(G)$ has the fractional extreme point $x$ given by $x(e)=1 / 2$ for $e \in C$ and $x(e)=1$ otherwise).

In what follows, we are going to discuss some structural properties of minimally nonperfectly-TEC graphs.

### 3.2. Structural properties of minimally nonperfectly-TEC graphs

Let $G=(V, E)$ be a minimally nonperfectly-TEC graph and let $x$ be a fractional extreme point of $P(G)$. Then there must exist a subset $\mathbf{C}^{*}(x)$ of $\mathbf{C}(x)$ and two edge subsets $E^{0}, E^{1}$ of $E$ such that $x$ is the unique solution of the system

$$
\begin{array}{ll}
x(\delta(W))=2 & \text { for all } \delta(W) \in \mathbf{C}^{*}(x) \\
x(e)=1 & \text { for all } e \in E^{1} .  \tag{3.1}\\
x(e)=0 & \text { for all } e \in E^{0} .
\end{array}
$$

We then have the following lemmas. The three first lemmas easily follow from Lemmas 3.1-3.3, respectively, the proofs are then omitted.

Lemma 3.7. $x(e)>0$ for every edge $e \in E$.


Fig. 2. Examples of minimally nonperfectly-TEC graphs.

Lemma 3.8. Let $W \subseteq V$ be a node subset such that $|W| \geqslant 2$. If $G(W)$ is 2-edge connected, then $x$ has a fractional component for at least one edge of $E(W)$.

Lemma 3.9. $G$ does not contain an edge whose endnodes are of degree two.

Let $G=(V, E)$ be a graph. A family $\mathbf{F}$ of subsets of $V$ is said to be nested if, for every $W, Z \in \mathbf{F}$, either $W \subset Z$ or $Z \subset W=\emptyset$. A family of edge cutsets $\{\delta(W), W \in \mathbf{F}\}$ is said to be laminar if the family $\mathbf{F}$ is nested.

In what follows, we show that the set of cuts defining $x$ in the system (3.1) can be chosen so that the family of its node sets is laminar.

Lemma 3.10. System 3.1 can be chosen so that $\mathbf{C}^{*}(x)$ is laminar.

Proof. Suppose there are two edge cutsets $\delta(W)$ and $\delta(Z)$ of $\mathbf{C}^{*}(x)$ such that $Z \cap W \neq \emptyset,(V \backslash Z) \cap W \neq \emptyset$ and $(V \backslash W) \cap Z \neq \emptyset$. Let $Z_{1}=Z \cap W, Z_{2}=(V \backslash Z) \cap W$, $Z_{3}=Z \cap(V \backslash W), Z_{4}=(V \backslash Z) \cap(V \backslash W)$. Without loss of generality, We may suppose that $Z_{4} \neq \emptyset$, otherwise one may replace $W$ by $W^{\prime}=V \backslash W$ and thus we would have $W^{\prime} \subset Z$. We have

$$
2=x(\delta(W))=x\left(\delta\left(Z_{1}, Z_{3}\right)\right)+x\left(\delta\left(Z_{1}, Z_{4}\right)\right)+x\left(\delta\left(Z_{2}, Z_{3}\right)\right)+x\left(\delta\left(Z_{2}, Z_{4}\right)\right)
$$

$$
\begin{equation*}
2=x(\delta(Z))=x^{*}\left(\delta\left(Z_{1}, Z_{2}\right)\right)+\left(\delta\left(Z_{1}, Z_{4}\right)\right)+x\left(\delta\left(Z_{2}, Z_{3}\right)\right)+x\left(\delta\left(Z_{3}, Z_{4}\right)\right) \tag{3.2}
\end{equation*}
$$

Since $Z_{1} \neq \emptyset$ and $Z_{4} \neq \emptyset$, we also have

$$
\begin{align*}
& x\left(\delta\left(Z_{1}\right)\right)=x\left(\delta\left(Z_{1}, Z_{2}\right)\right)+x\left(\delta\left(Z_{1}, Z_{3}\right)\right)+x\left(\delta\left(Z_{1}, Z_{4}\right)\right) \geqslant 2  \tag{3.4}\\
& x\left(\delta\left(Z_{4}\right)\right)=x\left(\delta\left(Z_{1}, Z_{4}\right)\right)+x\left(\delta\left(Z_{2}, Z_{4}\right)\right)+x\left(\delta\left(Z_{3}, Z_{4}\right)\right) \geqslant 2 . \tag{3.5}
\end{align*}
$$

By adding (3.2) and (3.3) and (3.4) and (3.5) we get

$$
\begin{align*}
& x\left(\delta\left(Z_{1}, Z_{2}\right)\right)+x\left(\delta\left(Z_{1}, Z_{3}\right)\right)+2 x\left(\delta\left(Z_{1}, Z_{4}\right)\right)+2 x\left(\delta\left(Z_{2}, Z_{3}\right)\right) \\
& \quad+x\left(\delta\left(Z_{2}, Z_{4}\right)\right)+x\left(\delta\left(Z_{3}, Z_{4}\right)\right)=4  \tag{3.6}\\
& x\left(\delta\left(Z_{1}, Z_{2}\right)\right)+x\left(\delta\left(Z_{1}, Z_{3}\right)\right)+2 x\left(\delta\left(Z_{1}, Z_{4}\right)\right) \\
& \quad+x\left(\delta\left(Z_{2}, Z_{4}\right)\right)+x\left(\delta\left(Z_{3}, Z_{4}\right)\right) \geqslant 4 \tag{3.7}
\end{align*}
$$

