Elicitation of a 2-additive bi-capacity through cardinal information on trinary actions

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Abstract. In the context of MultiCriteria Decision Aid, we present new properties of a 2-additive bi-capacity by using a bipolar Möbius transform. We use these properties in the identification of a 2-additive bi-capacity when we represent a cardinal information by a Choquet integral with respect to a 2-additive bi-capacity.

Keywords: MCDA, Preference modeling, bi-capacity, Choquet integral

1 Introduction

Multi-criteria decision analysis aims at representing the preferences of a decision maker (DM) over options. One possible model is the transitive decomposable one where an overall utility is determined for each option. The Choquet integral has been proved to be a versatile aggregation function to construct overall scores [3, 9, 10] and is based on the notion of the capacity or fuzzy measure. The definition of Choquet integral in this case is based on unipolar scales [13]. Because these scales are not always appropriate (See the motivating example in [13]), the bipolar general Choquet integral have been introduced [8, 12] for bipolar scales and in particular the bipolar Choquet integral w.r.t. a 2-additive capacity. Grabisch and Labreuche studied in [12] the expressions of a 2-additive bi-capacity according to their definition of 2-additivity via a Möbius transform. But, the identification of a 2-additive bi-capacities is not yet studied in the literature in details.

In this paper, we studied in details properties of a 2-additive bi-capacity by using the bipolar Möbius transform defined by Fujimoto et al. [8]. Hence we obtain here some simple expressions of monotonicity conditions of a 2-additive bi-capacity. We propose also an identification or elicitation of a 2-additive bi-capacity by asking to DM to express his preferences over a set of alternatives called here a set of trinary actions. A trinary action is an (fictitious) alternative representing a prototypical situation where on a given subset of at most two criteria, the attributes reach a satisfactory level **1** or unsatisfactory level $-\mathbf{1}$, while on the remaining ones, they are at a neutral level (neither satisfactory nor unsatisfactory) **0**.

After some basic notions and properties of a 2-additive capacity given in the next section, we present in Section 3 how to identify a 2-additive capacity by using a linear program.

2 Basic concepts

Let us denote by $N = \{1, \ldots, n\}$ a finite set of n criteria and $X = X_1 \times \cdots \times X_n$ the set of actions (also called alternatives or options), where $X_1, \ldots, X_n$ represent the point of view or attributes. For all $i \in N$, the function $u_i : X_i \to \mathbb{R}$ is called a utility function. Given an element $x = (x_1, \ldots, x_n)$, we set $U(x) = (u_1(x_1), \ldots, u_n(x_n))$. For a subset A of N and actions x and y, the notation $z = (x_A, y_{N-A})$ means that z is defined by $z_i = x_i$ if $i \in A$, and $z_i = y_i$ otherwise. We will often write ij, ijk instead of $\{i, j\}$ and $\{i, j, k\}$ respectively.

2.1 2-additive bi-capacities

Let us denote by $2^N := \{S \subseteq N\}$ the set of subsets of N and $3^N := \{(A,B) \in 2^N \times 2^N | A \cap B = \emptyset\}$ the set of couples of subsets of N with an empty intersection. We define on 3^N the following relation $\sqsubseteq$: for all (A_1, A_2), $(B_1, B_2) \in 3^N$

$$(A_1, A_2) \sqsubseteq (B_1, B_2) \Leftrightarrow [A_1 \subseteq B_1 \text{ and } B_2 \subseteq A_2]. \tag{1}$$

Definition 1 (Bi-capacity [12, 13]) *A function $\nu : 3^N \to \mathbb{R}$ is a bi-capacity on 3^N if it satisfies:*

1. $\nu(\emptyset, \emptyset) = 0$;
2. *For all (A_1, A_2), $(B_1, B_2) \in 3^N$,*

$$[(A_1, A_2) \sqsubseteq (B_1, B_2) \Rightarrow \nu(A_1, A_2) \leq \nu(B_1, B_2)].$$

In addition, a bi-capacity $\nu : 3^N \to \mathbb{R}$ is said to be

- normalized if
$$\nu(N, \emptyset) = 1 \text{ and } \nu(\emptyset, N) = -1;$$
- additive if for all $(A_1, A_2) \in 3^N$,
$$\nu(A_1, A_2) = \sum_{i \in A_1} \nu(i, \emptyset) + \sum_{j \in A_2} \nu(\emptyset, j).$$

Definition 2 (Möbius transform of a bi-capacity [11]) *Let ν a bi-capacity on 3^N. A Möbius transform of ν is a set function $m^\nu : 3^N \to \mathbb{R}$ such that for all $(A_1, A_2) \in 3^N$*

$$m^\nu(A_1, A_2) := \sum_{\substack{B_1 \subseteq A_1 \\ A_2 \subseteq B_2 \subseteq A_1{}^c}} (-1)^{|A_1 \setminus B_1| + |B_2 \setminus A_2|} \nu(B_1, B_2). \tag{2}$$

