On the separation of partition inequalities

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Abstract

Given a graph $G = V, E)$ with nonnegative weights $x(e)$ for each edge e, a partition inequality is of the form $x(\delta(V_1, \ldots, V_p)) \geq ap + b$. Here $\delta(V_1, \ldots, V_p)$ denotes the multicut defined by $(V_1, \ldots, V_p)$ of V. Partition inequalities arise as valid inequalities for optimization problems related to k-connectivity. In this paper, we will show that, if G decomposes into $G_1, \ldots, G_r$ by 1 and 2-node cutsets, then the separation problem for the partition inequalities on G can be solved by mean of a polynomial time combinatorial algorithm provided that such an algorithm exists for $\overline{G}_1, \ldots, \overline{G}_r$ where $\overline{G}_1, \ldots, \overline{G}_r$ are graphs related to $G_1, \ldots, G_r$. We will also discuss some applications.

Keywords: separation problem, partition inequalities, graph connectivity

1 Introduction

Given a graph $G = (V, E)$, a subset of nodes S whose cardinality is k, $k \geq 1$ is called a k-node cutset if the removal of S increases the numbers of connected components in G. If $\pi = (V_1, \ldots, V_p)$, $p \geq 2$ is a partition of V, we will denote by $\delta(V_1, \ldots, V_p)$ the set of edges with endnodes in different sets of the partition. We will also write $\delta(\pi)$ for $\delta(V_1, \ldots, V_p)$. For $T \subseteq E$, we will use $x(T)$ to denote $\sum_{e \in T} x(e)$. Given a and b, an inequality of the form

$$x(\delta(V_1, \ldots, V_p)) \quad \geq \quad ap + b, \tag{1}$$

is called a *partition inequality*. Partition inequalities arise as valid inequalities for network design and k-connectivity problems.
Given a vector x, the *separation problem* for inequalities (1) is to find an inequality of type (1) violated by x, if there is any. In [Baïou et al. (2000)], Baïou et al. show that the separation problem for inequalities (1) reduces to the minimization of a submodular function, and thus it is solvable in polynomial time.

In this paper, we consider the separation problem for inequalities (1) in the graphs that decompose by 1 and 2-node cutsets. We will show that the problem can be solved by mean of a polynomial combinatorial algorithm if such an algorithm exists for graphs related to the pieces.
If $a \leq 0$, the separation problem for inequalities (1), reduces to a minimum cut problem. Indeed, if a partition $(V_1, \ldots, V_p)$ with $p > 2$ induces a violated inequality, then any partition obtained by collapsing two elements also induces a violated inequality. So in the sequel we will suppose $a > 0$. In [Cunningham (1985)], Cunningham shows that if $b \leq -a$, the separation problem can be reduced to $|E|$ minimum cut problems. In [Barahona (1992)], Barahona proposes a reduction of the problem in this case to $|V|$ minimum cut problems. In this paper, we then consider the case $b > -a$.
If between two given nodes there are multiple edges $e_1, \ldots, e_t$ then by replacing the edges by only one edge and associated to this edge the sum of the weights of $e_1, \ldots, e_t$, then the separation problem for inequalities (1) in G is equivalent to the separation problem in the new graph. For this we will consider in the paper graphs without multiple edges.

Let $G = (V, E)$ be a graph. Given a partition $\pi = (V_1, \ldots, V_p)$ of V, we let G_π denote the graph obtained from G by contracting the elements V_i, $i = 1, \ldots, p$. We denote by $V_1, \ldots, V_p$ the nodes of G_π. A partition π_0 of V is said to be a *most violated partition* if for all violated partition π of V we have

$$ap_0 - x(\delta(\pi_0)) \geq ap - x(\delta(\pi)),$$

where p_0 and p are the number of elements of π_0 and π, respectively.
Given two sets $V_1, V_2 \subset V$ with $V_1 \cap V_2 = \emptyset$, we let $[V_1, V_2]$ denote the set of edges between v_1 and v_2.

2 Decomposition and separation

In this section we give some properties of the violated partitions. We first give a basic lemma.