As $x(e) \geqslant 0$ for all $e \in E$, by (3.6) and (3.7) it follows that

$$
x\left(\delta\left(Z_{2}, Z_{3}\right)\right)=0
$$

Now by considering $\delta\left(Z_{2}\right)$ and $\delta\left(Z_{3}\right)$ we similarly obtain that

$$
x\left(\delta\left(Z_{1}, Z_{4}\right)\right)=0
$$

Hence

$$
\begin{aligned}
& 2=x(\delta(Z))=x\left(\delta\left(Z_{1}, Z_{2}\right)\right)+x\left(\delta\left(Z_{3}, Z_{4}\right)\right) \\
& 2=x(\delta(W))=x\left(\delta\left(Z_{1}, Z_{3}\right)\right)+x\left(\delta\left(Z_{2}, Z_{4}\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& x\left(\delta\left(Z_{1}\right)\right)=x\left(\delta\left(Z_{1}, Z_{2}\right)\right)+x\left(\delta\left(Z_{1}, Z_{3}\right)\right) \geqslant 2, \\
& x\left(\delta\left(Z_{2}\right)\right)=x\left(\delta\left(Z_{1}, Z_{2}\right)\right)+x\left(\delta\left(Z_{2}, Z_{4}\right)\right) \geqslant 2, \\
& x\left(\delta\left(Z_{3}\right)\right)=x\left(\delta\left(Z_{1}, Z_{3}\right)\right)+x\left(\delta\left(Z_{3}, Z_{4}\right)\right) \geqslant 2, \\
& x\left(\delta\left(Z_{4}\right)\right)=x\left(\delta\left(Z_{2}, Z_{4}\right)\right)+x\left(\delta\left(Z_{3}, Z_{4}\right)\right) \geqslant 2 .
\end{aligned}
$$

This implies that

$$
\begin{aligned}
& x\left(\delta\left(Z_{1}, Z_{2}\right)\right)=x\left(\delta\left(Z_{1}, Z_{3}\right)\right)=1 \\
& x\left(\delta\left(Z_{2}, Z_{4}\right)\right)=x\left(\delta\left(Z_{3}, Z_{4}\right)\right)=1
\end{aligned}
$$

and hence the cuts $\delta\left(Z_{i}\right), i=1, \ldots, 4$ are all tight for $x$.
Therefore, $x(\delta(Z))=2$ and $x(\delta(W))=2$ are redondant with respect to the cuts $\delta\left(Z_{i}\right), i=1, \ldots, 4$ together with the trivial inequalities.

The following remark is easily seen to be true, it will be used frequently in the sequel.

Remark 3.11. Let $G=(V, E)$ be a graph. If $G$ is connected, then the graph obtained from $G$ by contracting the maximal node subsets including 2-edge connected subgraphs of $G$ is a tree. In what follows, we shall denote by $T(G)$ this tree.

Lemma 3.12. If $W \subset V$ is a node subset of $G$ such that $|\delta(W)|=2$, then either $|W|=1$ or $|V \backslash W|=1$.

Proof. Since $|\delta(W)|=2$, by inequalities (1.2)-(1.3) it follows that $x(e)=1$ for each of the two edges of $\delta(W)$. Also note that, without loss of generality, one can suppose that $x(\delta(W))=2$ is among the equations of the system (3.1). Hence by Lemma 3.10, the set $\mathbf{C}^{*}(x)$ can be chosen so that for every cut $\delta(Z)$ of $\mathbf{C}^{*}(x)$ either $W \subset Z$ or $Z \subset W$. Moreover, since $G$ is 2-edge connected, both graphs $G(W)$ and $G(V \backslash W)$ must be connected. Now let us assume that, on the contrary, we have $|W| \geqslant 2 \leqslant|\boldsymbol{V} \backslash W|$. We shall consider two cases:

Case 1: $G(W)$ is 2-edge connected.
Case 1.1: $x(e)=1$ for all $e \in E(V \backslash W)$. Then from Lemma 3.8, it follows that $G(V \backslash W)$ does not contain an induced 2-edge connected subgraph and hence it is acyclic. Since $G(V \backslash W)$ is connected, it is a tree. Now since $|\delta(W)|=2, G(V \backslash W)$ must be a path whose extremities are adjacent, respectively, to the edges of $\delta(W)$. Hence $G$ contains an edge whose endnodes are of degree two. But this contradicts Lemma 3.9.

Case 1.2: There exists an edge $e_{0} \in E(V \backslash W)$ with $0<x\left(e_{0}\right)<1$. Let $G^{\prime}=G / W$ and let $x^{\prime}$ be the solution induced by $x$ on $G^{\prime}$. Since $x^{\prime} \in P\left(G^{\prime}\right)$ and, by the minimality assumption, $G^{\prime}$ is perfectly-TEC, there must exist an integer extreme point $y^{\prime}$ of $P\left(G^{\prime}\right)$ such that $\mathbf{C}\left(x^{\prime}\right) \subseteq \mathbf{C}\left(y^{\prime}\right)$ and $\mathbf{T}\left(x^{\prime}\right) \subseteq \mathbf{T}\left(y^{\prime}\right)$. Let $\bar{x}$ be the solution given by

$$
\bar{x}(e)= \begin{cases}x(e) & \text { if } e \in E(W), \\ y^{\prime}(e) & \text { if } e \in E \backslash E(W) .\end{cases}
$$

Clearly, $\bar{x}(e) \in P(G)$. Moreover, we have that $\mathbf{T}(x) \subseteq \mathbf{T}(\bar{x})$ and $\mathbf{C}^{*}(x) \subseteq \mathbf{C}(\bar{x})$. In fact, this is clear for the first part of the statement. Now let $\delta(Z)$ be a cut of $\mathbf{C}^{*}(x)$. By our assumption, either $Z \subset W$ or $W \subset Z$.