When m^ν is given, it is possible to recover the original ν by the following expression:

$$\nu(A_1, A_2) := \sum_{(B_1, B_2) \sqsubseteq (A_1, A_2)} m^\nu(B_1, B_2), \; \forall (A_1, A_2) \in 3^N \tag{3}$$

Fujimoto [6, 8] has proposed another equivalent definition of a Möbius transform of a bi-capacities as follows:

Definition 3 (Bipolar Möbius transform of a bi-capacity) *Let ν a bi-capacity on 3^N. The (bipolar) Möbius transform of ν is a set function $b^\nu : 3^N \to \mathbb{R}$ defined by*

$$\begin{aligned} b^\nu(A_1, A_2) &:= \sum_{\substack{B_1 \subseteq A_1 \\ B_2 \subseteq A_2}} (-1)^{|A_1 \setminus B_1| + |A_2 \setminus B_2|} \nu(B_1, B_2) \\ &= \sum_{(\emptyset, A_2) \sqsubseteq (B_1, B_2) \sqsubseteq (A_1, \emptyset)} (-1)^{|A_1 \setminus B_1| + |A_2 \setminus B_2|} \nu(B_1, B_2) \end{aligned} \tag{4}$$

$\forall (A_1, A_2) \in 3^N$.

Conversely, for any $(A_1, A_2) \in 3^N$, it holds that

$$\nu(A_1, A_2) := \sum_{\substack{B_1 \subseteq A_1 \\ B_2 \subseteq A_2}} b^\nu(B_1, B_2). \tag{5}$$

There is a link, given by Fujimoto and Murofushi [7],between these two definitions of a Möbius transform of a bi-capacity:

Proposition 1 *Let ν be a bi-capacity on 3^N, m^ν the Möbius transform of ν, and b^ν the bipolar Möbius transform of ν. Then, it holds, for any $(A_1, A_2) \in 3^N$, that*

$$m^\nu(A_1, A_2) = (-1)^{|{A_1}^c \setminus A_2|} \sum_{{A_1}^c \setminus A_2 \subseteq C_2 \subseteq {A_1}^c} b^\nu(A_1, C_2) \tag{6}$$

and

$$b^\nu(A_1, A_2) = (-1)^{|A_2|} \sum_{C_2 \subseteq {A_1}^c \setminus A_2} m^\nu(A_1, C_2) \tag{7}$$

Proof. See [7].

If there is no confusion, we will use the notation m for m^ν and b for b^ν.

Bi-capacities on 3^N generally require $3^n - 1$ parameters. In order to reduce this number, Grabisch and Labreuche [11–13] proposed the notion of k-additivity of bi-capacity as follows:

Definition 4 *Given a positive integer $k < n$, a bi-capacity ν is said to be k-additive iff*

1. *$m^\nu(A_1, A_2) = 0$ whenever $|{A_2}^c| > k$;*
2. *There exists $(A_1, A_2) \in 3^N$ such that $|{A_2}^c| = k$ and $m^\nu(A_1, A_2) \neq 0$*

An alternative and equivalent concept of k-additivity is proposed by Fujimoto et al. [8] by using bipolar Möbius transform.

Proposition 2 *Given a positive integer $k < n$, a bi-capacity ν is k-additive iff*

1. *$b^\nu(A_1, A_2) = 0$ whenever $|A_1 \cup A_2| > k$;*
2. *There exists $(A_1, A_2) \in 3^N$ such that $|A_1 \cup A_2| = k$ and $b^\nu(A_1, A_2) \neq 0$*

Proof. See [8].

To avoid a heavy notation, for a bi-capacity ν, its Möbius transform m and its bipolar Möbius transform b, we use the following shorthand for all $i, j \in N$, $i \neq j$:

- $\nu_{i|} := \nu(i, \emptyset)$, $\nu_{|j} := \nu(\emptyset, j)$, $\nu_{i|j} := \nu(i, j)$, $\nu_{ij|} := \nu(ij, \emptyset)$, $\nu_{|ij} := \nu(\emptyset, ij)$,
- $m_{i|} := m(i, \emptyset)$, $m_{|j} := m(\emptyset, j)$, $m_{i|j} := m(i, j)$, $m_{ij|} := m(ij, \emptyset)$, $m_{|ij} := m(\emptyset, ij)$,
- $b_{i|} := b(i, \emptyset)$, $b_{|j} := b(\emptyset, j)$, $b_{i|j} := b(i, j)$, $b_{ij|} := b(ij, \emptyset)$, $b_{|ij} := b(\emptyset, ij)$.

