PREFERENCES AGGREGATION & DECISION THEORY (2)

A. ROLLAND, UNIVERSITE LUMIERE LYON II

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PLAN

FRAMEWORK

- a **Decision Maker (DM)** is facing a decision problem, i.e. the DM has to deal with multiple alternatives and has to compare themselves.
- alternatives are described on several **attributes**.
- a **criterion** is an attribute with a **preference relation**.
- criteria cannot **be reduced** to one criterion as they are potentially **in conflict**.

Framework

Example

attribute	Mountain bike	race bike
speed	20 km/h	35 km/h
robust	Good	Middle
price	500e	1000e

MODEL FOR MULTI-ATTRIBUTE DECISION

FORMAL MODEL : INPUTS

- a set of alternatives $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$
- a representation of the preferences on the values of each criterion $i \in N$ (utility function, qualitative preference relations $\succsim_i$...)
- a representation of the importance of each criterion or set of criteria (weights, importance relation...)

MODEL FOR MULTI-ATTRIBUTE DECISION

FORMAL MODEL : TREATMENTS

- a decision rule using informations on criteria and coalitions to discriminate the alternatives

$$\left.\begin{array}{l} x = (x_1, \ldots, x_n) \\ y = (y_1, \ldots, y_n) \end{array}\right\} \Rightarrow (x \succsim y) \text{ or } (y \succsim x)$$

SIMPLE METHODS : PARETO DOMINANCE

PARETO DOMINANCE

An alternative is preferred to another one if it is considered to be better on **all** the criteria.

$$x \succsim y \iff [\forall i \in N,\ x_i \succsim y_i]$$

EXAMPLE

criteria	bike A	bike B
speed	20 km/h	30 km/h
robust	Good	Good
price	600e	500e

$$B \succsim A \text{ and } A \not\succsim B \Rightarrow B \succ A$$

SIMPLE METHODS : PARETO DOMINANCE

PARETO

Not so interresting...

EXEMPLE

criteria	bike A	bike B
speed	15 km/h	30 km/h
robustness	very Good	Good
price	1600e	500e

$$B \not\succsim A \text{ and } A \not\succsim B \Rightarrow B \sim A!!$$

Simple method : weighted sum

Weighted sum

$$x \succsim y \iff \sum_i \omega_i . x_i \succsim \sum_i \omega_i . y_i$$

$\forall i \in N,\ x_i, y_i, \omega_i \in \mathbb{R},\ \forall i \in N$

Example (1)

criteria	**A**	**B**
speed	8/20	18/20
robustness	18/20	8/20

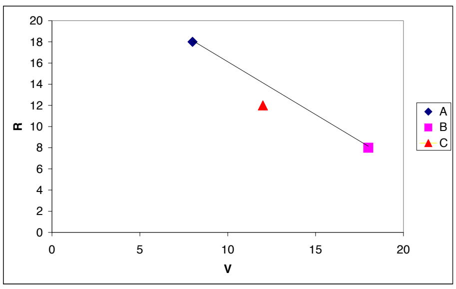
$$\omega_r > \omega_s \quad \Rightarrow \quad A \succ B$$
$$\omega_s > \omega_r \quad \Rightarrow \quad B \succ A$$

SIMPLE METHOD : WEIGHTED SUM

EXAMPLE (2)

criteria	**A**	**B**	**C**
speed	8/20	18/20	12/20
robustness	18/20	8/20	12/20

$$\forall \omega_r, \omega_s \in \mathbb{R}, \quad A \succsim C \text{ or } B \succsim C$$



SIMPLE METHOD : MAJORITY VOTE

MAJORITY VOTE

An alternative is preferred to another one if it is considered as better on **a majority** of criteria.

$$x \succsim y \iff |\{i \in N,\ x_i \succsim y_i\}| \succsim |\{i \in N,\ y_i \succsim x_i\}|$$