- If $Z \subset W$ then $\bar{x}(\delta(Z))=x(\delta(Z))=2$.
- If $W \subset Z$ then $\bar{x}(\delta(Z))=x^{\prime}(\delta(Z))=y^{\prime}(\delta(Z))=2$.

Thus, $\delta(Z) \in \mathbf{C}(\bar{x})$ and, consequently, every constraint of the system (3.1) is also satisfied by $\bar{x}$. Since $\bar{x} \neq x$, this is a contradiction.

Case 2. $G(W)$ is not 2 -edge connected.
Since $G(W)$ is connected, by Remark $3.11, G(W)$ can be reduced to a tree $T(G(W))$ by contracting the maximal 2-edge connected induced subgraphs of $G(W)$. Let $W_{1}, \ldots, W_{k}, k \geqslant 2$, be the sets of nodes of these subgraphs. Since $|\delta(W)|=2$, $T(G(W))$ must be a path. Suppose $W_{1}$ and $W_{k}$ are the extremity sets of $T(G(W))$. Thus, $W_{1}$ is adjacent to one edge of $\delta(W)$ and $W_{k}$ is adjacent to the other edge, hence

$$
\left|\delta\left(W_{i}\right)\right|=2 \text { for } i=1, \ldots, k .
$$

Since $G\left(W_{i}\right)$ is 2-edge connected and $\left|V \backslash W_{i}\right|>1$, for $i=1, \ldots, k$, from Case 1, it follows that $\left|W_{i}\right|=1$ for $i=1, \ldots, k$. But then $G$ contains an edge whose endnodes are of degree two, which contradicts Lemma 3.9 and our proof is complete.

Lemma 3.13. Let $\delta(W)$ be a tight cut for $x$ with $|\delta(W)|=3$. Then
(i) at least one of the graphs $G(W)$ and $G(V \backslash W)$ is 2-edge connected, and
(ii) at least one of the graphs $G(W)$ and $G(V \backslash W)$ is reduced to either one node or two nodes joined by one edge.

Proof. (i) First note that, as in Lemma 3.12, the graphs $G(W)$ and $G(V \backslash W)$ must be connected. Now if both $G(W)$ and $G(V \backslash W)$ are not 2-edge connected then from Lemmas 3.9 and 3.12 it follows that at least two edges of $\delta(W)$ must belong to 2-edge cutsets of $G$. Since $x(e)>0$ for all $e \in E$, this implies that $x(\delta(W))>2$, a contradiction. For the rest of the proof we suppose that $G(W)$ is 2-edge connected.
(ii) If $|W|=1$ or $|V \backslash W|=1$, then we are done. Thus, let us assume that $|W| \geqslant 2 \leqslant|V \backslash W|$. Also, as we did in Lemma 3.12, we may suppose that $\delta(W)$ is among the cuts of the system (3.1). Consequently, any further cut $\delta(Z)$ of (3.1) is such that either $W \subset Z$ or $Z \subset W$. We shall consider two cases.

Case 1. $G(V \backslash W)$ is 2-edge connected.
Since $\delta(W)$ is tight for $x$ and $|\delta(W)|=3$, from Lemma 3.7 it follows that there exists an edge $e_{0} \in \delta(W)$ such that $0<x\left(e_{0}\right)<1$. Let $G_{1}=G / W$ and $G_{2}=G /(V \backslash W)$. Obviously, the graphs $G_{1}$ and $G_{2}$ are 2-edge connected. Let $x^{1}$ and $x^{2}$ be the restrictions of $x$ on $G_{1}$ and $G_{2}$, respectively. Clearly, $x^{i} \in P\left(G_{i}\right)$ for $i=1,2$. Since $G_{1}$ and $G_{2}$ are perfectly-TEC, there is an integer extreme point $\tilde{x}^{i}$ of $P\left(G_{i}\right)$ such that $\mathbf{C}\left(x^{i}\right) \subseteq \mathbf{C}\left(\tilde{x}^{i}\right)$ and $\mathbf{T}\left(x^{i}\right) \subseteq \mathbf{T}\left(\tilde{x}^{i}\right)$, for $i=1,2$. Moreover, since $0<x^{i}\left(e_{0}\right)<1$, we may assume that $\tilde{x}^{i}\left(e_{0}\right)=0$ for $i=1,2$. Hence, we have $\tilde{x}^{i}(e)=1$ for all $e \in \delta(W) \backslash\left\{e_{0}\right\}$ for $i=1,2$. Now let us consider the solution $\bar{x}$ given by

$$
\bar{x}(e)= \begin{cases}\tilde{x}^{2}(e) & \text { if } e \in E(W), \\ \tilde{x}^{1}(e) & \text { if } e \in E(V \backslash W), \\ 1 & \text { if } e \in \delta(W) \backslash\left\{e_{0}\right\}, \\ 0 & \text { if } e=e_{0} .\end{cases}
$$

It is easy to see that $\bar{x}$ belongs to $P(G)$ and $\mathbf{T}(x) \subseteq \mathbf{T}(\bar{x})$. Moreover, we have that $\mathbf{C}^{*}(x) \subseteq \mathbf{C}(\bar{x})$. Indeed, consider a cut $\delta(Z)$ of $\mathbf{C}^{*}(x)$.

