Hop-Level Flow Formulation for the Hop Constrained Survivable Network Design Problem

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Abstract. The HSNDP consists in finding a minimum cost subgraph containing K edge-disjoint paths with length at most H joining each pair of vertices in a given demand set. The only formulation found in the literature that is valid for any K and any H is based on multi-commodity flows over suitable layered graphs (Hop-MCF) and has typical integrality gaps in the range of 5% to 25%. We propose a new formulation called Hop-Level-MCF (in this short paper only for the rooted demands case), having about H times more variables and constraints than Hop-MCF, but being significantly stronger. Typical gaps for rooted instances are between 0% and 6%. Some instances from the literature are solved for the first time.

1 Introduction

Let $G = (V, E)$ be an undirected graph with n vertices, numbered from 0 to $n-1$, and m edges with non-negative costs c_e, $e \in E$; $D \subseteq V \times V$ be a set of demands; and $K \geq 1$ and $H \geq 2$ be natural numbers. The Hop-constrained Survivable Network Design Problem (HSNDP) consists in finding a subgraph of G with minimum cost containing, for each demand $d = (u, v) \in D$, K edge-disjoint (u, v)-paths with at most H edges. If all demands have a common vertex, w.l.o.g. the vertex 0, we say that the demands are *rooted*, otherwise they are *unrooted*. When $|D| = 1$, the HSNDP is polynomial for $H \leq 3$ and NP-hard for $H \geq 4$ (see [1]). When the cardinality of D is not constrained, the problem is NP-hard even if D is rooted, $K = 1$ and $H = 2$ (see [3]). In this short paper we only consider the case of rooted demands.

2 Hop Multi-Commodity Flow Formulation (Hop-MCF)

An extended formulation was recently proposed for the general HSNDP [2]. As D here is assumed to be rooted, a demand $(0, d) \in D$ will be identified by its destination vertex d. Let $V' = V - \{0\}$ and $E' = E \setminus \delta(0)$, where $\delta(i)$ represents the set of edges adjacent to a vertex i. For each demand $d \in D$, define the hop layered directed graph $G_H^d = (V_H^d, A_H^d)$, where $V_H^d = \{(0,0)\} \cup \{(i,h) : i \in V'; 1 \leq h \leq H-1\} \cup \{(d,H)\}$. Assuming that G is a complete graph, $A_H^d =$

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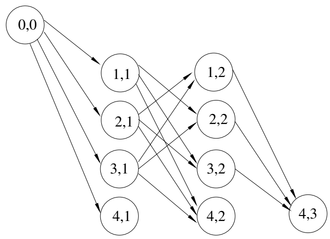

Fig. 1. Example of auxiliary graph G_H^d: G complete, $n = 5$, $d = 4$, and $H = 3$

$$\begin{aligned}&\{[0,j,1] = [(0,0),(j,1)] : j \in V'\}\\ \cup\ &\{[i,j,h] = [(i,h-1),(j,h)] : i,j \in V' - \{d\}, i \neq j; 2 \leq h \leq H-1\}\\ \cup\ &\{[i,d,h] = [(i,h-1),(d,h)] : i \in V' - \{d\}; 2 \leq h \leq H\}.\end{aligned}$$

Each arc in A_H^d is identified by a triple $[i,j,h]$, giving its origin, destination and hop. When G is not complete, if $(i,j) \notin E$, arcs of form $[i,j,h]$ and $[j,i,h]$ are omitted from A_H^d . Figure 1 depicts an example of such auxiliary network. For each $d \in D$, and for each arc $[i,j,h]$ in A_H^d, define binary flow variables f_{ij}^{dh}. For each edge (i,j) in E, define design binary variables x_{ij}. Let $\delta^-(i,h,d)$ and $\delta^+(i,h,d)$ denote, respectively, the set of arcs in A_H^d entering and leaving vertex (i,h). The Hop-MCF formulation follows:

$$\min \sum_{(i,j)\in E} c_{ij}x_{ij} \tag{1}$$

$$s.t.$$

$$\sum_{a\in\delta^-(i,h,d)} f_a - \sum_{a\in\delta^+(i,h,d)} f_a = 0 \quad d \in D; (i,h) \in V_H^d, i \notin \{0,d\} \tag{2}$$

$$\sum_{h=1}^{H} \sum_{a\in\delta^-(d,h,d)} f_a = K \quad d \in D \tag{3}$$

$$f_{0j}^{d1} \leq x_{0j} \quad d \in D; (0,j) \in \delta(0) \tag{4}$$

$$\sum_{h=2}^{H-1} (f_{ji}^{dh} + f_{ij}^{dh}) \leq x_{ij} \quad d \in D; (i,j) \in E' \setminus \delta(d) \tag{5}$$

$$\sum_{h=2}^{H} f_{jd}^{dh} \leq x_{jd} \quad d \in D; (j,d) \in \delta(d) \setminus \delta(0) \tag{6}$$