SIMPLE METHOD : MAJORITY VOTE

EXAMPLE

criteria	bike A	bike B
speed	20 km/h	30 km/h
robustness	Good	Middle
price	600e	500e

$$\left\{ \begin{array}{lcl} \{i \in N,\ B_i \succsim A_i\} & = & \{\text{speed},\ \text{price}\} \\ \{i \in N,\ A_i \succsim B_i\} & = & \{\text{robustness}\} \end{array} \right. \Rightarrow B \succ A$$

SIMPLE METHOD : MAJORITY VOTE

CONDORCET PARADOX

criteria	bike A	bike B	bike C
speed	14/20	12/20	13/20
robustness	15/20	16/20	17/20
price	13/20	15/20	12/20

Which bike do you choose ?

Multicriteria Decision Aiding

Difficulties

- multicriteria decision aiding is not so easy.
- Every method has advantages and inconveniences : there is no ”best method”
- natural and simple methods all have structural bias.

Paul Valéry

Ce qui est simple est faux. Ce qui est compliqué est inutilisable
What is simple is false. What is complex is useless.

PROBLEMS

PROBLEMS IN MULTICRITERIA DECISION THEORY [ROY96]

- Choice Problem : one has to choose the best alternative(s).
- Ranking Problem : one has to rank the alternatives from the best to the worst.
- Sorting Problem : one has to sort the alternatives into pre-defined categories (ordered or not)

MULTICRITERIA DECISION MODELS

TWO MAIN APPROACHES

- quantitative approach " aggregate then compare " (scoring)

$$x \succsim y \iff \psi(x_1, \ldots, x_n) \geq \psi(y_1, \ldots, y_n)$$

- qualitative approach " compare then aggregate" (outranking)

$$x \succsim y \iff \{j | x_j \succsim_j y_j\} \triangleright \{j | y_j \succsim_j x_j\}$$

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SOME USEFULL REFERENCES

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- D. Bouyssou, D. Dubois, M. Pirlot and H. Prade (Edts), Decision-making Process Concepts and Methods, ISTE & Wiley, 2009 (3 volumes)

PREFERENCES AGGREGATION OPERATORS

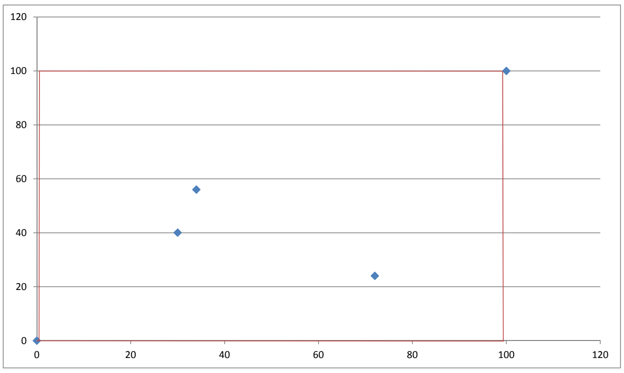
DIFFERENT METHODS

- Pareto Dominance **Pareto Dominance**
- Multi-objective optimization
- Utility functions

Notations

Example

In this part, $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ with $\mathcal{X}_1 = \mathcal{X}_2 = [0; 100]$

