Trading Time for Approximation

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Université Paris Dauphine

MTA SZTAKI – Sep 21 2016

Motivation – Sub-Exponential Approximation

(50 years in a slide)

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(50 years in a slide)

- Efficient = Poly-time ('60s)

Jack Edmonds

Juris Hartmanis

Richard Stearns

Motivation – Sub-Exponential Approximation

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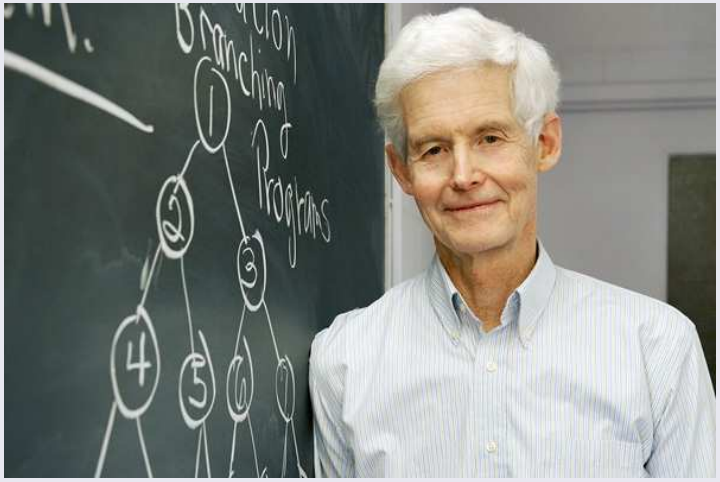
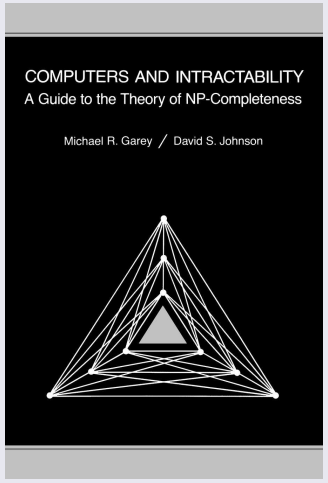
- Everything is NP-hard!(*) ('70s)



Stephen Cook

Richard Karp



Garey& Johnson

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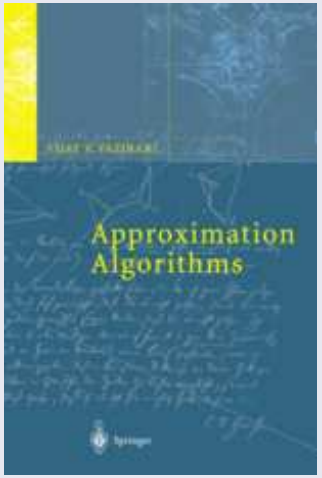
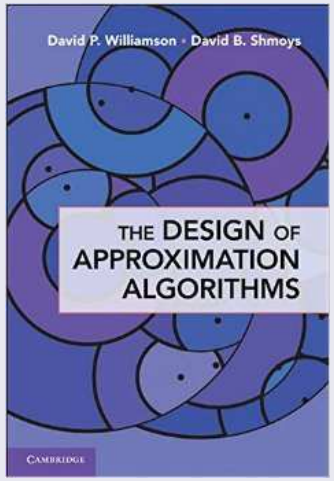
- So, we should approximate ('80s)

David Johnson



V. Vazirani



Williamson&Shmoys

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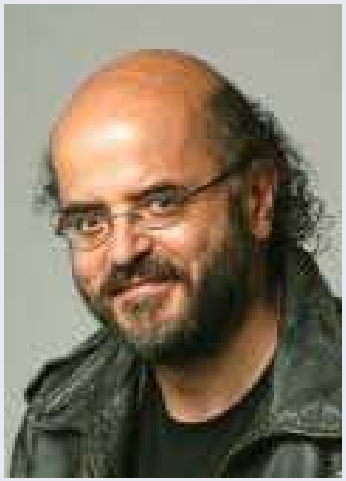
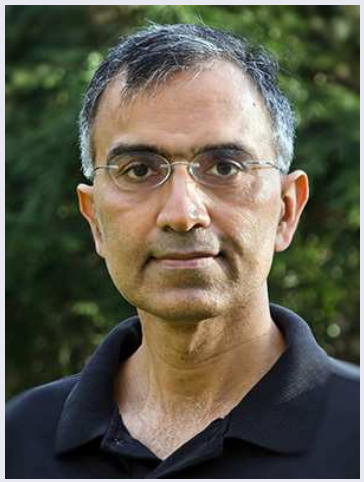


Christos Papadimitriou



Sanjeev Arora

Johan Håstad

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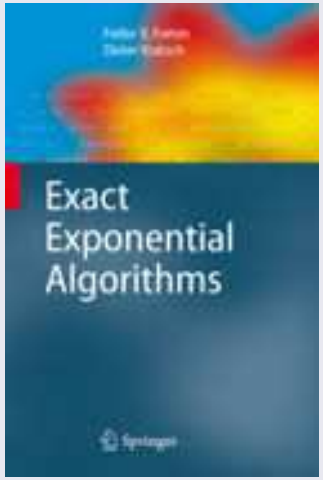
Mike Fellows

Russell Impagliazzo



Fomin&Kratsch

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Bottom line:

- Most problems hard to solve exactly in $2^{o(n)}$ time
- Most problems hard to approximate in $n^{O(1)}$ time

- $\rightarrow$ Perhaps we can approximate in $2^{o(n)}$ time?
- More broadly:
  - Does a **little extra time** give **much better approximation**?
  - Is a **good approximation** much faster than **solving exactly**?

