A characterization of the 2-additive Choquet integral

Brice Mayag
Thales R & T
University of Paris I
brice.mayag@thalesgroup.com

Michel Grabisch
University of Paris I
Centre d'Economie de la Sorbonne
106-112 Boulevard de l'Hôpital
75013 Paris, France
michel.grabisch@univ-paris1.fr

Christophe Labreuche
Thales R & T
RD 128
F-91767 Palaiseau cedex
France
christophe.labreuche@thalesgroup.com

Abstract

We present the necessary and sufficient conditions on ordinal preferential information provided by the decision maker for the existence of a 2-additive capacity such that the Choquet integral w.r.t. this capacity represents the preference of the decision maker. These conditions help the decision-maker to understand the main reasons of the recommendations made when inconsistencies occur in his preferences on the set of binary actions.

Keywords: Choquet integral, ordinal information, 2-additive capacity, binary actions.

1 Introduction

Multi-criteria decision analysis aims at representing the preferences of a decision maker over options. One possible model is the transitive decomposable one where an overall utility is determined for each option. The Choquet integral has been proved to be a versatile aggregation function to construct overall scores [3, 7, 11] and is based on the notion of the capacity or fuzzy measure. Used as an aggregation function, the Choquet integral arises as a generalization of the weighted sum, taking into account the interaction between criteria. Its expression according to the Shapley value [18] and the interaction index [16] is possible when the capacity is 2-additive [7, 6]. The use of the Choquet integral requires beforehand the determination of the capacity.

Most of identification approaches proposed in the literature are based on the resolution of an optimization problem, see [10] for a survey of these methods. For instance the Marichal and Roubens approach [14] computes, given a set of alternatives A and a set of criteria N, a capacity μ such that the Choquet integral w.r.t. μ represents the preference over A. In this method, it is supposed that the evaluations of each alternative over each criterion are known (on a numerical and normalized scale), and possibly partial preorders $\succeq_A$, $\succeq_N$, $\succeq_P$ over A, N, and the set of pairs of criteria, respectively. We also have the sign of interaction between some pairs of criteria.

Our approach in this paper can be seen as a particular case of the Marichal and Roubens approach. Compared to their method, we assume that the decision-maker provides an ordinal information on a particular set of alternatives, the set of binary actions. An ordinal information is a preferential information represented by a strict preference relation and an indifference relation. A binary action is a fictitious alternative which takes either the neutral value $\mathbf{0}$ for all criteria, or the neutral value $\mathbf{0}$ for all criteria except for one or two criteria for which it takes the satisfactory value $\mathbf{1}$. Under these hypotheses, we present the necessary and sufficient conditions on the ordinal information for the existence of a 2-additive capacity such that the Choquet integral w.r.t. this capacity represents the preference of the decision maker. The first condition

L. Magdalena, M. Ojeda-Aciego, J.L. Verdegay (eds): Proceedings of IPMU'08, pp. 1512–1518
Torremolinos (Málaga), June 22–27, 2008

concerns the existence of strict cycles in the ordinal information and the second condition, called MOPI condition, comes from the definition of a 2-additive capacity. Therefore, we have a characterization of a 2-additive Choquet integral which permits to deal with inconsistencies in the ordinal information given by the decision maker.

The basic material on the Choquet integral, binary actions, ordinal information and the ordinal 2-additive scale are given in the next section. In Section 3, we introduce some relations on the set of binary actions through notions of graph theory, in order to provide the conditions we need to solve our problem. Finally, we present in Section 3.3 our main results of the characterization of a 2-additive Choquet integral.

2 Basic concepts

In this paper, we will use the following notations:

- $N = \{1, ..., n\}$ is a set of n criteria. The set of attributes is denoted by $Y_1, ..., Y_n$. An option or an action x is identified to an element of $Y = Y_1 \times ... \times Y_n$ with $x = (x_1, ..., x_n)$.
- 2^N or $\mathcal{P}(N)$ is the set of all subsets of N.
- $\forall A \subseteq N,\ N - A = N \setminus A$.
- $\forall i, j \in N,\ N - i = N - \{i\}$ and $N - ij = N - \{i, j\}$.
- $\forall A \subseteq N,\ z = (x_A, y_{N-A})$ means that z is defined by $z_i = x_i$ if $i \in A$, and $z_i = y_i$ otherwise.
- DM is an individual or a decision-maker.