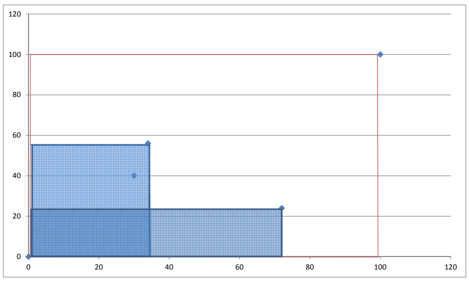


DOMINANCE

DEFINITION

An alternative $x \in \mathcal{X}$ is said to

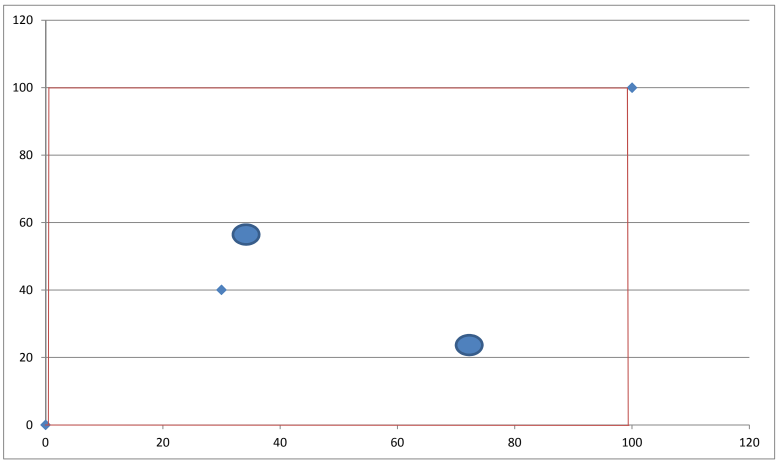
- **dominate** an alternative $y \in \mathcal{X}$ if $\forall i \in N, \ \ x_i \succsim_i y_i$
- **strictly dominate** an alternative $y \in \mathcal{X}$ if $\forall i \in N, \ \ x_i \succsim_i y_i$ et $\exists i \in N, x_i \succ_i y_i$



PARETO FRONT

DEFINITION

The Pareto front is the set of all the non-dominated alternatives



PARETO FRONT

PROPERTIES

- the optimum solution is necessary in the Pareto front
- by often the Pareto front is not very smaller that the set of alternatives

PREFERENCES AGGREGATION OPERATORS

DIFFERENT METHODS

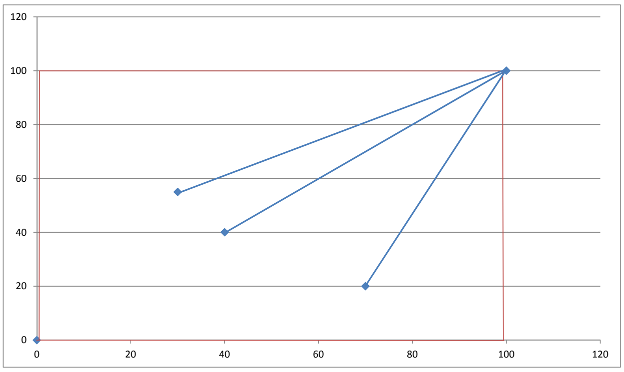
- Pareto Dominance
- **Multi-objective optimization**
- Utility functions

MULTI-OBJECTIVE OPTIMIZATION (1)

PRINCIPLE

Best alternative = nearest alternative from an “ideal point”

Usually, the ideal point if computed by taking the max (resp min) value on each criterion.



MULTI-OBJECTIVE OPTIMIZATION (2)

DISTANCES

- Manhattan distance : $\sum_i \mid x_i - y_i \mid$
- Euclidian distance : $\sqrt{\sum_i \mid x_i - y_i \mid^2}$
- Euclidian p-distance : $(\sum_i (x_i - y_i)^p)^{1/p}$
- Chebychev distance : ∞-distance : $max_i \mid x_i - y_i \mid$.

MULTI-OBJECTIVE OPTIMIZATION (3)

EXAMPLE

Ideal point : (100 ;100)

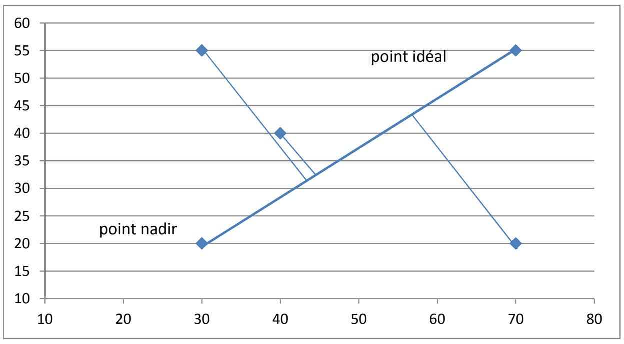
x	y	Manhattan	Euclidian	Chebychev
30	55	115	83.2	70
70	20	110	85.4	80
40	40	120	84.9	60

A distance : an optimum !