Dead on Arrival?

- Better than $3/2$ for TSP, $4/3$ for Max-3-DM, $7/8$ for Max-3-SAT,. . . , in **sub-exponential time**?

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Probably won't work

(at least for Max-3-SAT)

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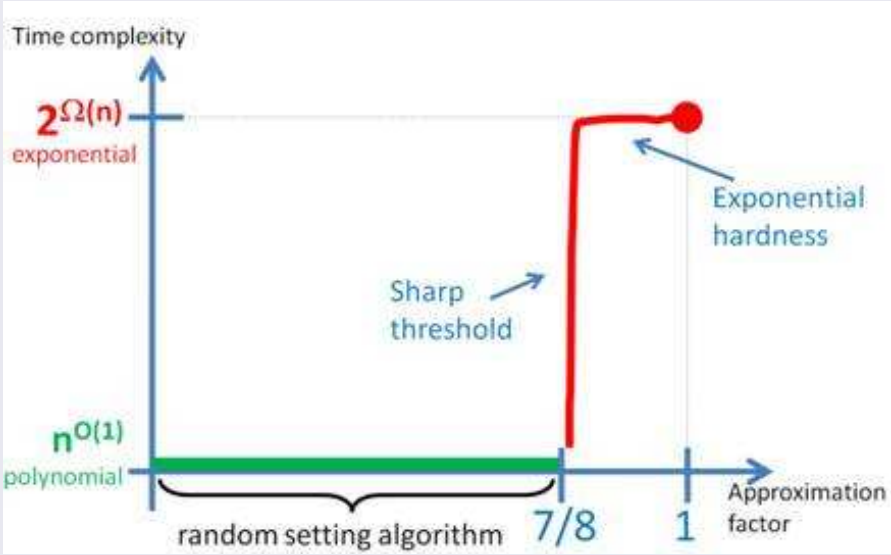

Almost-linear PCPs (Moshkovitz& Raz) and P-time hardness (Håstad) give tight inapproximability for Max-3-SAT even for $2^{n^{1-\epsilon}}$ time.

(Credit: Dana Moshkovitz)

Dead on Arrival?

- Better than $3/2$ for TSP, $4/3$ for Max-3-DM, $7/8$ for Max-3-SAT,. . . , in **sub-exponential time**?

  If this is the "normal" behavior of APX problems, what's the point of sub-exponential approximation?

  - Is this the "normal" behavior?
  - What about problems outside APX?
    - Independent Set?
    - Set Cover?
  - **What else** can we use the extra time for? (if anything)

Approximations for Hard Problems

Very Hard Problems

Joint work with:

Edouard Bonnet Vangelis Paschos

"Time-Approximation Trade-Offs for Inapproximable Problems", STACS '16

Very Hard Problems

- Consider problems which are really hard to approximate in polynomial time.
- Does extra time at least help with these?

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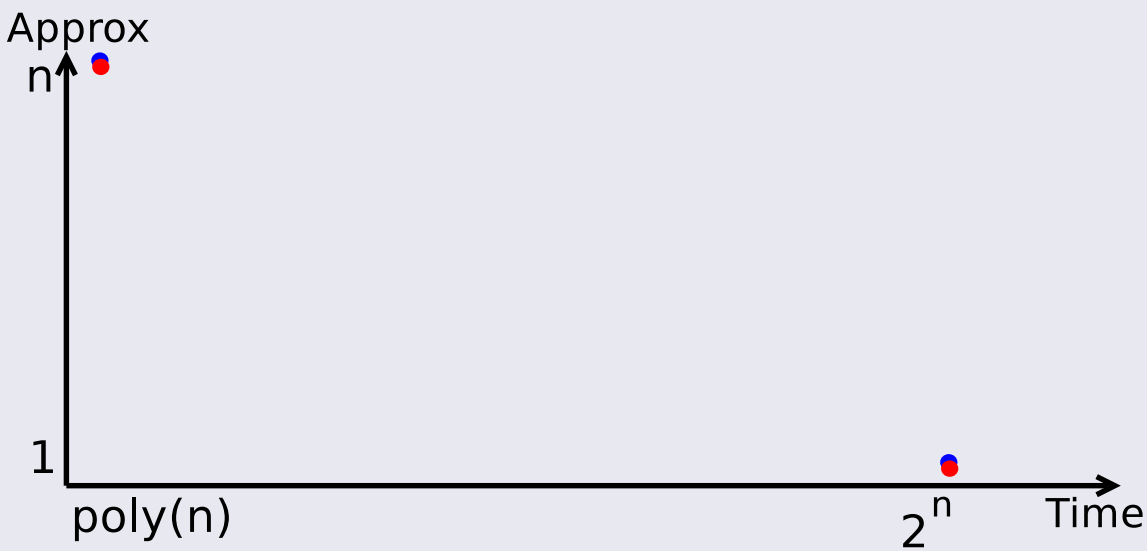

- Approximation ratio in poly time $n^{1-\epsilon}$
- No $2^{o(n)}$ exact algorithm