2.1 Binary Actions and ordinal information

We suppose that the DM can identify for each criterion two reference levels $\mathbf{1}_i$ and $\mathbf{0}_i$ in Y_i corresponding respectively to levels he considers satisfactory and neutral on the criterion i. A *binary action* is an element of the set $X = \{\mathbf{0}_N,\ (\mathbf{1}_i, \mathbf{0}_{N-i}),\ (\mathbf{1}_{ij}, \mathbf{0}_{N-ij}),\ i, j \in N\} = \{a_0,\ a_i,\ a_{ij},\ i, j \in N\} \subseteq Y$ where

- $\mathbf{0}_N = (\mathbf{1}_\emptyset, \mathbf{0}_N) = a_0$ is an action considered neutral on all criteria.
- $(\mathbf{1}_i, \mathbf{0}_{N-i}) = a_i$ is an action considered satisfactory on criterion i and neutral on the other criteria.
- $(\mathbf{1}_{ij}, \mathbf{0}_{N-ij}) = a_{ij}$ is an action considered satisfactory on criteria i and j and neutral on the other criteria.

Using the pairwise comparisons we suppose also that the DM gives a preferential information on X allowing to build the following two relations:

$P = \{(x, y) \in X \times X :$ DM strictly prefers x to $y\}$

$I = \{(x, y) \in X \times X :$ DM is indifferent between x and $y\}$

Note that P is asymmetric, I is reflexive and symmetric.

Definition 2.1. The *ordinal information on* X is the structure $\{P,\ I\}$.

> **"non-triviality axiom":** In this paper, we suppose that P is nonempty for any ordinal information $\{P,\ I\}$.

The map ϕ will indicate the bijection between X and $\mathcal{P}^2(N) = \{S \subseteq N : |S| \leq 2\}$ defined by for all $S \in \mathcal{P}^2(N)$, $\phi((\mathbf{1}_S, \mathbf{0}_{N-S})) := S$. We will denote by P^N and I^N the relations on $\mathcal{P}^2(N) \times \mathcal{P}^2(N)$ defined by: $\forall S, T \in \mathcal{P}^2(N),\ S\ P^N\ T \Leftrightarrow \phi^{-1}(S)\ P\ \phi^{-1}(T)$, and $S\ I^N\ T \Leftrightarrow \phi^{-1}(S)\ I\ \phi^{-1}(T)$.

2.2 Ordinal 2-additive scale

We introduce in this section the notion of ordinal 2-additive scale. Prior to that we introduce some general concepts on the Choquet integral, which will be useful in the sequel. For more details about fuzzy integrals, see [11, 7, 8, 9].

Definition 2.2. A *capacity* (or fuzzy measure) on N is a set function $\mu : 2^N \rightarrow [0,1]$ satisfying the three properties:

1. $\mu(\emptyset) = 0$
2. $\mu(N) = 1$
3. $\forall A, B \in 2^N$, $[A \subseteq B \Rightarrow \mu(A) \leq \mu(B)]$ (monotonicity).

The definition of a capacity requires in general 2^n coefficients, which are the values of μ for all different subsets of N. When monotonicity is not satisfied, μ is called a *nonmonotonic capacity.*

Definition 2.3. Let μ be a capacity on N. The *Möbius transform* of μ is a function $m : 2^N \rightarrow \mathbb{R}$ defined by $m(T) := \sum_{K \subseteq T} (-1)^{|T \setminus K|} \mu(K)$, $\forall T \in 2^N$.

Definition 2.4. A capacity μ is said to be *k-additive* ($1 \leq k \leq n$) [7] if its Möbius transform satisfies $\forall T \in 2^N$, $m(T) = 0$ if $|T| > k$, and there exists at least a subset $B \in 2^N$ such that $|B| = k$ and $m(B) \neq 0$.

Hence a capacity μ is *2-additive* if $\forall T \in 2^N$, $m(T) = 0$ if $|T| > 2$, and $\exists B \in 2^N$, $|B| = 2$ and $m(B) \neq 0$.