Lemma 1 *Let $\pi = (V_1, \ldots, V_p)$ be a most violated partition. Suppose that V_1 is a node cutset in G_π. Suppose that $V_2, \ldots, V_r$, $2 \leq r < p$, are such that there is no edge linking a node V_i to a node V_j where $i \in \{1, \ldots, r\}$ and $j \in \{r+1, \ldots, p\}$. Let $\pi_1 = (V_1 \cup (\cup_{i=r+1}^{p} V_i), V_2, \ldots, V_r)$ and $\pi_2 = (V_1 \cup (\cup_{i=1}^{r} V_i), V_{r+1}, \ldots, V_p)$. Then π_1 and π_2 are violated.*

Proof. Note that r and $p - r + 1$ are the cardinalities of π_1 and π_2. Suppose first that both partitions π_1 and π_2 are not violated. Hence $x(\delta(\pi_1)) \geq ar + b$ and $x(\delta(\pi_2)) \geq a(p - r + 1) + b$. By summing these inequalities, we obtain $x(\delta(\pi)) \geq a(r + p - r + 1) + 2b = a(p + 1) + 2b$. Since π is violated, it follows that $ap + b > a(p + 1) + 2b$. Hence $a < -b$, a contradiction.
Now suppose that exactly one of the partitions, say π_1, is violated. Thus $x(\delta(\pi_2)) \geq a(p - r + 1) + b$. Since π is a most violated partition, we also have $x(\delta(\pi_1)) - ar - b \geq x(\delta(\pi)) - ap - b$. By summing these two latter inequalities and making some simplifications, we obtain $a < -b$, yielding again a contradiction. □

Lemma 2 *If $\pi = (V_1, \ldots, V_p)$ is a most violated partitions of G, then $G(V_i)$ is connected for $i = 1, \ldots, p$.*

Proof. Suppose, for instance, that $G(V_1)$ is not connected and let (V_1^1, V_1^2) be a partition of V_1 such that $[V_1^1, V_1^2] = \emptyset$. Let $\pi' = (V_1', \ldots, V_{p+1}')$ be the partition given by

$$\begin{cases} V_i' = V_1^i & \text{for } i = 1, 2, \\ V_i' = V_{i-1} & \text{for } i = 3, \ldots, p+1. \end{cases}$$

Note that $\delta(\pi') = \delta(\pi)$. Moreover,$a(p + 1) + b - x(\delta(\pi')) > ap + b - x(\delta(\pi))$, which contradicts the fact that π is a most violated partitions. □

Lemma 3 *Let $G = (V, E)$ be a graph. Suppose that G contains a node cutset v_0. Let $G_1 = (U_1, E_1)$ and $G_2 = (U_2, E_2)$ be such that $U_1 \cap U_2 = \{v_0\}$, $U_1 \cup U_2 = V$, $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \emptyset$. If there is a violated partition in G, then at least one of the graphs G_1 and G_2 contains a violated partition.*

Proof. Let $\pi = (V_1, \ldots, V_p)$ be a most violated partitions in G. By Lemma 2, $G(V_i)$ is connected for $i = 1, \ldots, p$. W.l.o.g we may suppose that $v_0 \in V_1$. Then for $i = 2, \ldots, p$, V_i is contained either in U_1 or in U_2. Suppose that there are two elements among $V_2, \ldots, V_p$ such that one is in U_1 and the other in U_2. Then we may suppose that $V_2, \ldots, V_r \subseteq U_1$ and $V_{r+1}, \ldots, V_p \subseteq U_2$, for some $2 \leq r < p$. As V_1 is a node cutset in the graph G_π, by Lemma 1, the partition obtained from π by collapsing $V_1, V_2, \ldots, V_r$ $(V_1, V_{r+1}, \ldots, V_p)$ is violated. This implies that the partitions $((V_1 \backslash U_2) \cup \{v_0\}, V_2, \ldots, V_r)$ of G_1 and $((V_1 \backslash U_1) \cup \{v_0\}, V_{r+1}, \ldots, V_p)$ of G_2 are violated.
Now suppose that all the elements $V_2, \ldots, V_p$ are, for instance, in U_1. This means that $U_2 \subseteq V_1$. Consider the partition $\pi_1 = ((V_1 \backslash U_2) \cup \{v_0\}, V_2, \ldots, V_p)$ of G_1. As $x(\delta(\pi)) = x(\delta(\pi_1))$ and π is violated, π_1 is violated. $\square$