- If $Z \subset W$, then $2=x(\delta(Z))=x^{2}(\delta(Z))=\tilde{x}^{2}(\delta(Z))=\bar{x}(\delta(Z))$.
- If $W \subset Z$, then $2=x(\delta(Z))=x^{1}(\delta(Z))=\tilde{x}^{1}(\delta(Z))=\bar{x}(\delta(Z))$.

In both cases we have $\delta(Z) \in \mathbf{C}(\bar{x})$. Since $x \neq \bar{x}$, this contradicts the extremality of $x$.
Case 2. $G(V \backslash W)$ is not 2-edge connected.
Thus, $G(V \backslash W)$ is connected and consequently, by Remark 3.11, it can be reduced to a tree $T(G(V \backslash W))$ containing at least two pendant nodes. Let $r$ be the number of pendant nodes in $T(G(V \backslash W))$. Since $|\delta(W)|=3$, we should have $r \leqslant 3$.

- If $r=3$, then the three edges of $\delta(W)$ all belong to 2-edge cutsets of $G$, implying that $x(e)=1$ for all $e \in \delta(W)$ and thus $x(\delta(W))>2$, a contradiction.
- If $r=2$, then $T(G(V \backslash W))$ is a path. Let $S_{1}, S_{2}, \ldots, S_{k}, k \geqslant 2$ be the node subsets
of $G$ whose contractions yield $T(G(V \backslash W))$. Suppose that $S_{1}$ and $S_{k}$ correspond to the pendant nodes of $T(G(V \backslash W))$. Thus, we have that

$$
\delta\left(S_{i}\right) \cap \delta(W)=\emptyset \quad \text { for } i=2, \ldots, k-1,
$$

for otherwise, $\delta\left(S_{1}\right), \delta\left(S_{k}\right)$ would be both 2-edge cutsets and thus at least two edges of $\delta(W)$ would have $x(e)=1$. But by Lemma 3.7 it follows that $x(\delta(W))>2$, a contradiction. Consequently, one can, without loss of generality, assume that $\left|\delta\left(S_{1}\right) \cap \delta(W)\right|=1$ and $\left|\delta\left(S_{k}\right) \cap \delta(W)\right|=2$. From Lemma 3.12 it follows that the set $S_{1}, \ldots, S_{k-1}$ all are reduced to nodes.

Case 2.1: $k \geqslant 3$. Then $G$ contains an edge whose endnodes are of degree two, contradicting Lemma 3.9.

Case 2.2: $k=2$. If $\left|S_{2}\right|=1$, then $G(V \backslash W)$ consists of one edge and we are done. If $\left|S_{2}\right| \geqslant 2$, then let $G_{1}=G / W$ and $G_{2}=G / S_{2}$. Obviously, $G_{1}$ and $G_{2}$ are 2-edge connected. Let $x^{1}$ and $x^{2}$ be the restrictions of $x$ to $G_{1}$ and $G_{2}$, respectively. The solutions $x^{1}$ and $x^{2}$ are in $P\left(G_{1}\right)$ and $P\left(G_{2}\right)$, respectively. Moreover, since $G_{1}$ and $G_{2}$ are perfectly-TEC, there are two integer extreme points $y^{1}$ and $y^{2}$ of $P\left(G_{1}\right)$ and $P\left(G_{2}\right)$ such that

$$
\mathbf{C}\left(x^{i}\right) \subseteq \mathbf{C}\left(y^{i}\right) \text { and } \mathbf{T}\left(x^{i}\right) \subseteq \mathbf{T}\left(y^{i}\right) \text { for } i=1,2 .
$$

Since $x^{i}\left(\delta\left(S_{1}, W\right)\right)=x^{i}\left(\delta\left(S_{2}, W\right)\right)=1$, for $i=1,2$, we then have, $y^{i}\left(\delta\left(S_{1}, W\right)\right)=$ $y^{i}\left(\delta\left(S_{2}, W\right)\right)=1$. Since $\left|\delta\left(S_{2}, W\right)\right|=2$, we may assume that for some edge of $\delta\left(S_{2}, W\right)$, say $f$, we have $y^{1}(f)=y^{2}(f)=0$.

Now consider the solution $\bar{x}$ given by

$$
\bar{x}(e)= \begin{cases}y^{2}(e) & \text { if } e \in E(W), \\ y^{1}(e) & \text { if } e \in E\left(S_{2}\right), \\ 1 & \text { if } e \in \delta(W) \backslash\{f\}, \\ 0 & \text { if } e=f\end{cases}
$$

Clearly, $\bar{x} \in P(G)$. Moreover, we can see, in a similar way as for Case 1 , that $\mathbf{C}(x) \subseteq \mathbf{C}(\bar{x})$ and $\mathbf{T}(x) \subseteq \mathbf{T}(\bar{x})$. Since $x \neq \bar{x}$, we then have a contradiction and this ends the proof of our lemma.

## 4. Perfectly-TEC graphs and the class $\Gamma$

In this section we give sufficient conditions for a 3-edge connected graph to be perfectly-TEC. This can be seen as a first step towards a complete characterization of perfectly-TEC graphs. To this end we first state two lemmas.

Lemma 4.1. Let $G=(V, E)$ be a graph which is 3-edge connected and minimally nonperfectly-TEC. Let $W \subseteq V$ be a node subset such that $|\delta(W)|=3$. Then either $W$ or $V \backslash W$ is reduced to one node.