MULTI-OBJECTIVE OPTIMIZATION (4)

OTHER METHOD

Compute the ideal and anti-ideal points. Project the alternatives one the ideal - anti-ideal axis to determine the optimum.



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PREFERENCES AGGREGATION OPERATORS

DIFFERENT METHODS

- Pareto Dominance
- Multi-objective optimization
- **Utility functions**

UTILITY FUNCTIONS

GENERAL PRINCIPLE

$\Rightarrow$ **Aggregation** of the criteria values into a single criterion the **comparison** of the obtained scores.

$$x \succsim y \iff u(x) \geq u(y)$$

with $u(x) = f(x_1, \ldots x_n)$.

PROBLEMS

CODING THE CRITERIA VALUES

How to determine the la relation ”alternatives - consequences”, i.e. the set of values taken by each alternative on each criteria ? We associate to an alternative x from $\mathcal{X}$, n functions $g_1(x), \dots, g_n(x)$ into $\mathcal{X}_1, \dots, \mathcal{X}_n$.

CODING THE INTERACTIONS BETWEEN CRITERIA

To each alternative x in $\mathcal{X}$, we associate a number $U(x) = U(g(x)) = U(g_1(x), \dots, g_n(x))$ such that $a \succsim b \iff U(a) \geq U(b)$.

Coding the criteria values(1)

Principles

- functions g_i aim at focusing on criteria from attributes
- sets $\mathcal{X}_i$ are generaly subsets of $\mathbb{R}$
- scales of $\mathcal{X}_i$ must be carefully chosen to enable comparisons
- scales of $\mathcal{X}_i$ should have a signification for the decision maker

Coding the criteria values(2)

Ordinal scales

Differences between values have no importance (e.g. rank).
Can represent orders and pre-orders.

Cardinal scales

Differences between values do have an importance.

- interval scales : absolute differences between values are important. Functions g_i are invariant by translation $h_i(x_i) = g_i(x_i) + b_i$
- ratio scales : ratio between values are important. Functions g_i are invariant by similitude $h_i(x_i) = a.g_i(x_i)$.

AGGREGATION OPERATORS

We suppose here that all scales are the same, eg [0; 1].

AGGREGATION OPERATOR

Coding the interactions and substitutions between criteria.

AGGREGATION OPERATOR

- min/max
- weighted sum
- OWA
- Choquet integral

MIN-MAX OPERATORS

MIN

Min is a conjunctive operator : an alternative is good if all its values are good

MAX

Max is a disjunctive operator : an alternative is good if one of its values is good

MIN-MAX OPERATORS

Example :
Compute the min and max of the following alternatives

	c_1	c_2	c_3
x	5	4	6
y	3	7	5

WEIGHTED SUM

DEFINITION

$U(x) = \omega_1.g_1(x) + \ldots + \omega_n.g_n(x)$ with $\sum_i \omega_i = 1$

PROPERTIES

- trade-off between criteria are fix
- the weighted sum is totally compensatory

WEIGHTED SUM

Example :
Compute the min and max of the following alternatives

	c_1	c_2	c_3
weight	0.3	0.5	0.2
x	5	4	6
y	3	7	5

OTHERS MEANS

weighted mean	$\sum_i \omega_i x_i$
quadratic mean	$(\sum_i \omega_i x_i^2)^{1/2}$
geometric mean	$\prod_i x_i^{\omega_i}$
harmonic mean	$(\sum_i \omega_i \frac{1}{x_i})^{-1}$
Power α mean	$(\sum_i \omega_i x_i^{\alpha})^{1/\alpha}$

Others means

Properties

- $\alpha \rightarrow \infty \Rightarrow$ mean $\rightarrow$ max
- $\alpha \rightarrow -\infty \Rightarrow$ mean $\rightarrow$ min

Models : what for ?