Definition 2.5. Let μ be a capacity on N. The *interaction index* I of μ [7] is defined by: $\forall A \subseteq N$, $A \neq \emptyset$, $I(A) := \sum_{K \subseteq N \setminus A} \frac{(n-k-|A|)!k!}{(n-|A|+1)!} \sum_{L \subseteq A} (-1)^{|A|-|L|} \mu(K \cup L)$ with $|K| := k$.

This definition is a generalization of the Shapley [18] value and the interaction index of Murofushi and Soneda [16], the latter being defined as follows: $\forall i, j \in N$, $I_{ij} := I(\{i,j\})$.

Definition 2.6. Let μ be a capacity on N. The *Shapley index* for every $i \in N$ is defined by:

$$v_i = \sum_{K \subseteq N \setminus i} \frac{(n-k-1)!k!}{n!} (\mu(K \cup i) - \mu(K)).$$

The *Shapley value* of μ is the vector $v(\mu) = [v_1, ..., v_n]$. Clearly $v_i = I(\{i\})$, $\forall i \in N$.

Let μ be a capacity on N. We denote in this section by m its Möbius transform and I its interaction index.

Remark 1. If μ is 2-additive then we have $\forall i, j \in N$, $m(\{i\}) = \mu(\{i\})$ and $I(\{i,j\}) = m(\{i,j\}) = \mu(\{i,j\}) - \mu(\{i\}) - \mu(\{j\})$. For simplicity, we will use for a capacity μ and its Möbius transform m the following notations: $\mu_i := \mu(\{i\})$, $\mu_{ij} := \mu(\{i,j\})$, $m_i := m(\{i\})$, $m_{ij} := m(\{i,j\})$ $\forall i, j \in N$. Whenever we use i and j together, it always means that they are different.

Lemma 1.

1. *Let μ be a 2-additive capacity on N. We have $\forall K \subseteq N, |K| \geq 2$,*

$$\mu(K) = \sum_{\{i,j\} \subseteq K} \mu_{ij} - (|K|-2) \sum_{i \in K} \mu_i. \quad (1)$$

2. *If the coefficients μ_i and μ_{ij} are given for all $i, j \in N$, then the necessary and sufficient conditions that μ is a 2-additive capacity are:*

   (i)

$$\sum_{\{i,j\} \subseteq N} \mu_{ij} - (n-2) \sum_{i \in N} \mu_i = 1 \quad (2)$$

   (ii)

$$\mu_i \geq 0, \ \forall i \in N \quad (3)$$

   (iii) $\forall A \subseteq N$, $|A| \geq 2$ $\forall k \in A$

$$\sum_{i \in A \setminus \{k\}} (\mu_{ik} - \mu_i) \geq (|A|-2)\mu_k. \quad (4)$$

Proof. See [7] □

Definition 2.7. Let μ be a capacity on N and $a := (a_1, a_2, ..., a_n) \in \mathbb{R}^n_+$. The *Choquet integral* of a w.r.t μ is given by:

$$C_\mu(a) := a_{\tau(1)}\mu(N) + \sum_{i=2}^{n} (a_{\tau(i)} - a_{\tau(i-1)}) \mu(\{\tau(i), ..., \tau(n)\})$$

where τ is a permutation on N such that $a_{\tau(1)} \leq a_{\tau(2)} \leq ... \leq a_{\tau(n-1)} \leq a_{\tau(n)}$.

Remark 2.

1. If μ is a 2-additive capacity, then the Choquet integral of $a := (a_1, a_2, ..., a_n) \in \mathbb{R}^n_+$ can be written according to I as follows:

   $C_I(a) = \sum_{I_{ij}>0} (a_i \wedge a_j) I_{ij} + \sum_{I_{ij}<0} (a_i \vee a_j)|I_{ij}|$
   $+ \sum_{i=1}^{n} a_i (v_i - \frac{1}{2} \sum_{j \neq i} |I_{ij}|)$

   with the property $v_i - \frac{1}{2} \sum_{j \neq i} |I_{ij}| \geq 0$, $\forall i \in N$.