Proof. Since $G$ is not nonperfectly-TEC there must exist a fractional extreme point $x$ of $P(G)$. Also since $G$ is 3-edge connected, $W$ and $V \backslash W$ must both induce 2-edge connected subgraphs of $G$ (otherwise we would have $|\delta(W)| \geqslant 4$ ). If $\delta(W)$ is tight for $x$ then by Lemma 3.13 it follows that either $W$ or $V \backslash W$ is reduced to one node. If not, then there must exist an edge $e$ of $\delta(W)$ such that $0<x(e)<1$. Now the rest of the proof is along the same line as the proof of Case 1 of Lemma 3.13.

Lemma 4.2. Let $G=(V, E)$ be a minimally nonperfectly-TEC graph. Suppose that $G$ is 3-edge connected and not reducible to a graph of $\Gamma$. Suppose there are four subsets $V_{1}, \ldots, V_{4}$ of $V$ such that
(i) $V_{i} \cap V_{j}=\emptyset$ for $i=1, \ldots, 4 ; j=1, \ldots, 4, i \neq j$,
(ii) $\delta\left(V_{i}, V_{j}\right) \neq \emptyset$ for $i=1, \ldots, 4 ; j=1, \ldots, 4, i \neq j$,
(iii) $G\left(V_{i}\right)$ is 2-edge connected for $i=1, \ldots, 4$.

Then $G$ is reduced to $K_{4}$.

Proof. Let $S=V \backslash\left(\bigcup_{i=1, \ldots, 4} V_{i}\right)$. The lemma is trivial if $S=\emptyset$. Let us assume that $S \neq \emptyset$, and that $v=\sum_{i=1, \ldots, 4}\left|V_{i}\right|$ is maximum with respect to (i)-(iii).

Case 1: $G(S)$ is connected.
By Remark $3.11, G(S)$ can be reduced to a tree $T(G(S))$. Let $S_{1}, \ldots, S_{k} \subseteq S, k \geqslant 1$, be the pendant nodes of $T(G(S))$.

Claim 1. $\left|\delta\left(V_{i}, S\right)\right| \leqslant 1$ for $i=1, \ldots, 4$.
Proof. Indeed, if this is not the case, then there must exist two nodes $U_{1}, U_{2}\left(U_{1}\right.$ and $U_{2}$ may be the same) of $T(G(S))$ and $i_{0} \in\{1, \ldots, 4\}$ such that $\delta\left(V_{i_{0}}, U_{j}\right) \neq \emptyset$ for $j=1,2$. Let $P=\left(X_{1}=U_{1}, X_{2}, \ldots, X_{t}=U_{2}\right)$ be the (unique) path of $T(G(S))$ between $U_{1}$ and $U_{2}$. Let $V_{i_{0}}^{\prime}=V_{i_{0}} \cup X_{1} \cup \cdots \cup X_{t}$. Clearly, $V_{i_{0}}^{\prime}$ induces a 2-edge connected subgraph of $G$ and the sets $V_{i_{0}}^{\prime}, V_{i}^{\prime}=V_{i}$ for $i \in\{1, \ldots, 4\} \backslash\left\{i_{0}\right\}$ satisfy (i)-(iii). Moreover, we have $\sum_{i=1, \ldots, 4}\left|V_{i}^{\prime}\right|>v$, which contradicts the maximality of $v$.

Claim 2. $T(G(S))$ has at most two pendant nodes.

Proof. Assume the contrary. Let $S_{1}, S_{2}, S_{3}$ be three pendant nodes of $T(G(S))$. Since $G$ is 3-edge connected, it follows that $\left|\delta\left(V \backslash S, S_{i}\right)\right| \geqslant 2$ for $i=1, \ldots, 3$. But this implies that for some $j \in\{1, \ldots, 4\}$, we have $\left|\delta\left(V_{j}, S\right)\right| \geqslant 2$, contradicting Claim 1.

Now consider the case where $T(G(S))$ is reduced to one node (i.e. $G(S)$ is 2-edge connected). Since $|\delta(S)| \geqslant 3$, by Claim 1 it follows that $S$ is adjacent to at least three sets among $V_{1}, \ldots, V_{4}$. Suppose that $S$ is adjacent to, say $V_{1}, V_{2}, V_{3}$. Let $G^{\prime}$ be the graph obtained from $G$ by contracting the sets $V_{i}, i=1, \ldots, 4$ and $S$, and deleting the edges of $\delta\left(V_{1}, V_{2}\right)$ and $\delta\left(V_{1}, V_{3}\right)$. Clearly, the graph $G^{\prime}$ corresponds to the graph $J$ of Fig. 2, and thus it is not perfectly-TEC. Since $G \neq G^{\prime}$, we have a contradiction.