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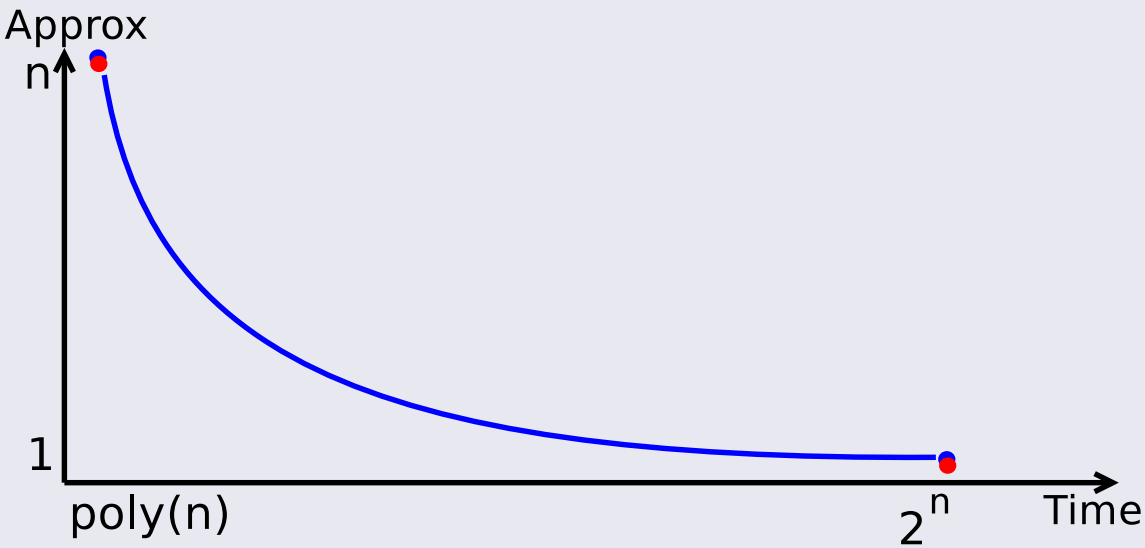

- Simple r-approximation in $2^{n/r}$

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Randomly sample n/r vertices, find their Max IS

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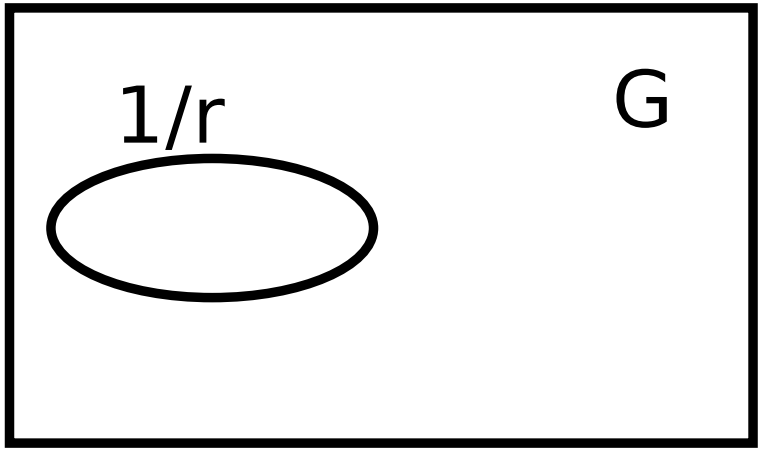

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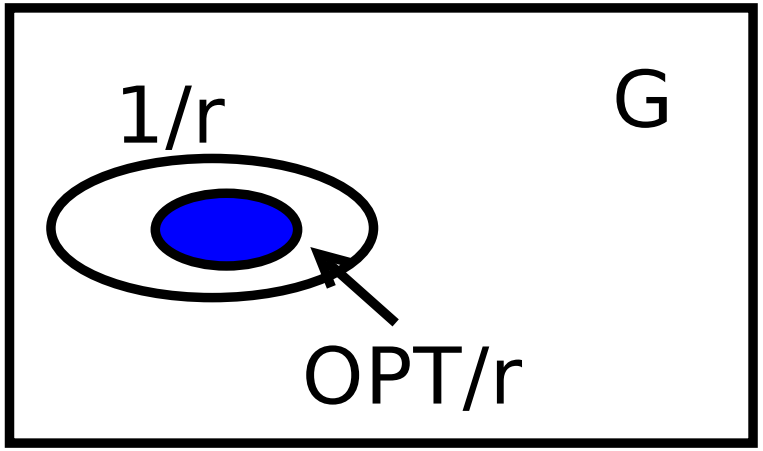

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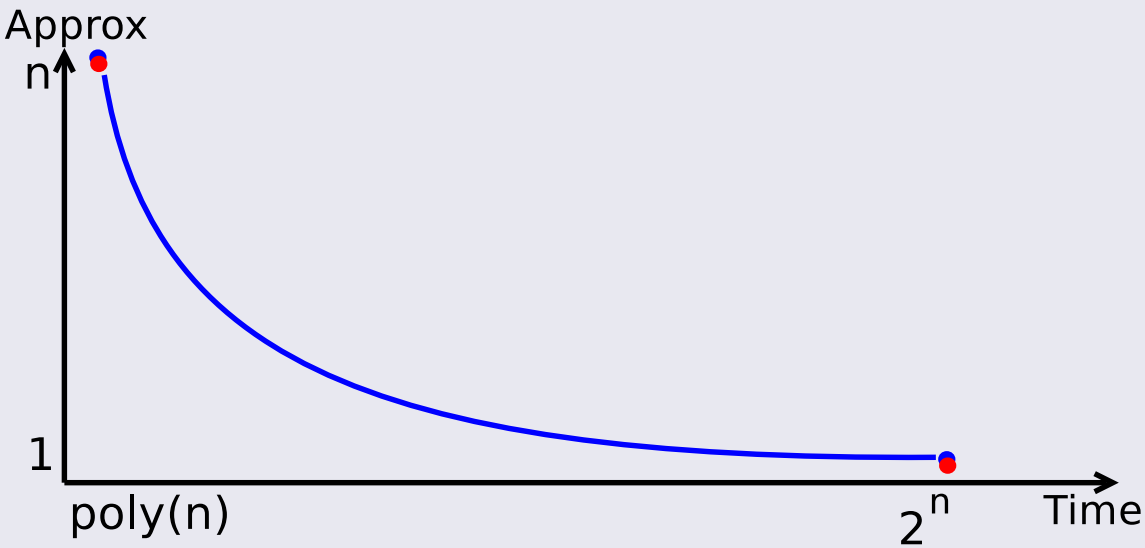