3 Hop-Level Multi-Commodity Flow Formulation (HL-MCF)

It is well-known that directed formulations of network design problems, when available, are much stronger that their undirected counterparts. The relative weakness of known HSNDP formulations, including Hop-MCF, is related to the impossibility of directing the solutions, since both orientations of an edge can be used in the paths for different demands. The proposed formulation tries to remedy this difficulty by introducing the concept of *solution level*. Given a solution T, we can partition V into $L+2$ levels, according to their distances to 0 in T. In the rooted case, L is set as equal to H, in the unrooted case (not presented here) L is usually greater than H. Level 0 only contains vertex 0; level l, $1 \leq l \leq L$, contains vertices with distance l; and level $L+1$ contains the vertices that are not connected to 0 in T. Besides variables x, HL-MCF also has:

- Binary variables w_i^l, $i \in V', 1 \leq l \leq L+1$, indicating that vertex i is in level l; constant w_0^0 is defined as 1.
- Binary variables $y_{ij}^{l_1 l_2}$ indicating that edge (i,j) belongs to T, i is in level l_1 and j in level l_2. For each $(0,j) \in \delta(0)$ there is a single variable y_{0j}^{01}. Each $e=(i,j) \in E'$ is associated with a set of $3(L-1)$ variables $\{y_{ij}^{ll} : 1 \leq l \leq L-1\} \cup \{y_{ij}^{l(l+1)}, y_{ji}^{l(l+1)} : 1 \leq l \leq L-1\}$.
- Binary flow variables $g_{ij}^{dhl_1 l_2}$ associated to $|D|$ auxiliary hop-level networks.

The x and (w,y) variables are linked by the following constraints:

$$\sum_{l=1}^{L+1} w_i^l = 1 \quad i \in V' \tag{7}$$

$$w_j^1 = y_{0j}^{01} = x_{0j} \quad (0,j) \in \delta(0) \tag{8}$$

$$\sum_{l=1}^{L-1} y_{ij}^{ll} + \sum_{l=1}^{L-1} (y_{ij}^{l(l+1)} + y_{ji}^{l(l+1)}) = x_{ij} \quad (i,j) \in E' \tag{9}$$

$$\begin{array}{l} y_{ij}^{11} + y_{ij}^{12} \leq w_i^1 \\ y_{ij}^{11} + y_{ji}^{12} \leq w_j^1 \end{array} \quad (i,j) \in E' \tag{10}$$

$$\begin{array}{l} y_{ij}^{ll} + y_{ij}^{l(l+1)} + y_{ji}^{(l-1)l} \leq w_i^l \\ y_{ij}^{ll} + y_{ji}^{l(l+1)} + y_{ij}^{(l-1)l} \leq w_j^l \end{array} \quad (i,j) \in E'; l = 2, \ldots, L-1 \tag{11}$$

$$\begin{array}{l} y_{ji}^{(L-1)L} \leq w_i^L \\ y_{ij}^{(L-1)L} \leq w_j^L \end{array} \quad (i,j) \in E' \tag{12}$$

$$w_i^l \leq \sum_{(j,i) \in \delta(i), j \neq 0} y_{ji}^{(l-1)l} \quad i \in V'; l = 2, \ldots, L \tag{13}$$

It can be checked that for any fixed binary solution x there is a single (w,y) solution that satisfies (7–13), that solution is binary and every vertex is assigned to a single level. However, a fractional x usually forces a (w,y) solution that splits vertices and edges into different levels. In order to profit from that splitting, for each $d \in D$, we define hop-level directed graphs $G_{HL}^d = (V_{HL}^d, A_{HL}^d)$, where $V_{HL}^d = \{(0,0,0)\} \cup \{(i,h,l) : i \in V'; 1 \leq h \leq H-1; h \leq l \leq L\} \cup \{(d,H,l) : 1 \leq l \leq L\}$, and $A_{HL}^d =$

$$\begin{aligned} &\{[0,j,1,0,1] = [(0,0,0),(j,1,1)] : j \in V'\} \\ \cup\ &\{[i,j,h+1,l,l'] = [(i,h,l),(j,h+1,l')] : i,j \in V', i \neq d; 1 \leq h \leq H-2; \\ &\quad 1 \leq l \leq h; max(l-1,1) \leq l' \leq l+1\} \\ \cup\ &\{[i,d,H,l,l'] = [(i,H-1,l),(d,H,l')] : i \in V', i \neq d; \\ &\quad 1 \leq l \leq L-1; max(l-1,1) \leq l' \leq l+1\} \end{aligned}$$