Consequently, $T(G(S))$ has exactly two pendant nodes and thus it is a path. Let $S_{1}, \ldots, S_{k}, k \geqslant 2$, be the nodes of $T(G(S))$ and suppose that $S_{1}$ and $S_{k}$ are the pendant
nodes of $T(G(S))$. Since $G$ is 3-edge connected and hence $\delta\left(V \backslash S, S_{i}\right) \geqslant 2$ for $i=1, k$, it follows from Claim 1 that each set $S_{i}, i=1, k$ is adjacent to exactly two nodes among $V_{1}, \ldots, V_{4}$. Also from Claim 1 it follows that $\delta\left(V \backslash S, S_{i}\right)=\emptyset$ for $i=2, \ldots, k-1$. But this implies that $T(G(S))$ contains no internal nodes and thus $k=2$. Since $\left|\delta\left(S_{1}\right)\right|=\left|\delta\left(S_{k}\right)\right|=3$, and $\left|V \backslash S_{i}\right| \geqslant 2$, for $i=1, k$, it thus follows by Lemma 4.1 that $\left|S_{1}\right|=\left|S_{k}\right|=1$. Thus we may assume, without loss of generality, that $S_{1}$ is adjacent to $V_{1}, V_{2}$ and $S_{2}$ is adjacent to $V_{3}, V_{4}$. Let $V_{1}^{\prime}=V_{1} \cup V_{2} \cup S_{1}, V_{2}^{\prime}=S_{2}, V_{3}^{\prime}=V_{3}$ and $V_{4}^{\prime}=V_{4}$. Clearly, the sets $V_{i}^{\prime}, i=1, \ldots, 4$ satisfy (i)-(iii). Moreover, we have $\sum_{i=1, \ldots, 4}\left|V_{i}^{\prime}\right|>v$, a contradiction.

Case 2: $G(S)$ is not connected.
Let $S_{1}, S_{2}, \ldots, S_{r} \subset S, r \geqslant 2$, be the node subsets defining the connected components of $G(S)$. From the proof of Case 1 it follows that $G\left(S_{i}\right)$ is 2-edge connected for $i=1, \ldots, r$. We claim that all the nodes $S_{i}$ have the same neighbour set among $V_{1}, V_{2}, V_{3}, V_{4}$. Indeed, assume that this does not hold and let us suppose, for instance, that $V_{1}$ is adjacent to, say, $S_{1}$ but not to $S_{2}$. By the 3-edge connectivity of $G$ together with Claim 1, it follows that $S_{2}$ is adjacent to $V_{2}, V_{3}$ and $V_{4}$ and $S_{1}$ is adjacent to at least two more sets among $V_{2}, V_{3}, V_{4}$. Without loss of generality we may suppose, for instance, that $S_{1}$ is adjacent to $V_{2}, V_{3}$. Let $V_{1}^{\prime}=V_{1}, V_{2}^{\prime}=V_{2}, V_{3}^{\prime}=V_{3} \cup V_{4} \cup S_{2}$ and $V_{4}^{\prime}=S_{1}$. Clearly, the sets $V_{i}^{\prime}, i=1, \ldots, 4$ satisfy (i)-(iii), but we have $\sum_{i=1, \ldots, 4}\left|V_{i}^{\prime}\right|>v$, a contradiction. Consequently, all the sets $S_{i}$ have the same neighbour set, say $N$. If $|N|=4$, then the sets $S_{i}, i=1, \ldots, r$ are all adjacent to $V_{1}, \ldots, V_{4}$. Since $r \geqslant 2$ one can easily obtain in this case four sets $V_{i}^{\prime}, i=1, \ldots, 4$ which satisfy (i) (iii) and such that $\sum_{i=1, \ldots .4}\left|V_{i}^{\prime}\right|>v$, contradicting the maximality of $v$. Thus, $|N|=3$. Let us assume, for instance, that $N=\left\{V_{1}, V_{2}, V_{3}\right\}$. From Claim 1 it follows that $\left|\delta\left(V_{4}\right)\right|=\left|\delta\left(S_{i}\right)\right|=3$ for $i=1, \ldots, r$. Moreover, by Lemma 4.1 we should have $\left|V_{4}\right|=\left|S_{i}\right|=1$ for $i=1, \ldots, r$. Now by contracting the sets $V_{1}, V_{2}, V_{3}$ we obtain a graph in $\Gamma$, a contradiction, which ends the proof of our lemma.

Now we are ready to state the main result of this section.
Theorem 4.3. Let $G=(V, E)$ be a 3-edge connected graph. Then $G$ is perfectly-TEC if
(i) $G \neq K_{4}$,
(ii) $G$ is not reducible to a graph in $\Gamma$,
(iii) Any graph obtained from $G$ by application of one of the operations $\theta_{1}, \theta_{2}, \theta_{3}$ is perfectly-TEC.

Proof. Suppose that, on the contrary, $G$ is not perfectly-TEC. Then by (iii) $G$ is minimally nonperfectly-TEC. From [25] it follows that $G$ is not series-parallel and, consequently, there are four subsets $V_{1}, \ldots, V_{4} \subset V$ such that

$$
\begin{array}{ll}
\text { ( } \alpha) V_{i} \cap V_{j}=\emptyset & \text { for } i=1, \ldots, 4 ; j=1, \ldots, 4, i \neq j, \\
\text { ( } \beta \text { ) } \delta\left(V_{i}, V_{j}\right) \neq \emptyset & \text { for } i=1, \ldots, 4 ; j=1, \ldots, 4, i \neq j, \\
(\gamma) G\left(V_{i}\right) \text { is connected } & \text { for } i=1, \ldots, 4 .
\end{array}
$$

It is not hard to see that at least one of the sets $V_{i}$ may be assumed to induce a 2-edge connected subgraph. If the four sets $V_{i}, i=1, \ldots, 4$ induce 2-edge connected subgraphs of $G$ then by Lemma 4.2, we have $G=K_{4}$, which contradicts (i). In what follows, we are going to discuss the case where only three sets among $V_{1}, \ldots, V_{4}$ induce 2-edge connected subgraphs, the cases where only one or two sets satisfy this, can be treated in similar way.

Thus let us assume, for instance, that
(iv) $G\left(V_{i}\right)$ is 2-edge connected for $i=1,2,3$.