2. Chateauneuf and Jaffray have established an equivalent expression for the Choquet integral using m in [2]:

   $$C_m(a) = \sum_{K \subseteq N} m(K) \min_{i \in K}\{a_i\}.$$

   In particular, if μ is a 2-additive capacity, we have

   $$C_m(a) = \sum_{i \in N} m_i\, a_i + \sum_{\{i,j\} \subseteq N} m_{ij} \min\{a_i, a_j\}.$$

Remark 3. The Choquet integral satisfies the following property [13, 11]: if μ is a capacity then

$$C_\mu(\mathbf{1}_A, \mathbf{0}_{N-A}) = \mu(A),\ \forall A \subseteq N.$$

Hence we have:

$C_\mu(a_0) = 0$, $C_\mu(a_i) = \mu_i$,

$C_\mu(a_{ij}) = \mu_{ij}$, $\forall i, j \in N$.

Definition 2.8. The ordinal information $\{P, I\}$ is *representable by a 2-additive Choquet integral* if there exists a 2-additive capacity μ such that:

1. $\forall x, y \in X,\ x\ P\ y \Rightarrow C_\mu(x) > C_\mu(y)$
2. $\forall x, y \in X,\ x\ I\ y \Rightarrow C_\mu(x) = C_\mu(y)$.

Definition 2.9. An *ordinal 2-additive scale* on X is a 2-additive capacity on N which satisfies the two conditions of Definition 2.8.

Using Remark 3, we get the following lemma:

Lemma 2. *A 2-additive capacity μ is a representation of P^N and I^N if and only if the following conditions are satisfied:*

1. $\forall S, T \in \mathcal{P}^2(N),\ S\ P^N\ T \Rightarrow \mu(S) > \mu(T)$

2. $\forall S, T \in \mathcal{P}^2(N),\ S\ I^N\ T \Rightarrow \mu(S) = \mu(T)$.

We end this section by defining on X a new binary relation M from preferences P and I given by the DM. The relation M completes the preferential information given by the DM, with respect to the natural monotonicity relation. In this way, we call M the *relation of monotonicity on the pairs of criteria.*

Definition 2.10. Let $\{P, I\}$ be an ordinal information on X. Let $x, y \in X$, $x\ M\ y$ if one of the following two conditions is satisfied:

1. $y = a_0$ and $not(x\ (P \cup I)\ a_0)$,
2. $\exists i, j \in N$ such that $[x = a_{ij},\ y = a_i]$ and $not[x\ (P \cup I)\ y]$.

Remark 4.

- The condition on μ derived from M corresponds to (3) and (4) for the sets of pairs with $|A| = 2$.
- Let μ be an ordinal 2-additive scale on X. By the definition of the relation M, we have for all $x, y \in X$, $x\ M\ y \Rightarrow \mu(\phi(x)) \geq \mu(\phi(y))$.

3 Treatment of ordinal information

In the previous section, the binary relation M has been added to the two relations P, I of preferential information given by DM. Now the problem we have to treat is: how to build a 2-additive ordinal scale using these three relations? To solve it, we study here some inconsistencies which can occur in the analysis of the ordinal information. The first one concerns cycles of $(P \cup I \cup M)$.

3.1 Cycles inside the ordinal information

For a general binary relation $\mathcal{R}$ on X and x, y elements of X, $\{x_1, x_2, \cdots, x_p\} \subseteq X$ is a *path* of $\mathcal{R}$ from x to y if $x = x_1\ \mathcal{R}\ x_2\ \mathcal{R} \cdots \mathcal{R}\ x_{p-1}\ \mathcal{R}\ x_p = y$. A path of $\mathcal{R}$ from x to x is called a *cycle* of $\mathcal{R}$. We will distinguish between strict and nonstrict cycles.

Definition 3.1.

1. A path $\{x_1, x_2, ..., x_p\}$ of $(P \cup I \cup M)$ is said to be a *strict path* from x to y if there exists i in $\{1, ..., p-1\}$ such that $x_i\ P\ x_{i+1}$.