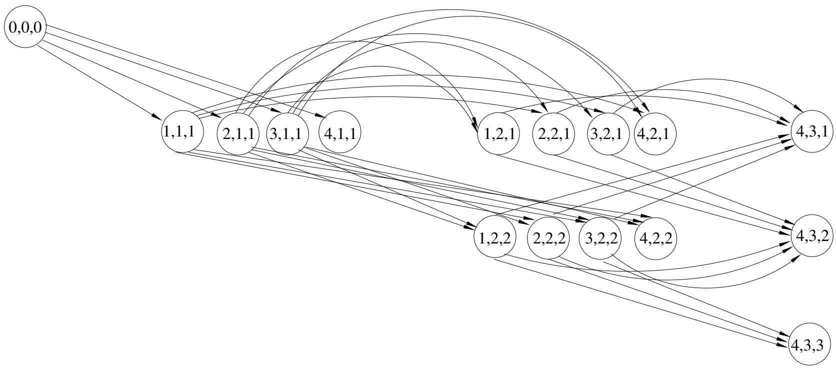

Fig. 2. Example of auxiliary graph G^d_{HL}: G complete, $n=5$, $d=4$, and $H=L=3$

Again, if G is not complete, the arcs corresponding to missing edges are removed. Each arc in A^d_{HL} is identified by a tuple $[i,j,h,l_1,l_2]$, giving its origin, destination, hop, origin level and destination level. For each such arc, we define a binary flow variable $g^{dhl_1l_2}_{ij}$. Let $\delta^-(i,h,l,d)$ and $\delta^+(i,h,l,d)$ denote, respectively, the set of arcs in A^d_H entering and leaving vertex (i,h,l). The new formulation HL-MCF is (1), subject to (7)–(13) and to the following constraints:

$$\sum_{a\in\delta^-(i,h,l,d)} g_a - \sum_{a\in\delta^+(i,h,l,d)} g_a = 0 \quad d\in D; (i,h,l)\in V^d_{HL}, i\notin\{0,d\} \tag{14}$$

$$\sum_{h=1}^{H}\sum_{a\in\delta^-(d,h,l,d)} g_a = K.w^l_d \quad d\in D;\ 1\le l\le L \tag{15}$$

$$g^{d101}_{0j} \le y^{01}_{0j} \quad d\in D; (0,j)\in\delta(0) \tag{16}$$

$$\sum_{h=l+1}^{H-1}(g^{dhll}_{ji}+g^{dhll}_{ij}) \le y^{ll}_{ij} \quad d\in D; (i,j)\in E'\setminus\delta(d); 1\le l\le L-2 \tag{17}$$

$$\sum_{h=l+2}^{H-1} g^{dh(l+1)l}_{ji} + \sum_{h=l+1}^{H-1} g^{dhl(l+1)}_{ij} \le y^{l(l+1)}_{ij} \quad d\in D; (i,j)\in E'\setminus\delta(d); 1\le l\le L-2 \tag{18}$$

$$\sum_{h=l+1}^{H} g^{dhll}_{jd} \le y^{ll}_{jd} \quad d\in D; (j,d)\in\delta(d)\setminus\delta(0); 1\le l\le L-2 \tag{19}$$

$$\sum_{h=l+1}^{H} g^{dhl(l+1)}_{jd} \le y^{l(l+1)}_{jd} \quad d\in D; (j,d)\in\delta(d)\setminus\delta(0); 1\le l\le L-2 \tag{20}$$

The HL-MCF formulation has $O(|D|.H.L.m)$ variables and $O(|D|.H.L.n)$ constraints, an increase by a factor of $L=H$ (in both dimensions) with respect to Hop-MCF. It can be proved that HL-MCF is at least as strong as Hop-MCF in terms of bounds provided by their linear relaxations.

4 Computational Experiments

The experiments were performed with CPLEX 12.1 MIP solver over a single core of an Intel i5 2.27GHz CPU. The first tests were on the rooted instances used in [2], complete graphs with 21 vertices associated to random points in a square, euclidean distances. The root vertex is in the center on instances TC-5 and TC-10, and on a corner on instances TE-5 and TE-10. The numbers 5 and 10 refer to the number of demands. We also added instances TC-20 and TE-20 with 20 demands. The average gaps of formulations Hop-MCF and HL-MCF are listed in Table 1.