Suppose that $v=\sum_{i=1, \ldots, 4}\left|V_{i}\right|$ is maximum with respect to (i)-(iv). Moreover, suppose that $V$ does not contain four subsets $V_{1}, \ldots, V_{4}$ satisfying the hypotheses of Lemma 4.2, otherwise we would have $G=K_{4}$.

Claim 1. For every pendant node $U$ of $T\left(G\left(V_{4}\right)\right)$ we have $\left|\delta\left(U, V \backslash V_{4}\right)\right|=2$, and $U$ is adjacent to exactly two sets among $V_{1}, \ldots, V_{3}$.

Proof. From the assumption above, $U$ can be adjacent to at most two sets among $V_{1}, \ldots, V_{3}$. If $U$ is adjacent to only one of these sets, then, since $|\delta(U)| \geqslant 3$ we have $\left|\delta\left(U, V \backslash V_{4}\right)\right| \geqslant 2$, and thus there is $k \in\{1, \ldots, 3\}$ such that $\left|\delta\left(V_{k}, U\right)\right| \geqslant 2$. Thus the subset $V_{k} \in U$ induces a 2-edge connected subgraph. Let $V_{k}^{\prime}=V_{k} \in U, V_{j}^{\prime}=V_{j}$, for $j \in\{1, \ldots, 3\} \backslash\{k\}, V_{4}^{\prime}=V_{4} \backslash U$. Clearly the sets $V_{i}^{\prime}, i=1, \ldots, 4$ satisfy (i)-(iv). Moreover, we have $\sum_{i=1, \ldots, 4}\left|V_{i}^{\prime}\right|>v$, a contradiction.

Claim 2. If $T\left(G\left(V_{4}\right)\right)$ contains more than two pendant nodes, then it is a star (i.e. a tree with only one nonpendant node).

Proof. Assume the contrary, that is, $T\left(G\left(V_{4}\right)\right)$ contains more than two pendant nodes, but it is not a star. Then there are two pendant nodes, say $U_{1}, U_{2}$ of $T\left(G\left(V_{4}\right)\right)$ such that the subgraph $T\left(G\left(V_{4}\right)\right) \backslash P$ is connected and contains at least two nodes, where $P$ is the (unique) path of $T\left(G\left(V_{4}\right)\right)$ between $U_{1}$ and $U_{2}$. From Claim 1, we may assume that $U_{1}$ is adjacent to $V_{1}$ and $V_{2}$. We claim that $U_{2}$ is also adjacent to $V_{1}$ and $V_{2}$. Indeed, if this is not the case, then by Claim 1 we may assume, without loss of generality, that $U_{2}$ is adjacent to $V_{1}$ and $V_{3}$. Since $T\left(G\left(V_{4}\right)\right)$ contains more than two pendant nodes, there must exist a further pendant node, say $U_{3}$, of $T\left(G\left(V_{4}\right)\right) \backslash P$ different from $U_{1}$ and $U_{2}$. Now, if $U_{3}$ is adjacent to $V_{1}$ and $V_{2}$ then let $V_{1}^{\prime}=V_{1} \cup V_{3} \cup U_{2}, V_{2}^{\prime}=V_{2}, V_{3}^{\prime}=U_{3}, V_{4}^{\prime}=V_{4} \backslash\left(U_{2} \cup U_{3}\right)$. If not, then consider the sets $V_{1}^{\prime}=V_{1} \cup V_{2} \cup U_{1}, V_{2}^{\prime}=V_{3}, V_{3}^{\prime}=U_{3}, V_{4}^{\prime}=V_{4} \backslash\left(U_{1} \cup U_{3}\right)$. In both cases, the sets $V_{i}^{\prime}, i=1, \ldots, 4$ satisfy (i)-(vi), and $\sum_{i=1, \ldots, 4}\left|V_{i}^{\prime}\right|>v$, a contradiction. Thus $U_{1}$ and $U_{2}$ are both adjacent to $V_{1}$ and $V_{2}$. Consequently, Claim 1 together with $(\beta)$ imply that $V_{3}$ is adjacent to a node of $T\left(G\left(V_{4}\right)\right) \backslash\left(U_{1} \cup U_{2}\right)$. Consider the case where $V_{3}$ is adjacent to a node of $T\left(G\left(V_{4}\right)\right) \backslash P$. Since by Claim 1, each pendant node of $T\left(G\left(V_{4}\right)\right) \backslash P$ is adjacent to exactly two nodes among $V_{1}, V_{2}, V_{3}$, it follows that $U_{3}$ is adjacent to at least one node among $V_{1}, V_{2}$. Let us assume, without loss of generality, that $U_{3}$ is adjacent to $V_{2}$. Set $V_{1}^{\prime}=V_{1} \cup P, V_{2}^{\prime}=V_{2}, V_{3}^{\prime}=V_{3}, V_{4}^{\prime}=V_{4} \backslash P$. It is easy


Fig. 3.
to see that $V_{i}^{\prime}, i=1, \ldots, 4$, satisfy (i)-(iv). Since $\sum_{i=1, \ldots, 4}\left|V_{i}^{\prime}\right|>v$, we have a contradiction.

Thus $\delta\left(V_{3}, U_{3}\right)=\emptyset$ and $V_{3}$ is adjacent to a node of $P$. From Claim 1 it thus follows that $U_{3}$ is adjacent to $V_{1}$ and $V_{2}$ (see Fig. 3).