2. A strict path of $(P \cup I \cup M)$ from x to x is called a *strict cycle* of $(P \cup I \cup M)$.

3. A cycle $(x_1, x_2, ..., x_p)$ of $(P \cup I \cup M)$ is said to be a *nonstrict cycle* if it is not strict.

A strict cycle is designated by some authors ([17, 15]) as a cycle which contains an asymmetric preference or an asymmetric relation called here P. An ordinal information $(P \cup I \cup M)$ must not contain any strict cycle, but may contain nonstrict cycles, i.e, a cycle of $(I \cup M)$. Hence, we easily deduce the following:

Proposition 1. *Let μ be an ordinal 2-additive scale on X, and $x_1, x_2, \cdots, x_p$ be elements of X.*

If $(x_1, x_2, \cdots, x_p)$ is a nonstrict cycle of $(P \cup I \cup M)$ then $\mu(\phi(x_1)) = \mu(\phi(x_2)) = ... = \mu(\phi(x_p))$.

Proof. $(x_1, x_2, \cdots, x_p)$ is a nonstrict cycle of $(P \cup I \cup M)$ means $x_1\ (I \cup M)\ x_2\ (I \cup M) \cdots (I \cup M)\ x_{p-1}\ (I \cup M)\ x_p\ (I \cup M)\ x_1$. So using the definition of an ordinal 2-additive scale on X and the Remark 4, we have $\mu(\phi(x_1)) \geq \mu(\phi(x_2)) \geq ... \geq \mu(\phi(x_p)) \geq \mu(\phi(x_1))$. □

Proposition 1 shows that, with an ordinal 2-additive scale μ, all binary actions inside a non-strict cycle of $(P \cup I \cup M)$ have the same value by μ. Consequently the detection of these nonstrict cycles seems to be necessary if we want to find an algorithm which computes an ordinal 2-additive scale. To do it, we use the transitive closure TC of $(P \cup I \cup M)$ defined as follows:

Definition 3.2. $\forall x, y \in X$

1. $x\ TC\ y$ if there exists a path of $(P \cup I \cup M)$ from x to y.

2. $x\ TC_P\ y$ if there exists a strict path of $(P \cup I \cup M)$ from x to y.

The transitive closure is well-known in the literature. It is used in graph theory to find the strongly connected components of a graph (see [5, 4, 1, 12]). We define the binary relation $\sim$ which allows us to detect the nonstrict cycles of $(P \cup I \cup M)$ by:

$$\forall x, y \in X,\ x \sim y$$

$$\Leftrightarrow x = y \text{ or } \begin{cases} x\ TC\ y \text{ and } \textit{not}(x\ TC_P\ y) \\ \text{and} \\ y\ TC\ x \text{ and } \textit{not}(y\ TC_P\ x) \end{cases}$$

$\sim$ is obviously an equivalence relation on X. We will denote by $(X \backslash \sim)$ the set of all equivalence classes of $\sim$, and by $\overline{x}$ the equivalence class of an element x of X. The following lemma uses the results of Proposition 1.

Lemma 3. *If μ is an ordinal 2-additive scale on X, then $\forall \bar{x} \in (X \backslash \sim)$, $\forall y, z \in \bar{x}$, $\mu(\phi(y)) = \mu(\phi(z))$.*

Proof. The elements of $\bar{x}$ form a nonstrict cycle of $(P \cup I \cup M)$. Hence, by Proposition 1, we have $\forall y, z \in \bar{x}$, $\mu(\phi(y)) = \mu(\phi(z))$. □

3.2 MOPI property

In order to introduce the fundamental property called MOPI, let us consider an example with $N = \{1, 2, 3, 4\}$. Suppose that the DM says : $a_{12}\ I\ a_3$, $a_{13}\ I\ a_2$ and $a_1\ P\ a_0$. Using the relation M, we have $a_{12}\ M\ a_2\ I\ a_{13}\ M\ a_3\ I\ a_{12}$. So $(a_{12}, a_2, a_{13}, a_3, a_{12})$ forms a nonstrict cycle of $(P \cup I \cup M)$. If μ is an ordinal 2-additive scale of $\{P, I\}$ then we will have $\mu_{12} = \mu_{13} = \mu_2 = \mu_3$ and $\mu_1 > 0$. Hence, we will have a contradiction of monotonicity constraint $\mu_{12} + \mu_{13} \geq \mu_1 + \mu_2 + \mu_3$ of a 2-additive capacity with the subset $A = \{1, 2, 3\}$, $k = 1$ (see (4) in Lemma 1). To formalize this type of inconsistency, we use the following definitions.