- Formulation HL-MCF is very strong when $K = 1$ or $H = 2$, all problems are solved to optimality with almost no branching. The rooted case with $K = 1$ is equivalent to the hop-constrained Steiner tree problem, for which very strong formulations are already known [3]. But for $H = 2$ and $K > 1$, HL-MCF is much stronger than any other known formulation.
- When $K = 2$ and $H = 3$, HL-MCF is much stronger than Hop-MCF, the decreased gaps more than compensate for having to solve larger LPs. For example, instance TE-20 can be solved to optimality in 162 seconds using HL-MCF, but can not be solved in 1 hour with Hop-MCF.
- When $K = 2$ and $H = 4$ or when $K = 3$ and $H = 3$, HL-MCF is significantly stronger than Hop-MCF. However, the decreased gaps and smaller enumeration trees are roughly compensated by the burden of the larger LPs; the overall results are comparable.
- In the remaining three cases, HL-MCF is only slightly stronger than Hop-MCF, which performs much better on solving those instances to optimality.

Table 1. Average percentage gaps on instances TC-5, TC-10, TC-20, TE-5, TE-10, TE-20

	K	$H = 2$	$H = 3$	$H = 4$	$H = 5$
Hop	1	14.99	23.91	25.82	26.94
HL		0.00	0.00	0.83	1.93
Hop	2	12.92	13.13	13.05	9.60
HL		0.40	2.83	5.62	5.08
Hop	3	7.38	7.64	7.00	6.01
HL		0.08	3.27	5.20	5.27

The remaining experiments were performed on rooted instances from [4], defined over complete graphs with up to 40 vertices, for cases $K = 2$ and $H = 2,3$. Table 2 compares the performance of CPLEX MIP solver over formulations Hop-MCF and HL-MCF, in terms of duality gap, number of nodes in the branch tree and total time to solve the instance. The root gaps of the branch-and-cut in [4] for some instances are also presented.

Table 2. Results on the rooted instances from Huygens et al.[4] ($K = 2$)

		$H = 2$							$H = 3$						
n	$\|D\|$	HL-MCF			Hop-MCF			[4]	HL-MCF			Hop-MCF			[4]
		Gap	Nds	T(s)	Gap	Nds	T(s)	Gap	Gap	Nds	T(s)	Gap	Nds	T(s)	Gap
20	5	0.00	1	0.12	6.30	2	0.03	0.9	3.23	6	2.75	8.15	12	0.66	5.2
	10	0.00	1	0.12	12.73	22	0.44	6.8	4.09	16	10.6	12.52	181	26.0	6.8
	15	0.00	1	0.13	12.90	55	0.40	9.2	3.82	141	76.6	16.33	6788	1635	-
30	8	0.22	2	1.05	9.28	5	0.99	3.3	5.28	14	25.2	10.75	53	8.45	5.3
	15	0.00	1	0.37	14.27	36	0.44	7.4	4.23	86	217	18.36	7026	2110	-
	22	0.08	1	0.99	18.31	685	8.29	-	3.37	1608	6754	22.01	-	>24h	-
40	10	0.00	1	0.97	12.64	22	0.46	7.4	2.24	5	100	8.84	117	43.7	7.4
	20	0.00	1	1.09	14.96	263	5.47	-	6.23	6819	21866	18.06	-	>24h	-
	30	0.00	1	0.51	16.60	8456	198	-	4.11	23014	83150	20.98	-	>24h	-
Avg.		0.03	1	0.59	13.11	1061	23.8	5.8	4.07	3523	12466	15.11	-	-	6.18

References

1. Bley, A., Neto, J.: Approximability of 3- and 4-hop bounded disjoint paths problems. In: Eisenbrand, F., Shepherd, F.B. (eds.) IPCO 2010. LNCS, vol. 6080, pp. 205–218. Springer, Heidelberg (2010)
2. Botton, Q., Fortz, B., Gouveia, L., Poss, M.: Benders decomposition for the hop-constrained network design problem, Optimization on-line, report ULB (2010)
3. Gouveia, L., Simonetti, L., Uchoa, E.: Modeling hop-constrained and diameter-constrained minimum spanning tree problems as Steiner tree problems over layered graphs. Mathematical Programming, Online first (2009)
4. Huygens, D., Labbé, M., Mahjoub, A.R., Pesneau, P.: Two-edge connected hop-constrained network design problem: valid inequalities and branch-and-cut. Networks 49, 116–133 (2007)