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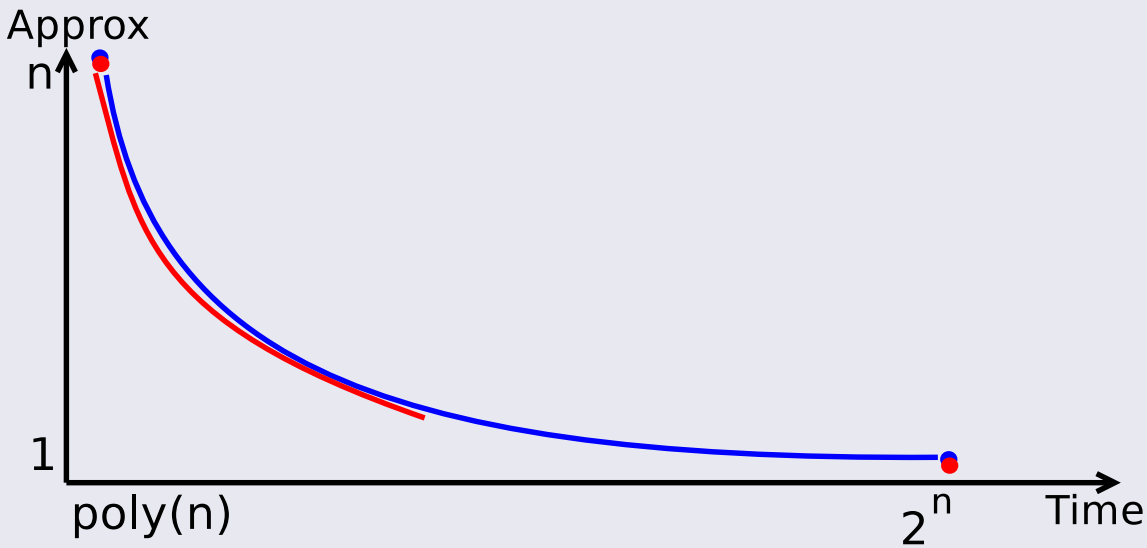

- For any $0 < r < \sqrt{n}$ there is no r-approximation for IS in time $2^{(n/r)^{1-\epsilon}}$. [Chalermsook, Laekhanukit, Nanongkai] (FOCS'11)

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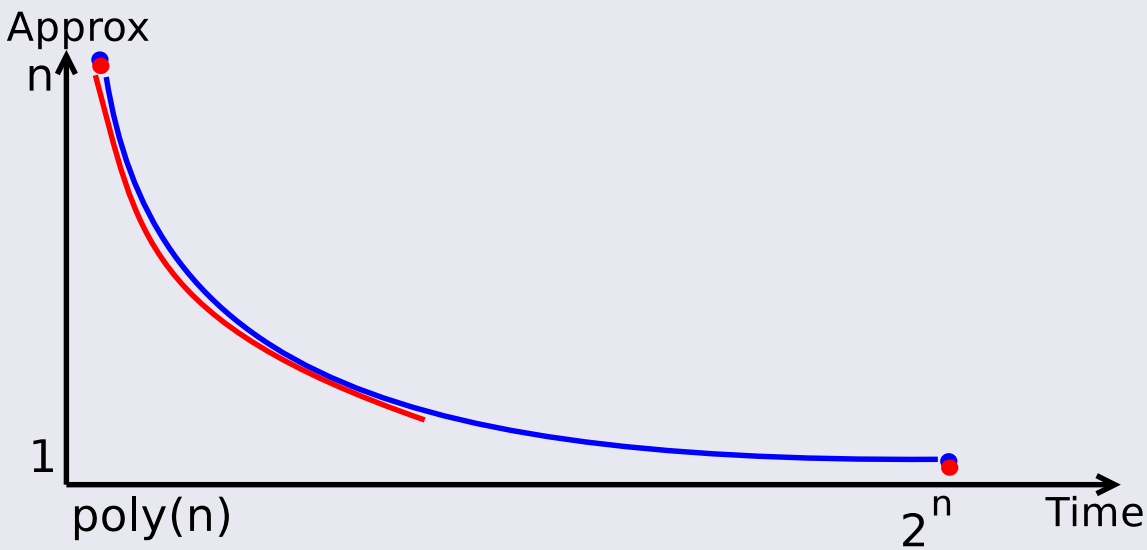

- Proof idea: Reduce 3-SAT with n vars to IS with rn vertices.
- Either $\mathrm{OPT} \approx rn$ or $\mathrm{OPT} \approx n$.

Very Hard Problems

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- Does extra time at least help with these?
- Example: Independent Set

Can we **completely** determine the trade-off curve for other problems?

- How much time do we need for r-approximation?
- What is the best approximation in time T?

Maximum Induced Path

Theorem: For any parameter $r > 0$, there is no r-approximation for Max Induced Path running in time $2^{o(n/r)}$.