Let $K \subseteq N$ and $|K| = k \geq 2$. Let i be a fixed element of K. Let us consider the multiset or bag K^i of X in which a repetition of the element a_i is allowed.

$$K^i = \{\underbrace{a_i, a_i, ..., a_i}_{(k-2)\text{ times}}\} \cup (\bigcup_{j \in K\setminus\{i\}} \{a_j\}).$$

K^i is a set of $2k-3$ elements, and it will be called (K,i)-*multiplied set* of i.

Example 1. *$N = \{1,2,3,4\}$ and $i = 1$ fixed.* $K = \{1,2,3,4\}$, $k = 4$: $\{1,2,3,4\}^1 = \{a_1, a_1, a_2, a_3, a_4\}$.

A multiset [19] (or bag) is a generalization of a set. Within set theory, a multiset can be formally defined as a pair (A, m) where A is some set and $m : A \to \mathbb{N}$ is a function from A to the set $\mathbb{N}$. The set A is called the underlying set of elements. For each a in A the multiplicity of a is the number $m(a)$.

Definition 3.3. Let $K \subseteq N$ such that $|K| = k \geq 3$. Let i be a fixed element of K. Let us set $K \setminus \{i\} := \{j_1, j_2, ..., j_{k-1}\}$.

1. We call *Monotonicity of Preferential Information in K w.r.t. i* the following property (denoted by (K,i)-**MOPI**):

$$\left.\begin{array}{l} a_{ij_1} \sim a_{l_1} \\ a_{ij_2} \sim a_{l_2} \\ a_{ij_3} \sim a_{l_3} \\ \vdots \\ a_{ij_{k-1}} \sim a_{l_{k-1}} \\ \{a_{l_1}, a_{l_2}, ..., a_{l_{k-1}}\} \subseteq K^i \end{array}\right\} \Rightarrow$$

$[not(a_{l_h}\ TC_P\ a_0),\ \forall a_{l_h} \in K^i \setminus \{a_{l_1}, a_{l_2}, ..., a_{l_{k-1}}\}]$

   If the property (K,i)-**MOPI** is satisfied then the elements $a_{l_h} \in K^i \setminus \{a_{l_1}, a_{l_2}, ..., a_{l_{k-1}}\}$ are called *Neutral Binary Actions of K w.r.t. i*. The set of all such elements is denoted by (K,i)-NBA.

2. We say that K satisfies the property of *Monotonicity of Preferential Information* (MOPI) if $\forall i \in K$, (K,i)-MOPI is satisfied.

Example 2. *Let $N = \{1,2,3,4\}$ and $i = 1$ fixed. For $K = \{1,2,3\}$, $k = 3$ and $\{1,2,3\}^1 = \{a_1, a_2, a_3\}$, the property $(\{1,2,3\},1)$-MOPI reads as follows:*

- $\left\{\begin{array}{l} a_{12} \sim a_1 \\ a_{13} \sim a_2 \end{array}\right. \Rightarrow not(a_3\ TC_P\ a_0)$
- $\left\{\begin{array}{l} a_{12} \sim a_2 \\ a_{13} \sim a_1 \end{array}\right. \Rightarrow not(a_3\ TC_P\ a_0)$
- $\left\{\begin{array}{l} a_{12} \sim a_1 \\ a_{13} \sim a_3 \end{array}\right. \Rightarrow not(a_2\ TC_P\ a_0)$
- $\left\{\begin{array}{l} a_{12} \sim a_3 \\ a_{13} \sim a_1 \end{array}\right. \Rightarrow not(a_2\ TC_P\ a_0)$
- $\left\{\begin{array}{l} a_{12} \sim a_3 \\ a_{13} \sim a_2 \end{array}\right. \Rightarrow not(a_1\ TC_P\ a_0)$
- $\left\{\begin{array}{l} a_{12} \sim a_2 \\ a_{13} \sim a_3 \end{array}\right. \Rightarrow not(a_1\ TC_P\ a_0).$

3.3 The main results

We give below our theorem of characterization of consistent ordinal information $\{P, I\}$ which permits to build an ordinal 2-additive scale.