Whenever we use i and j together, it always means that they are different.

Using the above definitions, we propose the following properties of a 2-additive bi-capacity ν and its bipolar Möbius transform b:

Proposition 3 *1. Let ν be a 2-additive bi-capacity and b its bipolar Möbius transform. For any $(A_1, A_2) \in 3^N$ we have:*

$$\nu(A_1, A_2) = \sum_{i \in A_1} b_{i|} + \sum_{j \in A_2} b_{|j} + \sum_{\substack{i \in A_1 \\ j \in A_2}} b_{i|j} + \sum_{\{i,j\} \subseteq A_1} b_{ij|} + \sum_{\{i,j\} \subseteq A_2} b_{|ij} \tag{8}$$

2. If the coefficients $b_{i|}$, $b_{|j}$, $b_{i|j}$, $b_{ij|}$, $b_{|ij}$ are given for all $i, j \in N$, then the necessary and sufficient conditions to get a 2-additive bi-capacity generated by (8) are: for any $(A, B) \in 3^N$ and $k \in A$,

$$b_{k|} + \sum_{j \in B} b_{k|j} + \sum_{i \in A \setminus k} b_{ik|} \geq 0 \tag{9}$$

$$b_{|k} + \sum_{j \in B} b_{j|k} + \sum_{i \in A \setminus k} b_{|ik} \leq 0 \tag{10}$$

3. The inequalities (9) *and* (10) *can be rewritten in terms of bi-capacity* ν *as follows: for any* $(A, B) \in 3^N$ *and* $k \in A$*, such that* $|B| + |A| \geq 2$*,*

$$\sum_{j \in B} \nu_{k|j} + \sum_{i \in A \setminus k} \nu_{ik|} \geq (|B| + |A| - 2)\nu_{k|} + \sum_{j \in B} \nu_{|j} + \sum_{i \in A \setminus k} \nu_{i|} \quad (11)$$

$$\sum_{j \in B} \nu_{j|k} + \sum_{i \in A \setminus k} \nu_{|ik} \leq (|B| + |A| - 2)\nu_{|k} + \sum_{j \in B} \nu_{j|} + \sum_{i \in A \setminus k} \nu_{|i} \quad (12)$$

Proof. 1. Because ν is 2-additive, the proof of the equation (8) is given by using the relation (5) between ν and b.
2. The proof of the second point of the proof is based on the expression of $\nu(A_1, A_2)$ given in (8) and on these equivalent monotonicity properties (which are easy to check): $\forall (A, B) \in 3^N$ and $\forall A \subseteq A'$,
   (i) $\nu(A, B) \leq \nu(A', B) \Leftrightarrow \nu(A \setminus k, B) \leq \nu(A, B)\ \forall k \in A$;
   (ii) $\nu(B, A') \leq \nu(B, A) \Leftrightarrow \nu(B, A) \leq \nu(B, A \setminus k)\ \forall k \in A$.
3. The inequalities (11) and (12) are obtained by using the relation (5) between ν and b.

Hence, Proposition 3 shows that the computation of a 2-additive bi-capacity ν can be done by knowing only the values of ν on the elements $(i, \emptyset)$, $(\emptyset, i)$, (i, j), $(ij, \emptyset)$, $(\emptyset, ij)$ for all $i, j \in N$ such that the inequalities (11) and (12), which correspond to the 2-additive monotonicity of a bi-capacity, are satisfied. In addition, we can add these normalized conditions:

$$\nu_{N|} = \sum_{i \in N} b_{i|} + \sum_{\{i,j\} \subseteq N} b_{ij|} = 1 \quad (13)$$
$$\nu_{|N} = \sum_{i \in N} b_{|i} + \sum_{\{i,j\} \subseteq N} b_{|ij} = -1$$

2.2 Choquet integral w.r.t. a 2-additive bi-capacity

Definition 5 (Grabisch and Labreuche [12]) *Let* ν *be a bi-capacity on* 3^N *and* $x = (x_1, \dots, x_n) \in \mathbb{R}^n$*. The expression of Choquet of* x *w.r.t.* ν *is given by*

$$\mathcal{C}_\nu(x) := \sum_{i=1}^{n} |x_{\sigma(i)}| \Big[\nu(N_{\sigma(i)} \cap N^+, N_{\sigma(i)} \cap N^-) - \nu(N_{\sigma(i+1)} \cap N^+, N_{\sigma(i+1)} \cap N^-) \Big], \quad (14)$$

where

- $N^+ = \{i \in N | x_i \geq 0\}$ *and* $N^- = N \setminus N^+$*;*
- $N_{\sigma(i)} := \{\sigma(i), \dots, \sigma(n)\}$ *and* σ *is a permutation on* N *such that* $|x_{\sigma(i)}| \leq |x_{\sigma(i+1)}| \leq \dots \leq |x_{\sigma(n)}|$.