Problem: Find maximum number of vertices that **induce** a path

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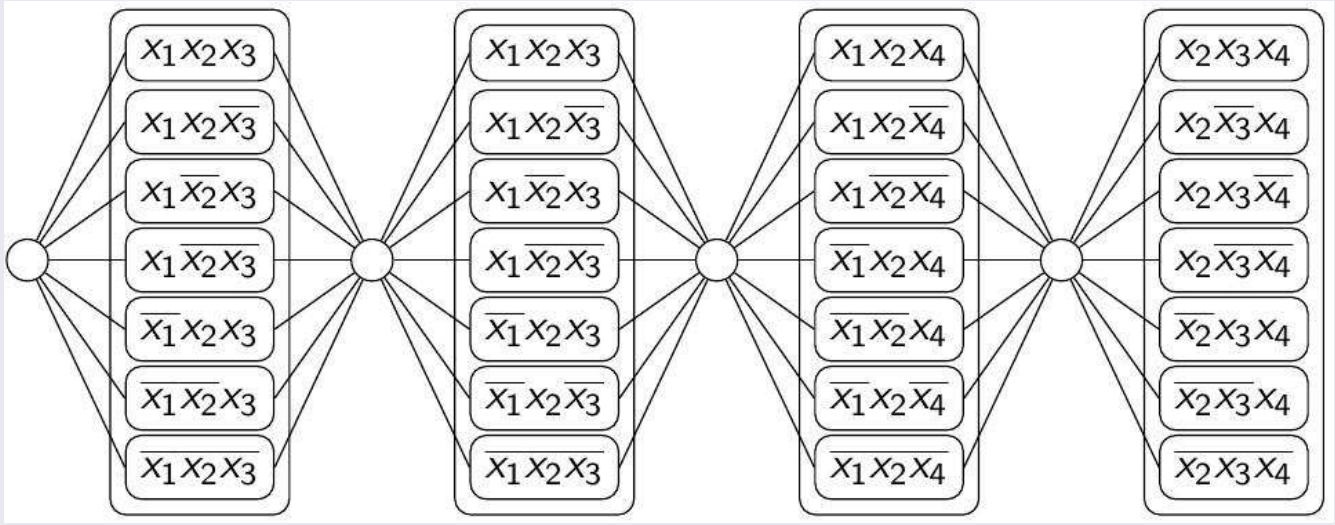

Reduction from 3-SAT

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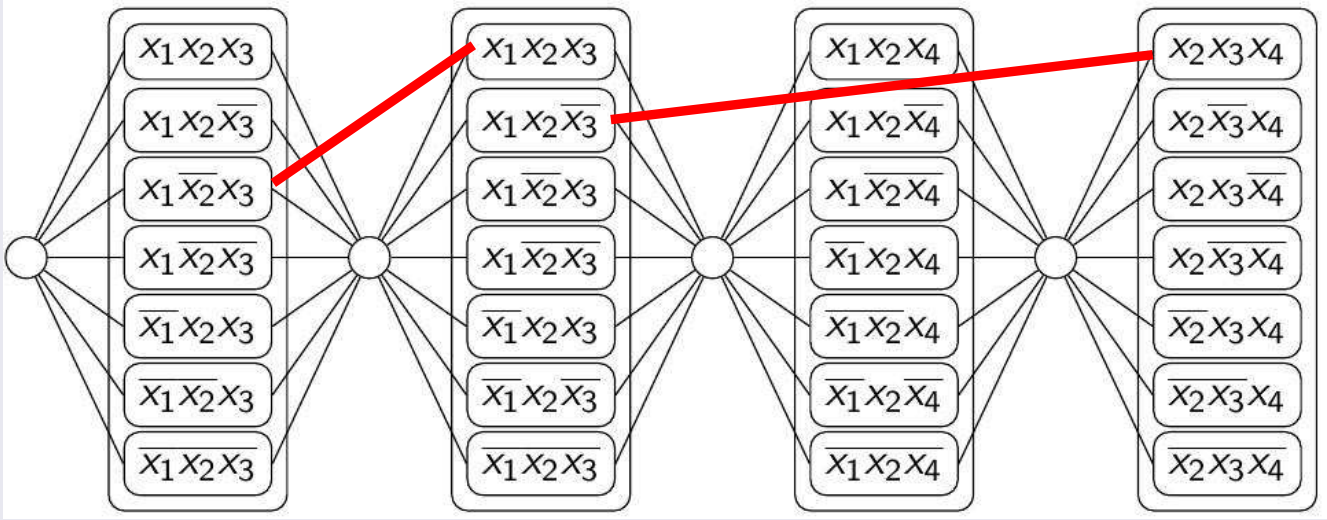

Reduction from 3-SAT
One node for every clause assignment
Extra edges for inconsistent assignments

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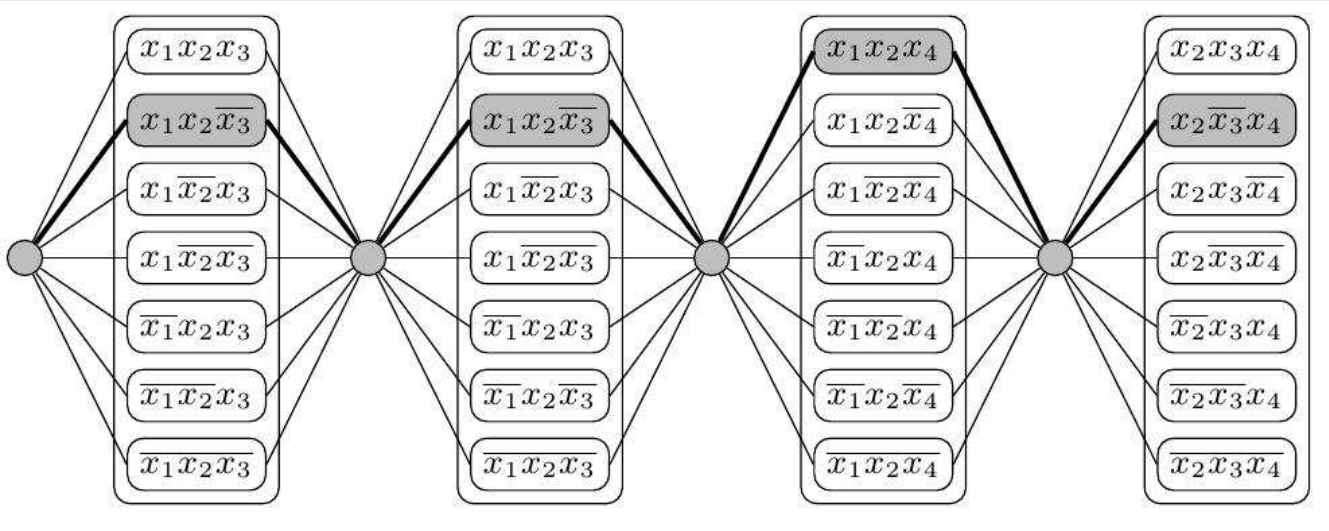