3 2-node cutsets

Consider a graph $G = (V, E)$ that decomposes into $G_1 = (U_1, E_1)$ and $G_2 = (U_2, E_2)$ by a 2-node cutset $\{v_1, v_2\}$, such that $U_1 \cap U_2 = \{v_1, v_2\}$, $U_1 \cup U_2 = V$, $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \emptyset$. Let $G'_1 = (U'_1, E'_1)$ be the graph obtained from G_1 by contracting v_1 and v_2. Let u be the node that arises from the contraction. The following is easily seen to be true.

Lemma 4 *Let $(S_1, \ldots, S_k)$ be a partition of U'_1 and suppose that $u \in S_1$. Let $\overline{S}_1 = (S_1 \backslash \{u\}) \cup U_2$. If $(S_1, \ldots, S_k)$ is violated, then $(\overline{S}_1, S_2, \ldots, S_k)$ is a violated partition in G.*

In what follows, we suppose that there is no violated partition in G'_1. We will show that, in this case, determining a violated partition in G if there is any, reduces to determining a violated partition in the graph obtained from G_2 by adding an edge between v_1 and v_2.
Let $G'_2 = (U_2, E'_2)$ be the graph obtained from G_2 by adding an edge e_0 between v_1 and v_2. Consider the problem

$$\text{minimize} \quad \{x(\delta(S_1, \ldots, S_k)) - ak\}. \tag{2}$$

where $(S_1, \ldots, S_k)$ is a partition of U_1. We will denote by $\pi_1^* = (S_1^*, \ldots, S_{p_1^*}^*)$ an optimal solution of (2). Let $\sigma = x(\delta(S_1^*, \ldots, S_{p_1^*}^*)) - a(p_1^* - 2)$ and $x'_2 \in I\!R^{E'_2}$ be defined as

$$x'_2(e) = \begin{cases} x(e) & \text{if } e \in E_2, \\ \sigma & \text{if } e = e_0. \end{cases}$$

We have the following lemmas.

Lemma 5 *If $\pi = (V_1, \ldots, V_p)$ is a most violated partitions of G, then $x([V_i, V_j]) \leq a$ for $i, j = 1, \ldots, p, i \neq j$.*

Proof. Suppose, for instance, that $x([V_1, V_2]) > a$. Let $\pi' = (V'_1, \ldots, V'_{p-1})$ be the partition given by

$$\begin{cases} V'_1 = V_1 \cup V_2, & \\ V'_i = V_{i+1}, & \text{for } i = 2, \ldots, p-1. \end{cases}$$

We have

$$\begin{aligned} a(p-1) + b - x(\delta(\pi')) &= a(p-1) + b - x(\delta(\pi)) + x([V_1, V_2]) \\ &> a(p-1) + b - x(\delta(\pi)) + a \\ &= ap + b - x(\delta(\pi)), \end{aligned}$$

contradicting the fact that π is a most violated partition. $\square$

Lemma 6 *If there exists a violated partition in G with at least three elements, then there exists a violated partition in G'_2 with respect to x'_2.*

Proof. Let $\pi = (V_1, \ldots, V_p)$ be a most violated partitions in G with $p \geq 3$. Let us consider first the case where v_1 and v_2 are in the same element of π, say V_1. As G'_1 does not contain violated partition, at least one of the elements of π intersect U_2. Since π is a most violated partition and hence by Lemma 2, $G(V_i)$ is connected for $i = 1, \ldots, p$, it follows that for $i = 2, \ldots, p$, either $V_i \subset U_1$ or $V_i \subset U_2$. Hence, we may suppose that $V_2, \ldots, V_q$, for some $2 \leq q \leq p$, are the elements of π in U_2. We claim that $p = q$. In fact, if not then V_1 would be a node cutset in G'_π, and by Lemma 1, it follows that the partition $(\cup_{i=1}^{q} V_i, V_{q+1}, \ldots, V_p)$ of G_1 is violated. As $v_1, v_2 \in V_1$, this implies that G'_1 contains a violated partition. A contradiction. Consequently $q = p$, that is $U_1 \subset V_1$. Let $\pi_2 = (V'_1, V_2, \ldots, V_q)$ where $V'_1 = V_1 \cap U_2$. Note that π_2 is a partition of G'_2. Since this partition has the same number of elements and same weight as π, we have that π_2 is violated.