We have also the following equivalent expression of Choquet integral w.r.t. ν, given by Fujimoto and Murofushi [7]:

$$\mathcal{C}_\nu(x) = \sum_{(A_1, A_2) \in 3^N} b(A_1, A_2) \Big(\bigwedge_{i \in A_1} x_i^+ \ \wedge \bigwedge_{j \in A_2} x_j^- \Big) \quad (15)$$

where $\begin{cases} x_i^+ = x_i & \text{if } x_i > 0 \\ x_i^+ = 0 & \text{if } x_i \leq 0 \end{cases}$ and $\begin{cases} x_i^- = x_i & \text{if } x_i < 0 \\ x_i^- = 0 & \text{if } x_i \geq 0 \end{cases}$

Therefore the Choquet integral of x w.r.t. a 2-additive bi-capacity ν is given by:

$$\mathcal{C}_\nu(x) = \sum_{i=1}^{n} b_{i|}\ x_i^+ + \sum_{i=1}^{n} b_{|i}\ x_i^- + \sum_{\{i,j\} \subseteq N} b_{i|j}\ (x_i^+ \wedge x_j^-) \quad (16)$$
$$+ \sum_{\{i,j\} \subseteq N} b_{ij|}\ (x_i^+ \wedge x_j^+) + \sum_{\{i,j\} \subseteq N} b_{|ij}\ (x_i^- \wedge x_j^-)$$

3 Elicitation of a 2-additive bi-capacity

3.1 The set of trinary actions and relations

We assume that the DM is able to identify for each criterion i three reference levels:

1. A reference level $\mathbf{1}_i$ in X_i which he considers as good and completely satisfying if he could obtain it on criterion i, even if more attractive elements could exist. This special element corresponds to the *satisficing level* in the theory of bounded rationality of Simon [16].
2. A reference level $\mathbf{0}_i$ in X_i which he considers neutral on i. The neutral level is the absence of attractiveness and repulsiveness. The existence of this neutral level has roots in psychology [17], and is used in bipolar models [18].
3. A reference level $-\mathbf{1}_i$ in X_i which he considers completely unsatisfying.

We set for convenience $u_i(\mathbf{1}_i) = 1$, $u_i(\mathbf{0}_i) = 0$ and $u_i(-\mathbf{1}_i) = -1$.

We call a *trinary action or trinary alternative*, an element of the set

$$\mathcal{T} = \{(\mathbf{1}_\emptyset, -\mathbf{1}_\emptyset),\ (\mathbf{1}_i, -\mathbf{1}_\emptyset),\ (\mathbf{1}_\emptyset, -\mathbf{1}_j),\ (\mathbf{1}_i, -\mathbf{1}_j),\ (\mathbf{1}_{ij}, -\mathbf{1}_\emptyset),\ (\mathbf{1}_\emptyset, -\mathbf{1}_{ij}),\ i,j \in N\} \subseteq X$$

where

- $(\mathbf{1}_\emptyset, -\mathbf{1}_\emptyset) =: a_{0|0}$ is an action considered neutral on all criteria.
- $(\mathbf{1}_i, -\mathbf{1}_\emptyset) =: a_{i|}$ is an action considered satisfactory on criterion i and neutral on the other criteria.
- $(\mathbf{1}_\emptyset, -\mathbf{1}_j) =: a_{|j}$ is an action considered unsatisfactory on criterion j and neutral on the other criteria.
- $(\mathbf{1}_i, -\mathbf{1}_j) =: a_{i|j}$ is an action considered satisfactory on criteria i, unsatisfactory on j and neutral on the other criteria.
- $(\mathbf{1}_{ij}, -\mathbf{1}_\emptyset) := a_{ij|}$ is an action considered satisfactory on criteria i and j and neutral on the other criteria.
- $(\mathbf{1}_\emptyset, -\mathbf{1}_{ij}) := a_{|ij}$ is an action considered unsatisfactory on criteria i and j and neutral on the other criteria.

The number of binary actions is $1+n+\dfrac{n\times(n-1)}{2} = 1+\dfrac{n\times(n+1)}{2}$. On the other hand, the number of trinary actions is: $1+2\times n+\dfrac{2\times n\times(n-1)}{2} = 1+2\times n^2$. Roughly speaking there are 4 times as much trinary actions for 2 additive bi-capacities compare to the 2 additive capacities.