Reduction from 3-SAT
One node for every clause assignment
Extra edges for inconsistent assignments
Consistent Global Assignment $\rightarrow$ Induced Path

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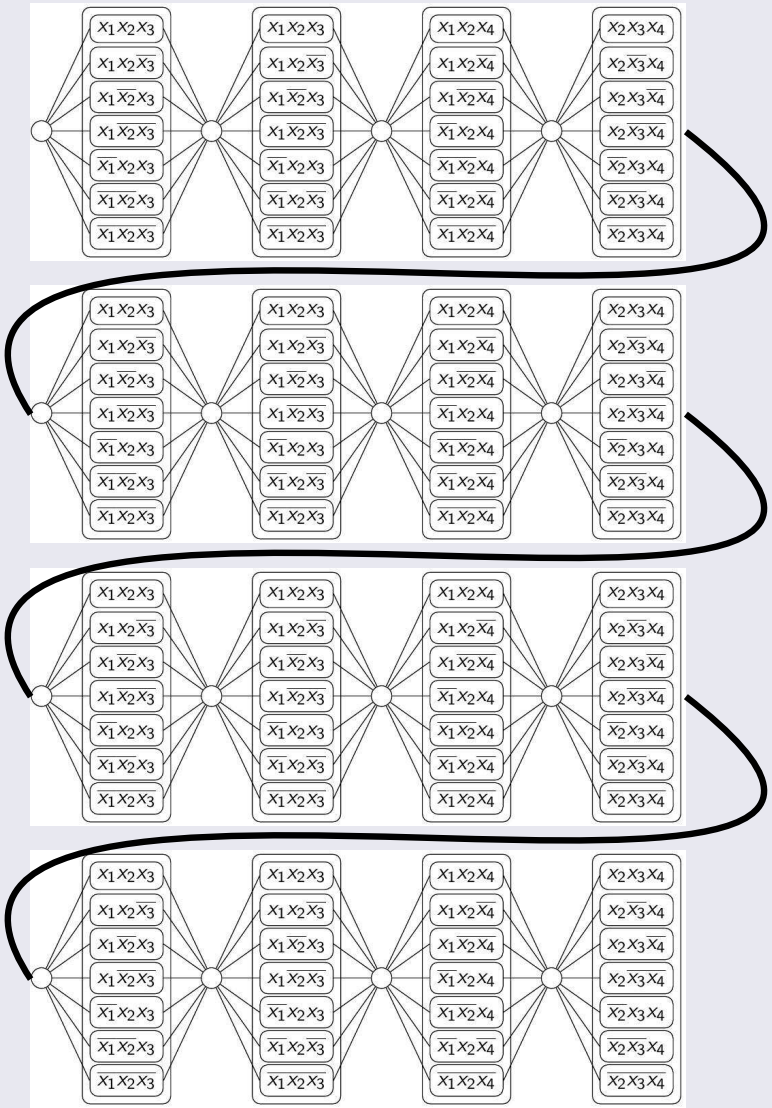

Take r copies

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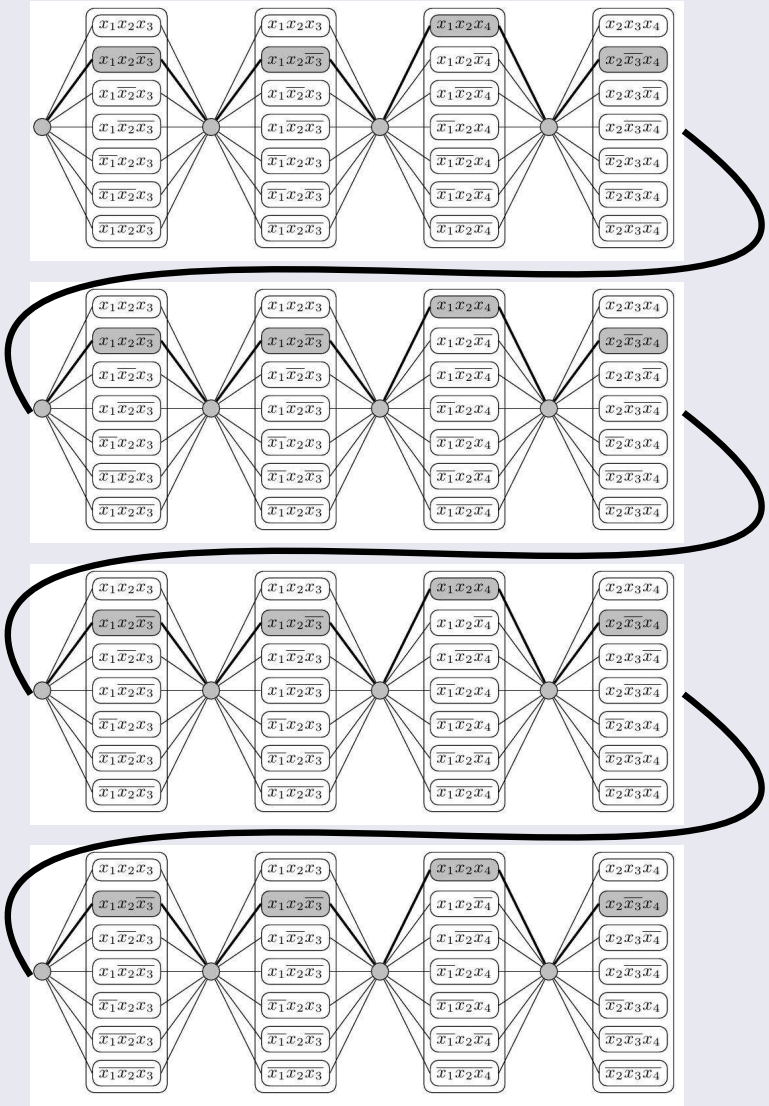