Now suppose that the nodes v_1 and v_2 are in two different elements. Assume for instance that $v_1 \in V_1$ and $v_2 \in V_2$. Let $V_3, \ldots, V_q$ be the elements of π in G_1 and $V_{q+1}, \ldots, V_p$ those in G_2. Consider the partition $\pi_2 = (V'_1, V'_2, \ldots, V'_{p_2})$, with $p_2 = p - q + 2$, of U_2 given by

$$\begin{cases} V'_1 = (V_1 \setminus U_1) \cup \{v_1\}, & \\ V'_2 = (V_2 \setminus U_1) \cup \{v_2\}, & \\ V'_i = V_{q+i-2}, & \text{for } i = 3, \ldots, p_2. \end{cases}$$

We have

$$\begin{aligned} x'_2(\delta(\pi_2)) &= x(\delta(\pi)) - x(\delta(V_1, V_2, \ldots, V_q)) + x'_2(e_0) \\ &< ap + b - x(\delta(V_1, V_2, \ldots, V_q)) + x(\delta(\pi_1^*)) - a(p_1^* - 2) \\ &= a(p_2 + q - 2) + b - x(\delta(V_1, V_2, \ldots, V_q)) + x(\delta(\pi_1^*)) - a(p_1^* - 2). \end{aligned}$$

Since π_1^* is the partition of G_1 which minimizes (2), we have $aq - x(\delta(V_1, V_2, \ldots, V_q)) \leq ap_1^* - x(\delta(\pi_1^*))$. Thus

$$\begin{aligned} x'_2(\delta(\pi_2)) &< a(p_2 - 2) + b + (ap_1^* - x(\delta(\pi_1^*))) + x(\delta(\pi_1^*)) - a(p_1^* - 2) \\ &= ap_2 + b. \end{aligned}$$

Therefore, the partition inequality associated with π_2 is violated with respect to x'_2. $\square$

Lemma 7 *If $\pi'_2 = (V'_1, \ldots, V'_{p_2})$ is a most violated partitions in G'_2 and $e_0 \in \delta(\pi'_2)$, then π_1^* is violated. Moreover there are two different elements of π_1^* such that v_1 belongs to one of the elements and v_2 to the other.*

Proof. Suppose that π_1^* is not violated. Then

$$x(\delta(\pi_1^*)) \geq ap_1^* + b \tag{3}$$

Since π'_2 is violated in G'_2, by Lemma 5, it follows that $a \geq x'_2(e_0) = x(\delta(\pi_1^*)) - a(p_1^* - 2)$. Thus $x(\delta(\pi_1^*)) \leq a(p_1^* - 1)$. By summing this inequality and (3), we obtain $b \leq -a$, a contradiction.
So π_1^* is violated. Moreover as G'_1 does not contain violated partition, we have that v_1 and v_2 are in different elements. $\square$

By Lemmas 6 and 7, there exists a violated partition in G if and only if there exists a violated partition in G'_2 with respect to x'_2.

The following lemma is easy to prove.