Using the expression (16) of the Choquet integral w.r.t. a 2-additive bi-capacity ν, we get the following consequences:
$C_\nu(U(a_{0|0})) = 0$, $C_\nu(U(a_{i|})) = \nu_{i|}$,
$C_\nu(U(a_{|j})) = \nu_{|j}$, $C_\nu(U(a_{i|j})) = \nu_{i|j}$,
$C_\nu(U(a_{ij|})) = \nu_{ij|}$ and $C_\nu(U(a_{|ij})) = \nu_{|ij}$.

To entirely determine the 2-additive bi-capacity, as shown by Proposition 3, it should be sufficient to get some preferential information from the DM only on trinary actions. We assume that, given two trinary actions x and y the DM is able to judge the difference of attractiveness between x and y when he strictly prefers x to y. Like in MACBETH [2, 4], 2-additive MACBETH [15] and GRIP [5], MCDA methodologies, the difference of attractiveness[3] will be provided under the form of semantic categories d_s, $s = 1, \ldots, q$ defined so that, if $s < t$, any difference of attractiveness in the class d_s is smaller than any difference of attractiveness in the class d_t. If there is no ambiguity, a category d_s will be simply designated by s.

Under these hypotheses, the preferences given by the DM is expressed by the following relations:

[3] MACBETH approach uses the following six semantic categories: d_1 = very weak, d_2 = weak, d_3 = moderate, d_4 = strong, d_5 = very strong, d_6 = extreme.

- $P = \{(x, y) \in \mathcal{T} \times \mathcal{T} :$ the DM strictly prefers x to $y\}$,
- $I = \{(x, y) \in \mathcal{T} \times \mathcal{T} :$ the DM is indifferent between x and $y\}$,
- For the semantic categories "d_s", "d_t", $s, t \in \{1, ..., q\}$, $s \leq t$,
  $P_{st} = \{(x, y) \in P$ such that the DM judges the difference of attractiveness between x and y as belonging from the class "d_s" to the class "d_t" $\}$. When $s < t$, P_{st} expresses some hesitation.

We will suppose always P nonempty ("non-trivial axiom") and use the notation $\mathbb{N}_{st} = \{s, s+1, \ldots, t-1, t\}$ for $s \leq t$.

Remark 1. In this paper, the relation $P \cup I$ is not necessarily complete.

Definition 6 *1. The ordinal information on $\mathcal{T}$ is the structure $\{P, I\}$.*
2. The cardinal information on $\mathcal{T}$ is the structure $\{P, I, \{P_{st}\}_{s \leq t}\}$.

3.2 The representation and the linear program to solve

A cardinal information $\{P, I, \{P_{st}\}_{s \leq t}\}$ is said to be *representable by a Choquet integral w.r.t. a 2-additive bi-capacity* $\nu : 3^N \rightarrow \mathbb{R}$ if the following conditions are satisfied: $\forall x, y, z, w \in \mathcal{T}$, $\forall s, t, u, v \in \{1, \ldots, q\}$ such that $u \leq v < s \leq t$,

$$x \; I \; y \Rightarrow C_\nu(U(x)) = C_\nu(U(y)), \tag{17}$$

$$x \; P \; y \Rightarrow C_\nu(U(x)) > C_\nu(U(y)), \tag{18}$$

$$\left.\begin{array}{r}(x, y) \in P_{st} \\ (z, w) \in P_{uv}\end{array}\right\} \Rightarrow C_\nu(U(x)) - C_\nu(U(y)) > C_\nu(U(z)) - C_\nu(U(w)) \tag{19}$$

De Corte [2] proved that the previous conditions are equivalent to the existence of q thresholds $\sigma_1, \ldots, \sigma_q$ such that:

$$\forall (x, y) \in I : C_\nu(U(x)) = C_\nu(U(y)), \tag{20}$$

$$\forall s, t \in \mathbb{N}_{1q}, s \leq t, \forall (x, y) \in P_{st} : \sigma_s < C_\nu(U(x)) - C_\nu(U(y)), \tag{21}$$

$$\forall s, t \in \mathbb{N}_{1(q-1)}, s \leq t, \forall (x, y) \in P_{st} : C_\nu(U(x)) - C_\nu(U(y)) < \sigma_{t+1}, \tag{22}$$

$$0 < \sigma_1 < \sigma_2 < \cdots < \sigma_q \tag{23}$$

In order to identify a 2-additive bi-capacity ν such that a cardinal information $\{P, I, \{P_{st}\}_{s \leq t}\}$ on $\mathcal{T}$ is representable by C_ν, we use the following linear program (PL) where the variables to determine are $\nu(i, \emptyset)$, $\nu(\emptyset, i)$, $\nu(i, j)$, $\nu(ij, \emptyset)$ and $\nu(\emptyset, ij)$ for all $i, j \in N$:

$$\min C_\nu(U(x_0))$$

$$C_\nu(U(x)) = C_\nu(U(y)), \forall (x, y) \in I \tag{24}$$

$$\sigma_i + d_{\min} \leq C_\nu(U(x)) - C_\nu(U(y)), \forall (x, y) \in P_{ij}, \forall i, j \in \mathbb{N}_{1q}, \; i \leq j \tag{25}$$

$$C_\nu(U(x)) - C_\nu(U(y)) \leq \sigma_{j+1} - d_{\min}, \forall (x, y) \in P_{ij}, \forall i, j \in \mathbb{N}_{1(q-1)}, i \leq j \tag{26}$$

$$d_{\min} \leq \sigma_1 \tag{27}$$

$$\sigma_{i-1} + d_{\min} \leq \sigma_i, \forall i \in \{2, \ldots, q\} \tag{28}$$

$$\forall (A, B) \in 3^N, \forall k \in A \text{ such that } (|A| + |B| - 2) \geq 0$$

$$\sum_{j \in B} \nu_{k|j} + \sum_{i \in A \setminus k} \nu_{ik|} \geq (|B| + |A| - 2)\nu_{k|} + \sum_{j \in B} \nu_{|j} + \sum_{i \in A \setminus k} \nu_{i|} \tag{29}$$

$$\sum_{j \in B} \nu_{j|k} + \sum_{i \in A \setminus k} \nu_{|ik} \leq (|B| + |A| - 2)\nu_{|k} + \sum_{j \in B} \nu_{j|} + \sum_{i \in A \setminus k} \nu_{|i} \tag{30}$$

where x_0 is an alternative of $\mathcal{T}$ arbitrarily chosen, and $d_{\min}$ an arbitrary strictly positive constant. The variables σ_i in this linear program are thresholds of categories.

Example 1 *$N = \{1,2,3\}$, $q = 6$,*

$3^N = \{(\emptyset,\emptyset), (\emptyset,N), (N,\emptyset), (1,\emptyset), (2,\emptyset), (3,\emptyset), (\emptyset,1), (\emptyset,2), (\emptyset,3), (1,2), (1,3), (2,3), (2,1), (3,1), (3,2), (12,\emptyset), (23,\emptyset), (13,\emptyset), (\emptyset,12), (\emptyset,23), (\emptyset,13), (12,3), (23,1), (3,12), (2,13), (1,23), (13,2)\}$

$\mathcal{T} = \{a_{0,0}; a_{1|}; a_{2|}; a_{3|}; a_{|1}; a_{|2}; a_{|3}; a_{1|2}; a_{2|1}; a_{1|3}; a_{3|1}; a_{3|2}; a_{2|3}; a_{13|}; a_{12|}; a_{23|}; a_{|12}; a_{|23}; a_{|13}\}$

Let us suppose that the DM gives the following preferences: $I = \{(a_{1|3}; a_{3|1}); (a_{2|}, a_{|3})\}$; $P_3 = \{(a_{23|}, a_{|1})\}$; $P_{24} = \{(a_{1|}, a_{|3})\}$;

To look for a 2-additive capacity such that the cardinal information $\{I, P_3, P_{24}\}$ is representable by C_μ, we solve the following linear program:

$$\begin{aligned}
&\min C_\nu(U(a_{0,0}))\\
&C_\nu(U(a_{1|3})) - C_\nu(U(a_{3|1})) = 0 \Leftrightarrow \nu(1,3) - \nu(3,1) = 0\\
&C_\nu(U(a_{2|})) - C_\nu(U(a_{|3})) = 0 \Leftrightarrow \nu(2,\emptyset) - \nu(\emptyset,3) = 0\\
&\sigma_3 + 0.01 \leq C_\nu(U(a_{23|})) - C_\nu(U(a_{|1})) \Leftrightarrow \sigma_3 + 0.01 \leq \nu(23,\emptyset) - \nu(\emptyset,1)\\
&C_\nu(U(a_{23|})) - C_\nu(U(a_{|1})) \leq \sigma_4 - 0.01 \Leftrightarrow \nu(23,\emptyset) - \nu(\emptyset,1) \leq \sigma_4 - 0.01\\
&\sigma_2 + 0.01 \leq C_\nu(U(a_{1|})) - C_\nu(U(a_{|3})) \Leftrightarrow \sigma_2 + 0.01 \leq \nu(1,\emptyset) - \nu(\emptyset,3)\\
&C_\nu(U(a_{1|})) - C_\nu(U(a_{|3})) \leq \sigma_5 - 0.01 \Leftrightarrow \nu(1,\emptyset) - \nu(\emptyset,3) \leq \sigma_5 - 0.01\\
&0.01 \leq \sigma_1; \sigma_{i-1} + 0.01 \leq \sigma_i, \forall i \in \{2,\dots,6\}\\
&\nu(1,2) \geq \nu(\emptyset,2); \nu(1,3) \geq \nu(\emptyset,3);\\
&\nu(2,3) \geq \nu(\emptyset,3); \nu(2,1) \geq \nu(\emptyset,1); \nu(3,2) \geq \nu(\emptyset,2); \nu(3,1) \geq \nu(\emptyset,1);\\
&\nu(1,2) \leq \nu(1,\emptyset); \nu(2,1) \leq \nu(2,\emptyset); \nu(1,3) \leq \nu(1,\emptyset); \nu(3,1) \leq \nu(3,\emptyset);\\
&\nu(3,2) \leq \nu(3,\emptyset); \nu(2,3) \leq \nu(2,\emptyset); \nu(\emptyset,12) \leq \nu(\emptyset,1); \nu(\emptyset,12) \leq \nu(\emptyset,2);\\
&\nu(\emptyset,13) \leq \nu(\emptyset,1); \nu(\emptyset,13) \leq \nu(\emptyset,3); \nu(\emptyset,23) \leq \nu(\emptyset,2); \nu(\emptyset,23) \leq \nu(\emptyset,3);\\
&\nu(12,\emptyset) \geq \nu(1,\emptyset); \nu(12,\emptyset) \geq \nu(2,\emptyset); \nu(13,\emptyset) \geq \nu(1,\emptyset); \nu(13,\emptyset) \geq \nu(3,\emptyset);\\
&\nu(23,\emptyset) \geq \nu(2,\emptyset); \nu(23,\emptyset) \geq \nu(3,\emptyset); \nu(\emptyset,1) + \nu(\emptyset,2) \leq 0; \nu(1,\emptyset) + \nu(2,\emptyset) \geq 0;\\
&\nu(\emptyset,1) + \nu(\emptyset,3) \leq 0; \nu(1,\emptyset) + \nu(3,\emptyset) \geq 0; \nu(\emptyset,2) + \nu(\emptyset,3) \leq 0; \nu(2,\emptyset) + \nu(3,\emptyset) \geq 0;\\
&\nu(1,2) + \nu(1,3) \geq \nu(1,\emptyset) + \nu(\emptyset,2) + \nu(\emptyset,3); \nu(2,1) + \nu(2,3) \geq \nu(2,\emptyset) + \nu(\emptyset,1) + \nu(\emptyset,3);\\
&\nu(3,1) + \nu(3,2) \geq \nu(3,\emptyset) + \nu(\emptyset,1) + \nu(\emptyset,2); \nu(2,1) + \nu(2,3) \geq \nu(2,\emptyset) + \nu(\emptyset,1) + \nu(\emptyset,3);\\
&\nu(2,1) + \nu(3,1) \leq \nu(\emptyset,1) + \nu(2,\emptyset) + \nu(3,\emptyset); \nu(1,2) + \nu(3,2) \leq \nu(\emptyset,2) + \nu(1,\emptyset) + \nu(3,\emptyset);\\
&\nu(2,3) + \nu(1,3) \leq \nu(\emptyset,3) + \nu(1,\emptyset) + \nu(2,\emptyset); \nu(2,3) + \nu(1,3) \leq \nu(\emptyset,3) + \nu(1,\emptyset) + \nu(2,\emptyset);\\
&\nu(1,3) + \nu(12,\emptyset) \geq \nu(1,\emptyset) + \nu(\emptyset,3) + \nu(2,\emptyset); \nu(2,3) + \nu(12,\emptyset) \geq \nu(2,\emptyset) + \nu(\emptyset,3) + \nu(1,\emptyset);\\
&\nu(1,2) + \nu(13,\emptyset) \geq \nu(1,\emptyset) + \nu(\emptyset,2) + \nu(3,\emptyset); \nu(3,2) + \nu(13,\emptyset) \geq \nu(3,\emptyset) + \nu(\emptyset,2) + \nu(1,\emptyset);\\
&\nu(2,1) + \nu(23,\emptyset) \geq \nu(2,\emptyset) + \nu(\emptyset,1) + \nu(3,\emptyset); \nu(3,1) + \nu(23,\emptyset) \geq \nu(3,\emptyset) + \nu(\emptyset,1) + \nu(2,\emptyset);\\
&\nu(3,1) + \nu(\emptyset,12) \leq \nu(\emptyset,1) + \nu(3,\emptyset) + \nu(\emptyset,2); \nu(3,2) + \nu(\emptyset,12) \leq \nu(\emptyset,2) + \nu(3,\emptyset) + \nu(\emptyset,1);\\
&\nu(2,1) + \nu(\emptyset,13) \leq \nu(\emptyset,1) + \nu(2,\emptyset) + \nu(\emptyset,3); \nu(2,3) + \nu(\emptyset,13) \leq \nu(\emptyset,3) + \nu(2,\emptyset) + \nu(\emptyset,1);\\
&\nu(1,2) + \nu(\emptyset,23) \leq \nu(\emptyset,2) + \nu(1,\emptyset) + \nu(\emptyset,3); \nu(1,3) + \nu(\emptyset,23) \leq \nu(\emptyset,3) + \nu(1,\emptyset) + \nu(\emptyset,2);
\end{aligned}$$