Take r copies
Assignment $\rightarrow$ Induced Path
Size = rn
Otherwise $\leq n$
Graph size = $O(rn)$

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Proof: Try out all subsets of size $\leq r$.

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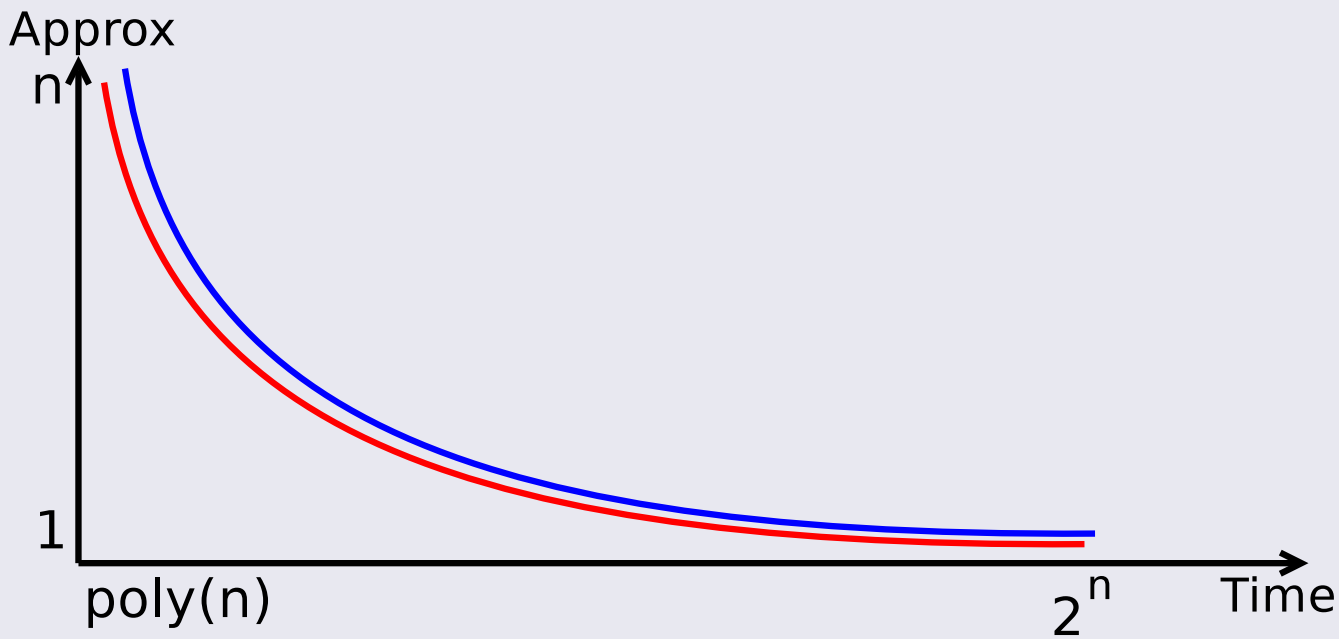

Can we make these match?

Maximum Minimal Vertex Cover

Theorem: For any parameter $r > 0$, there is no r-approximation for Max Minimal VC running in time $2^{(n/r^2)^{1-\epsilon}}$.
Theorem: For any parameter $r > 0$, there is an r-approximation for Max Minimal VC running in time $2^{O(n/r^2)}$.