Prescriptive approach

To **help** a decision maker by the proposal of a solution obtained by a model

Descriptive Approach

To **describe** a decision maker's preferences by the chosen model.

Elicitation

The elicitation of the decision maker's preferences consists in **obtaining parameters** of a decisional model which explain the past decisions in order to help in the future decisions.

Elicitation of the parameters

Option 1 : explicit elicitation

- explain the model to the decision maker
- let him choose the parameters

Option 2 : implicit elicitation

- present some (fictitious) alternatives and ask the decision maker to compare them
- deduct the parameters with optimization program

ELICITATION OF THE PARAMETERS

OPTION 2 : IMPLICIT ELICITATION

- present some (fictitious) alternatives and ask the decision maker to compare them
- deduct the parameters with optimization program

Need to find both :

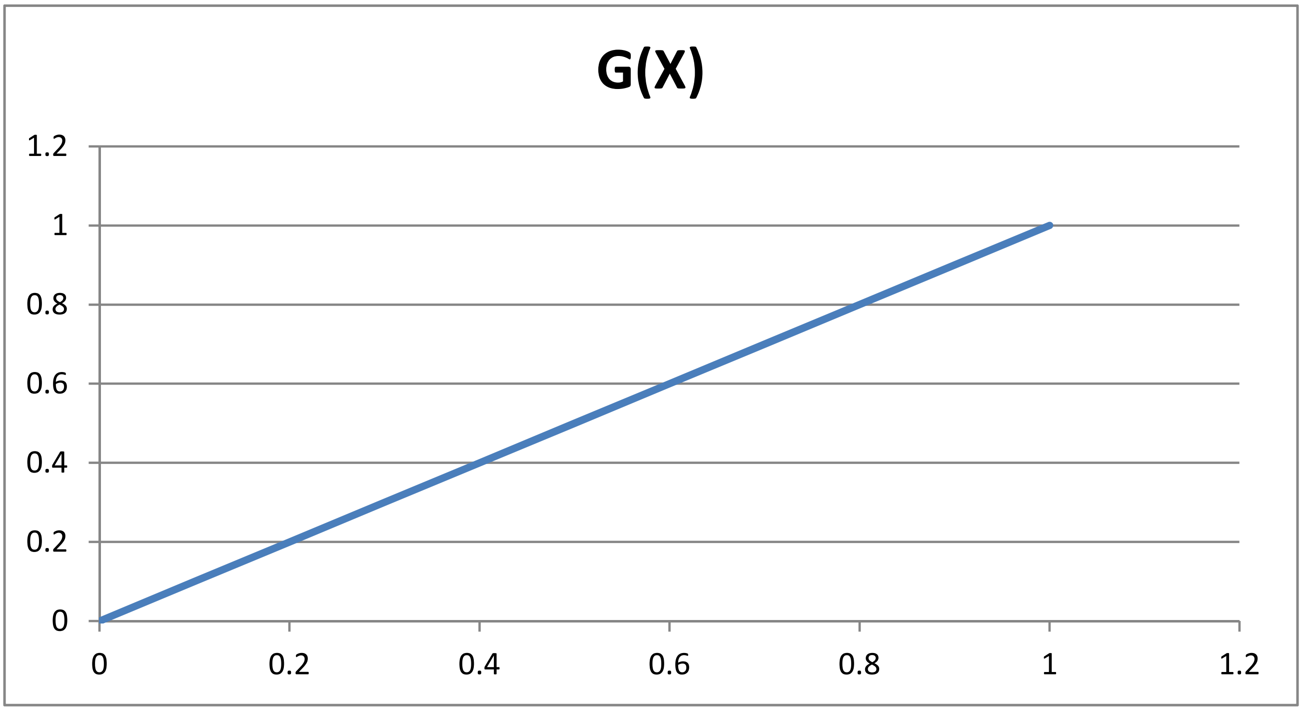
- values of the utility functions
- values of the trade-off between criteria.