If (PL) is feasible, then a 2-additive bi-capacity is computed by using the equation (8) in Proposition 3. Of course it can exist several bi-capacities compatible with the cardinal information provided by the DM. To compute a robust bi-capacity, one can use the approach of Angilella et al. [1] based on the concept of possible and necessary relation. To deal with inconsistencies when (PL) is infeasible, we can use the interactive algorithm of Mayag et al. [14] which generates recommendations for the DM to retrieve consistent cardinal information. This simple and intuitive algorithm identifies the minimal number of constraints we can relax in this infeasible linear problem. In order to retrieve consistent information, the preferences associated to these constraints are relaxed by an augmentation or diminution of categories.

Because the identification of a 2-additive bi-capacity is an interesting problem to investigate, we will look for in the future works necessary and sufficient conditions such that a cardinal information is representable by a Choquet integral w.r.t. a 2-additive bi-capacity.

References

1. S. Angilella, S. Greco, and B. Matarazzo. Non-additive robust ordinal regression: A multiple criteria decision model based on the Choquet integral. *European Journal of Operational Research*, 41(1):277–288, 2009.
2. C. A. Bana e Costa, J.-M. De Corte, and J.-C. Vansnick. On the mathematical foundations of MACBETH. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 409–437. Springer, 2005.
3. G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953.
4. J.M. De Corte. *Un logiciel d'Exploitation d'Information Préférentielles pour l'Aide à la Décision. Bases Mathématiques et Algorithmiques.* PhD thesis, University of Mons-Hainaut, Mons, 2002.
5. J. R. Figueira, S. Greco, and R. Slowinski. Building a set of additive value functions representing a reference preorder and intensities of preference : Grip method. *European Journal of Operational Research*, 195(2):460–486, 2009.
6. K. Fujimoto. New characterizations of k-additivity and k-monotonicity of bi-capacities. In *SCIS-ISIS 2004, 2nd Int. Conf. on Soft Computing and Intelligent Systems and 5th Int. Symp. on Advanced Intelligent Systems*, Yokohama, Japan, September 2004.
7. K. Fujimoto and T. Murofushi. Some characterizations of k-monotonicity through the bipolar möbius transform in bi-capacities. *J. of Advanced Computational Intelligence and Intelligent Informatics*, 9(5):484–495, 2005.
8. K. Fujimoto, T. Murofushi, and M. Sugeno. k-additivity and $\mathcal{C}$-decomposability of bi-capacities and its integral. *Fuzzy sets and Systems*, 158:1698–1712, 2007.
9. M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.
10. M. Grabisch and C. Labreuche. Fuzzy measures and integrals in MCDA. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 565–608. Springer, 2005.
11. M. Grabisch and Ch. Labreuche. Bi-capacities. Part I: definition, Möbius transform and interaction. *Fuzzy Sets and Systems*, 151:211–236, 2005.
12. M. Grabisch and Ch. Labreuche. Bi-capacities. Part II: the Choquet integral. *Fuzzy Sets and Systems*, 151:237–259, 2005.
13. M. Grabisch and Ch. Labreuche. A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid. *4OR*, 6:1–44, 2008.
14. B. Mayag, M. Grabisch, and C. Labreuche. An interactive algorithm to deal with inconsistencies in the representation of cardinal information. In Eyke Hüllermeier, Rudolf Kruse, and Frank Hoffmann, editors, *Information Processing and Management of Uncertainty in Knowledge-Based Systems*, volume 80 of *Communications in Computer and Information Science*, pages 148–157. Springer, 2010.
15. B. Mayag, M. Grabisch, and C. Labreuche. A characterization of the 2-additive Choquet integral through cardinal information. *Fuzzy Sets and Systems*, 184(1):84–105, 2011.
16. H. Simon. Rational choice and the structure of the environment. *Psychological Review*, 63(2):129–138, 1956.
17. P. Slovic, M. Finucane, E. Peters, and D. G. MacGregor. The affect heuristic. In T. Gilovitch, D. Griffin, and D. Kahneman, editors, *Heuristics and biases: the psychology of intuitive judgment*, pages 397–420. Cambridge University Press, 2002.
18. A. Tversky and D. Kahneman. Advances in prospect theory: cumulative representation of uncertainty. *J. of Risk and Uncertainty*, 5:297–323, 1992.