Lemma 8 *Let $\pi'_2 = (V'_1, \ldots, V'_{p_2})$ be a violated partition in G'_2 with respect to x'_2. Suppose that v_1 and v_2 are in the same element of π'_2, say V'_1. Then the partition $\pi = (V'_1 \cup U_1, V'_2, \ldots, V'_{p_2})$ is a violated partition in G.*

Now suppose that G'_2 contains a violated partition, and let $\pi'_2 = (V'_1, \ldots, V'_{p_2})$ be a most violated one. Suppose that $e_0 \in \delta(\pi'_2)$ and let us assume, for instance, that e_0 is between V'_1 and V'_2. Then by Lemma 7, π_1^* is violated and v_1 and v_2 belong to different elements of π_1^*, say S_1^* and S_2^*, respectively. Let $\pi = (V_1, \ldots, V_p)$ with $p = p_1^* + p_2 - 2$, be the partition of G such that

$$\begin{cases} V_1 = V_1' \cup S_1^*, & \\ V_2 = V_2' \cup S_2^*, & \\ V_i = S_i^*, & \text{for } i = 3, \ldots, p_1^*, \\ V_i = V'_{i-p_1^*}, & \text{for } i = p_1^* + 3, \ldots, p_1^* + p_2. \end{cases}$$

Lemma 9 *π is a most violated partition in G.*

Proof. We first show that π is violated. Note that $x(\delta(\pi)) = x(\delta(\pi_1^*)) + x(\delta(\pi_2'))$. We have

$$\begin{aligned} ap - x(\delta(\pi)) &= a(p_1^* + p_2 - 2) - x(\delta(\pi_1^*)) - x(\delta(\pi_2')) \\ &= ap_2 - \sigma - x(\delta(\pi_2')) \\ &= ap_2 - x_2'(\delta(\pi_2')). \end{aligned}$$

Since π_2' is violated, π is so.
Let $\hat{\pi} = (\hat{V}_1, \ldots, \hat{V}_{\hat{p}})$ be a partition of G. Let $\hat{\pi}_1$ and $\hat{\pi}_2$ be the restrictions of $\hat{\pi}$ on G_1 and G_2, respectively. Let $\hat{p_1}$ and $\hat{p_2}$ be the number of elements of $\hat{\pi}_1$ and $\hat{\pi}_2$, respectively.We will discuss three cases.
Case 1 There is $i \in \{1, \ldots, \hat{p}\}$ such that $U_1 \subseteq \hat{V}_i$. Thus

$$\begin{aligned} a\hat{p} - x(\delta(\hat{\pi})) &= a\hat{p} - x(\delta(\hat{\pi}_2)) \\ &\leq ap_2 - x(\delta(\pi_2')) \\ &= ap - x(\delta(\pi)). \end{aligned}$$

The second inequality comes from the fact that π_2' is a most violated partition in G_2'.
Case 2 There is no $i \in \{1, \ldots, \hat{p}\}$ such that $U_1 \subseteq \hat{V}_i$ and v_1 and v_2 are in the same element of $\hat{\pi}$. As G_1' does not contain a violated partition, it follows that $x(\delta(\hat{\pi}_1)) \geq a\hat{p}_1 + b$. This implies that

$$\begin{aligned} a\hat{p} - x(\delta(\hat{\pi})) &= a(\hat{p}_1 + \hat{p}_2 - 1) - x(\delta(\hat{\pi}_1)) - x(\delta(\hat{\pi}_2)) \\ &\leq a(\hat{p}_1 + \hat{p}_2 - 1) - a\hat{p}_1 - b - x(\delta(\hat{\pi}_2)) \\ &= a\hat{p}_2 - a - b - x(\delta(\hat{\pi}_2)) \\ &\leq a\hat{p}_2 - x(\delta(\hat{\pi}_2)) \\ &= ap - x(\delta(\pi)). \end{aligned}$$

The third inequality comes from the fact that $-b < a$.
Case 3 v_1 and v_2 are in two different elements. We have

$$\begin{aligned} a\hat{p} - x(\delta(\hat{\pi})) &= a(\hat{p}_1 + \hat{p}_2 - 2) - x(\delta(\hat{\pi}_1)) - x(\delta(\hat{\pi}_2)) \\ &\leq a(p_1^* + \hat{p}_2 - 2) - x(\delta(\pi_1^*)) - x(\delta(\hat{\pi}_2)) \\ &= a\hat{p}_2 - \sigma - x(\delta(\hat{\pi}_2)). \end{aligned}$$