How to find values of utilities ?

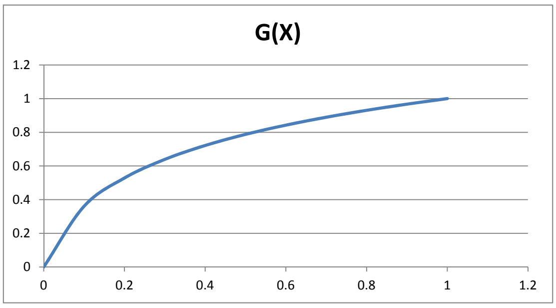
Just ask ! Questions are like :

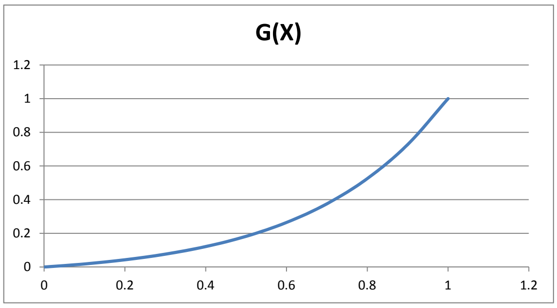
- How much should you pay to have the value 5 better than 4 on the criterion *i* ?
- Do you prefer to have 3 on criterion *i* or (0 or 5) with probability $p = 0.5$?

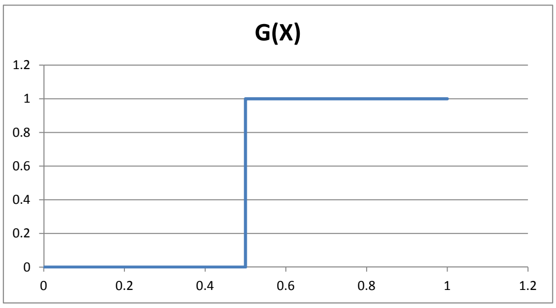
UTILITIES CAN BE LIKE...

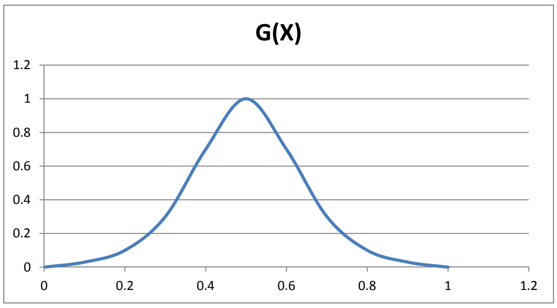


UTILITIES CAN BE LIKE…









HOW TO FIND VALUES OF WEIGHT ?

Just ask ! Questions are like :
"Do you prefer $(10, 6)$ or $(6, 10)$? or $(9, 7)$?"
Then solve the inequalities system

UTA METHOD [JACQUET-LAGRÈZE AND SISKOS 82]

Data :

- a set of alternatives X
- a pre-order on X

Model :

- $u(x) = \sum_i \omega_i u_i(x_i)$
- with $\sum_i \omega_i = 1$ and $0 \leq u_i(x_i) \leq 1$ with u_i non decreasing function
- $v(x) = u(x) + \delta(x)$

UTA METHOD (2)

Linear program :

$$Min \sum_{x \in X} \delta(x)$$

with the constraints :

$$\begin{cases} \sum_i \omega_i = 1 \\ 0 \leq u_i(x_i) \leq 1 \\ v(x) \geq v(y) + \epsilon \iff x \succ y \\ v(x) = v(y) \iff x \sim y \end{cases}$$

In order to conclude...

UTILITY FUNCTIONS

ADVANTAGES

- transitivity of the preference relation
- elicitation protocols exist and are (almost) efficient

INCONVENIENCES

- model hard to understand, parameters sometimes no coherent
- need of a huge amount of information (alternatives and criteria...)

UTILITY FUNCTIONS

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