How to model interactions between criteria in MCDA?

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Chapter 2
Outline

1. Introduction
2. The 2-additive Choquet integral
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1. Introduction

2. The 2-additive Choquet integral
The context: MultiCriteria Decision Aid

Aim: to help a decision-maker (DM) to select one or more alternatives among several alternatives evaluated on $|N|$ criteria often contradictory.

⇒ We need to construct a preference relation over the set of all alternatives $X$. 

Introduction
Example ("Star academy")

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**Problem**: Give a ranking of all the six students.

Maybe a simple problem if we use the weighted sum as aggregation function. 

but how to determine the weight of each criterion in this case?

**It is not an easy task!**
Notations

- $N$ is a set of $n$ criteria

- $X_1, \ldots, X_n$ represent the set of points of view or attributes

- An alternative or option $x = (x_1, \ldots, x_n)$ is identified to an element of $X = X_1 \times \cdots \times X_n$
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\[ X' = \{a, b, c, d\} \]

- If two students are good in Song and Music, then the jury prefers strictly the student who have a best evaluation in Choreographie \( \Rightarrow b \succ_{X'} a; \)

- If two students are bad in Song, then the jury prefers strictly the student who have a best evaluation in Music \( \Rightarrow c \succ_{X'} d. \)
Outline

1. Introduction
2. The 2-additive Choquet integral
Definition (2-additive Choquet integral)

For any $x := (x_1, ..., x_n) \in X$, the expression of the 2-additive Choquet is:

$$C_{\mu}((u(x_1), \ldots, u(x_n))) = \sum_{i=1}^{n} v_i u(x_i) - \frac{1}{2} \sum_{\{i,j\} \subseteq N} I_{ij} |u(x_i) - u(x_j)|$$

Where

- $v_i =$ the importance of the criterion $i$ (≡ Shapley index);
  $$I_{ij} = \mu_{ij} - \mu_i - \mu_j.$$  \hspace{1cm} (2)

- $l_{ij} =$ the interaction index between criteria $i$ and $j$.
  $$v_i = \mu_i + \frac{1}{2} \sum_{k \in N \setminus i} I_{ik}.$$  \hspace{1cm} (3)
The 2-additive Choquet integral

The 2-additive monotonicity conditions

\[
\sum_{\{i,j\} \subseteq N} \mu(\{i,j\}) - (n - 2) \sum_{i \in N} \mu(\{i\}) = 1 \quad \text{(normality)}
\]

\[
\mu(\{i\}) \geq 0, \ \forall i \in \mathcal{N} \quad \text{(nonnegativity)}
\]

\[
\forall A \subseteq \mathcal{N}, \ |A| \geq 2, \ \forall k \in A
\]

\[
\sum_{i \in A \setminus \{k\}} (\mu(\{i,k\}) - \mu(\{i\})) \geq (|A| - 2)\mu(\{k\}) \quad \text{(monotonicity)}.
\]

Notations

\[
\forall i, j \in \mathcal{N}, \ i \neq j, \ \mu_\emptyset = \mu(\emptyset), \ \mu_i = \mu(\{i\}) \ \text{and} \ \mu_{ij} = \mu(\{i,j\})
\]
Interpretation of $I_{ij}$

- $I_{ij} = 0 \Rightarrow$ independence between $i$ and $j$;
- $I_{ij} > 0 \Rightarrow$ complementary among $i$ and $j$;
- $I_{ij} < 0 \Rightarrow$ substitutability among $i$ and $j$;
Interest of the 2-additive model

The 2-additive Choquet integral

1. is very used in many applications such that
   - the evaluation of discomfort in sitting position (see Grabisch et al. (2002));
   - the construction of performance measurement systems model in a supply chain context (see Berrah and Clivillé (2007), Clivillé et al. (2007));
   - complex system design (Labreuche and Pignon (2007));

2. offers a good compromise between flexibility of the model and complexity;

3. requires to be able to compare any element of one point of view with any element of any other point of view (commensurateness between criteria);

4. The only way to construct the utility functions with the Choquet integral uses the reference levels (Grabisch and Labreuche (2003)).
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To apply the Choquet integral, we need commensurate scales
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$X' = \{ a, b, c, d \}$. If we consider the 2-additive capacity $\mu : 2^N \to [0, 1]$ defined such that: $\mu(N) = 1$, $\mu(\emptyset) = 0$, $\mu(\{2\}) = \mu(\{3\}) = \mu(\{2, 3\}) = \mu(\{1, 3\}) = \frac{1}{2}$, $\mu(\{1\}) = 0$, $\mu(\{1, 2\}) = 1$, then we have $l_{12} = \frac{1}{2}$, $l_{13} = 0$, $l_{23} = -\frac{1}{2}$, $v_1 = \frac{1}{4}$, $v_2 = \frac{1}{2}$, $v_3 = \frac{1}{4}$:

\[
\begin{align*}
C_\mu(a) &= 7 \cdot v_1 + 17 \cdot v_2 + 14 \cdot v_3 - \frac{1}{2}(l_{12} |7 - 17| + l_{13} |7 - 14| + l_{23} |17 - 14|) = 12 \\
C_\mu(b) &= 9 \cdot v_1 + 17 \cdot v_2 + 12 \cdot v_3 - \frac{1}{2}(l_{12} |9 - 17| + l_{13} |9 - 12| + l_{23} |17 - 12|) = 13 \\
C_\mu(c) &= 7 \cdot v_1 + 8 \cdot v_2 + 14 \cdot v_3 - \frac{1}{2}(l_{12} |7 - 8| + l_{13} |7 - 14| + l_{23} |8 - 14|) = 10.5 \\
C_\mu(d) &= 9 \cdot v_1 + 8 \cdot v_2 + 12 \cdot v_3 - \frac{1}{2}(l_{12} |9 - 8| + l_{13} |9 - 12| + l_{23} |8 - 12|) = 10
\end{align*}
\]

Hence $b \succ_{X'} a$ and $c \succ_{X'} d$. 