Since $T\left(G\left(V_{4}\right)\right) \backslash P$ contains at least two nodes, there must exist a node in $T\left(G\left(V_{4}\right)\right) \backslash P$ of degree $\leqslant 2$, say $U_{4}, U_{4} \neq U_{3}$, such that $\left(T\left(G\left(V_{4}\right)\right) \backslash\left(P \cup P^{\prime}\right)\right.$ is either empty or a connected graph, where $P^{\prime}$ is the (unique) path of $T\left(G\left(V_{4}\right)\right)$ between $U_{3}$ and $U_{4}$. Since $\left|\delta\left(U_{4}\right)\right| \geqslant 3$, there must exist a set among $V_{1}, \ldots, V_{3}$ which is adjacent to $U_{4}$. If $V_{3}$ (resp. $V_{i}$, for some $i \neq 3$ ) is adjacent to $U_{4}$, consider the sets $V_{1}^{\prime}=V_{1} \cup P, V_{2}^{\prime}=V_{2}, V_{3}^{\prime}=V_{3}, V_{4}^{\prime}=V_{4} \backslash P\left(\right.$ resp. $V_{1}^{\prime}=V_{i} \cup P^{\prime}, V_{2}^{\prime}=V_{j}$, where $\{j\}=\{1,2\} \backslash\{i\}, V_{3}^{\prime}=V_{3}, V_{4}^{\prime}=V_{4} \backslash P^{\prime}$. We have that $V_{i}^{\prime}, i=1, \ldots, 4$ satisfy (i)-(iv) and $\sum_{i=1, \ldots 4}\left|V_{i}^{\prime}\right|>v$, a contradiction and our claim is proved.

Thus, $T\left(G\left(V_{4}\right)\right)$ is either a star or contains only two pendant nodes. Consider the latter case. Then $T\left(G\left(V_{4}\right)\right)$ is a path $L$. Let $U_{1}$ and $U_{2}$ be the endnodes of $L$. From Claim 1, we may without loss of generality, assume that, $U_{1}$ is adjacent to $V_{1}$ and $V_{2}$, but not to $V_{3}$. If $V_{3}$ is adjacent to $U_{2}$, then we may also assume that one of the sets $V_{1}, V_{2}$, say $V_{2}$ is adjacent to $U_{2}$ (see Fig. 4(a)). Thus, $\left|\delta\left(U_{1}\right)\right|=\left|\delta\left(U_{2}\right)\right|=3$ and thus from Lemma 4.1 the sets $U_{1}, U_{2}$ are reduced to nodes. Furthermore, $\delta\left(V_{i}, L \backslash\left\{U_{1}, U_{2}\right\}\right)=\emptyset$, for $i=1,3$ holds, otherwise one can easily obtain a contradiction with the maximality of $v$. Now let $G^{\prime}$ be the graph obtained from $G$ by contracting the sets $V_{i}, i=1, \ldots, 3$, deleting the edges in $\delta\left(V_{2}, L \backslash U_{2}\right)$ and replacing the resulting multiple edges by single edges. This graph is either the graph $J$ of Fig. 2 or obtained from $K_{4}$ by subdivisions of one edge (see Fig. 4(b)). In both cases the resulting graph is not perfectly-TEC. Since $G \neq G^{\prime}$ we have a contradiction.

Now suppose that $T\left(G\left(V_{4}\right)\right)$ is a star and let $U_{1}, \ldots, U_{k}, k \geqslant 3$ be its pendant nodes. Let $S=V_{4} \backslash\left(\bigcup_{i=1, \ldots, k} U_{i}\right)$. Clearly, $G(S)$ is 2-edge connected. If all the pendant nodes do not have the same neighbour set then it is not hard to see that in this case one can obtain four sets $V_{i}^{\prime}, i=1, \ldots, 4$ satisfying (i)-(iv) with $\sum_{i=1, \ldots, 4}\left|V_{i}^{\prime}\right|>v$, which contradicts the maximality of $v$. If all the nodes $U_{i}$ have the same neighbour set, say


Fig. 4.


Fig. 5.
$\left\{V_{1}, V_{2}\right\}$, since by Claim 1 a pendant node cannot be adjacent to more than two nodes among $V_{i}, i=1, \ldots, 3$, this together with ( $\beta$ ) imply that $V_{3}$ is adjacent to $S$ (see Fig. 5). Moreover, $\left|\delta\left(V_{3}\right)\right|=3$ holds. If not, we would have for some $V_{j}, j \neq 3$, either $\left|\delta\left(V_{j}, V_{3}\right)\right| \geqslant 2$ or $\left|\delta\left(V_{3}, S\right)\right| \geqslant 2$. In both cases one can easily obtain four sets $V_{i}^{\prime}$, $i=1, \ldots, 4$ which contradict the maximality of $v$.

Consequently, $\left|\delta\left(V_{3}\right)\right|=3$ and $\left|\delta\left(U_{i}\right)\right|=3$ for $i=1, \ldots, k$. From Lemma 4.1 it follows that $V_{3}$ and $U_{i}, i=1, \ldots, k$, are all reduced to nodes. Let $G^{\prime}$ be the graph obtained from $G$ by contracting the nodes $V_{1}, V_{2}, S$ and replacing the resulting multiple edges by single edges. Clearly, $G^{\prime}$ is a graph in $\Gamma$, a contradiction, which ends the proof of our theorem.

As mentioned before, this theorem can be seen as a first step toward a complete characterization of perfectly-TEC graphs. Indeed, that characterization may need some reduction operations such as the operations $\theta_{1}, \theta_{2}, \theta_{3}$. Also, it may be that further operations or/and excluded configurations have to be defined for that characterization.

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