As π_2' is the most violated partition in G_2', it follows that

$$\begin{aligned} a\hat{p} - x(\delta(\hat{\pi})) &\leq ap_2 - x(\delta(\pi_2')) \\ &= ap - x(\delta(\pi)) \end{aligned}$$

In all cases, we obtain that $a\hat{p} - x(\delta(\hat{\pi})) \leq ap - x(\delta(\pi))$, that is to say π is more violated than $\hat{\pi}$. As π is an arbitrary partition, this implies that π is a most violated one. □

By Lemma 9, if we know a most violated partition in G_2' with respect to x_2', such that v_1 and v_2 belong to different sets of the partition, then we can extend this partition to a most violated one in G.

4 A polynomial combinatorial algorithm

The previous lemmas lead to a technic for separating the partition inequalities in the graphs that decompose by 1 and 2-node cutsets. If G has a node cutset and G decomposes into H_1 and H_2 then by Lemma 3, looking for a violated partition in G reduces to looking for a violated partition in each of the graphs H_1 and H_2. We may thus suppose that G decomposes only by 2-node cutsets. For separating the inequalities in G, we recursively apply the procedure below.

If G has a 2-node cutset v_1, v_2 and G decomposes into G_1 and G_2, then apply the following.
Solve the separation problem for inequalities (1) in G_1'.
If a violated partition is found, then extend the found partition to a violated partition in G according to Lemma 4 and stop.
If not, then
solve problem (2) for G_1.
Consider the graph G_2' obtained from G_2 by adding an edge e_0 between v_1 and v_2 and the weight vector x_2'.
Solve the separation problem in G_2' with respect to x_2' and extend the found violated partition, if there is any, to a violated partition in G according to either Lemma 9 or Lemma 8 depending on whether e_0 belongs to the partition or not.

If G decomposes into $G_1, \ldots, G_r$ by 1 and 2-node cutset and $\overline{G}_1, \ldots, \overline{G}_r$ are the graphs obtained from $G_1, \ldots, G_n$ by adding an edge between every pair of nodes consisting of a 2-node cutset, the above procedure yields a polynomial time combinatorial algorithm for separating inequalities (1) in G, if such an algorithm exists for $\overline{G}_1, \ldots, \overline{G}_r$. Moreover by Lemma 9, the algorithm may give a most violated inequality.

5 Applications

A graph $G = (V, E)$ is called *k-edge connected* (where k is a positive integer), if between any pair of nodes $i, j \in V$, there are at least k edge-disjoint paths.
Given a weight function $c : E \mapsto \mathbb{R}$, *the k-connected spanning subgraph problem* (kECSP) is to find a k-edge connected subgraph $H = (V, F)$ of G, spanning all the nodes in V such that $\sum_{e \in F} w(e)$ is minimum.
A *homeomorph of* K_4 (the complete graph on 4 nodes) is a graph obtained from K_4 where its edges are subdivised into paths by inserting new nodes of degree two. A graph is called *series-parallel* if it does not contain a homeomorph of K_4 as a subgraph. In [Didi Biha and Mahjoub (1996)], Didi Biha and Mahjoub show that the following inequalities are valid for the polytope associated with the kECSP when k is odd.

$$x(\delta(V_1, \ldots, V_p)) \geq \left\lceil \frac{k}{2} \right\rceil p - 1, \text{ for all partition } \pi = (V_1, \ldots, V_p) \quad (4)$$
$$\text{such that } G_\pi \text{ is series-parallel}$$

These inequalities are called *SP-partition inequalities*. Series-parallel graphs decomposable by node and 2-node cutsets into graphs consisting of paths. Since the separation problem for inequality (4) can be easily solved into these graphs, we have that this problem can be solved in polynomial time using a combinatorial algorithm in series-parallel graphs.

Graphs with no W_4 (the wheel on 5 nodes) as a minor decompose by 1 and 2-node cutsets. Each piece is either an edge or a diamant (the graph $K_4 - e$) [Halin (1981)]. It is clear that the separation problem of (1) can be solved in polynomial time (by enumeration) in the pieces. Then the problem can be solved using a polynomial combinatorial algorithm on the W_4-free graphs.

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