Theorem 1. *There exists an ordinal 2-additive scale on X if and only if the following conditions are satisfied:*

1. *$(P \cup I \cup M)$ contains no strict cycle*
2. *Any subset K of N such that $|K| = k \geq 3$ satisfies the MOPI property.*

Theorem 1 shows that, only two types of inconsistencies occur in an ordinal information given by a DM in order to compute an ordinal 2-additive scale. Therefore the conditions of Theorem 1 characterize the 2-additive Choquet integral. When $I = \emptyset$ in the ordinal information, we have the following result:

Corollary 1.

For any ordinal information $(P \cup I \cup M)$ such that $I = \emptyset$, there exists an ordinal 2-additive scale on X if and only if $(P \cup M)$ has no strict cycle.

Furthermore any ordinal information with empty indifference for which $(P \cup M)$ has no strict cycle, can be represented by a 2-additive capacity with nonnegative interactions.

By Corollary 1, we have only one type of inconsistencies when the DM has no indifference in his ordinal information. In this case, we can compute a 2-additive capacity with nonnegative interactions between two any criteria.

References

[1] D. Bouyssou. Monotonicity of 'ranking by choosing'. *Social Choice and Welfare*, 23:249–273, 2004.

[2] A. Chateauneuf and J.Y. Jaffray. Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Mathematical Social Sciences*, 17:263–283, 1989.

[3] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953.

[4] P. De Donder, M. Le Breton, and M. Truchon. Choosing from a weighted tournament. *Mathematical Social Sciences*, 40:85–109, 2000.

[5] M. Gondran and M. Minoux. *Graphes et algorithmes*. Eyrolles, 1986.

[6] M. Grabisch. Alternative representations of discrete fuzzy measures for decision making. *Int. J. of Uncertainty, Fuzziness, and Knowledge Based Systems*, 5:587–607, 1997.

[7] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[8] M. Grabisch. The Möbius transform on symmetric ordered structures and its application to capacities on finite sets. *Discrete Mathematics*, 287 (1-3):17–34, 2004.

[9] M. Grabisch. L'utilisation de l'intégrale de Choquet en aide multicritère à la décision. *Newsletter of the European Working Group "Multicriteria Aid for Decisions"*, Vol 3 No 14:5–10, Fall 2006.

[10] M. Grabisch, I. Kojadinovic, and P. Meyer. A review of capacity identification methods for Choquet integral based multi-attribute utility theory — applications of the Kappalab R package. *Eur. J. of Operational Research, to appear.*

[11] M. Grabisch and C. Labreuche. Fuzzy measures and integrals in MCDA. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis*, pages 563–608. Kluwer Academic Publishers, 2004.

[12] J.-F. Laslier. *Tournament solutions and majority voting*. Springer-Verlag, 1997.

[13] J.L. Marichal. An axiomatic approach of the discrete Choquet integral as tool to aggregate interacting criteria. *IEEE Tr. on Fuzzy systems*, 8(6):800–807, 2000.

[14] J.L. Marichal and M. Roubens. Determination of weights of interacting criteria from a reference set. *European journal of Operational Research*, 124:641–650, 2000.

[15] C. Mousset. *Familles de structures de préférence non complètes*. PhD thesis, University of Mons-Hainaut, Mons, 2004.

[16] T. Murofushi and S. Soneda. Techniques for reading fuzzy measures (iii): interaction index. In *9th Fuzzy System Symposium*, pages 693–696, Sapporo, Japan, May 1993. Japanese.

[17] M. Pirlot and Ph. Vincke. *Semiorders*. Kluwer academic publishers, 1997.

[18] L.S. Shapley. A value for n-person games. *Contributions to the Theory of games*, pages 307–317, 1953.

[19] P. Stanley. *Enumerative Combinatorics*, volume 1 and 2. Cambridge University Press, 1997 and 1999.