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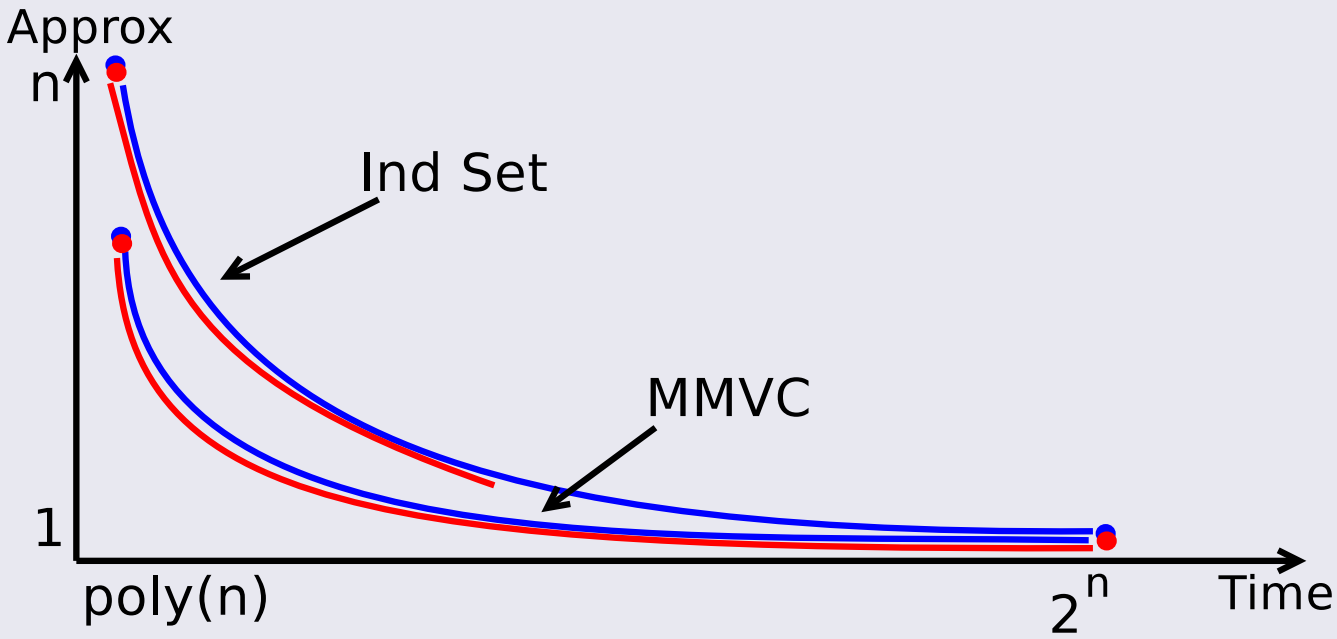

Summary for hard problems

- Extra time can buy us much better approximation!
  - But only for really hard problems.
- We can completely determine the optimal trade-off!
  - But only for really hard problems.
- Can the extra time buy us anything else?

Trading Time for Generality

An island of Tractability

Joint work with:

Dimitris Fotakis

Vangelis Paschos

"Sub-Exponential Approximation Schemes for CSPs: from Dense to Almost-Sparse", STACS '16

An island of Tractability

- We **cannot** get better than 7/8 for Max-3-SAT in sub-exp time (under ETH).
- We will therefore try to get something else:

An island of tractability:

- Max-k-CSP admits a PTAS (a $(1-\epsilon)$-approximation for all $\epsilon > 0$) for **dense** instances
- (Arora, Karger, Karpinski '99), (de la Vega '96)

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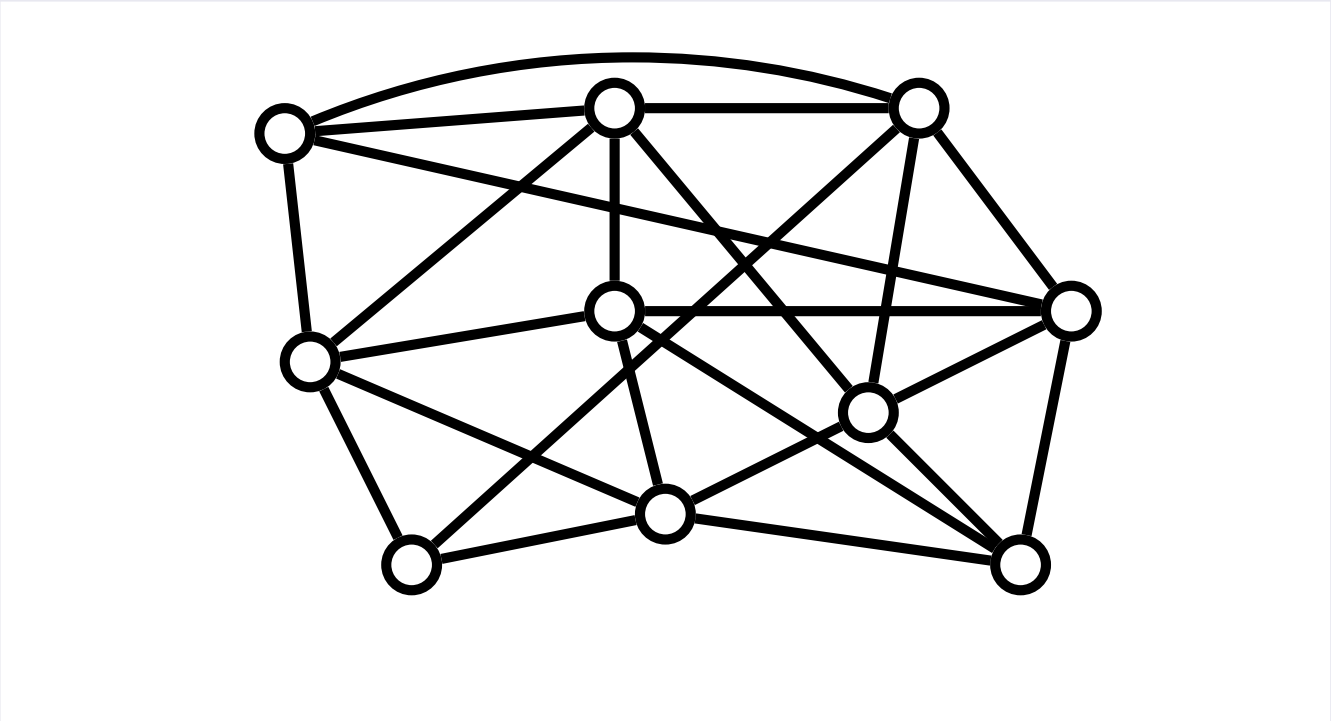
Strategy: We use our extra time to extend this island of tractability.

Basic scheme (Max Cut)



We are given a dense graph for which we want to find a large cut

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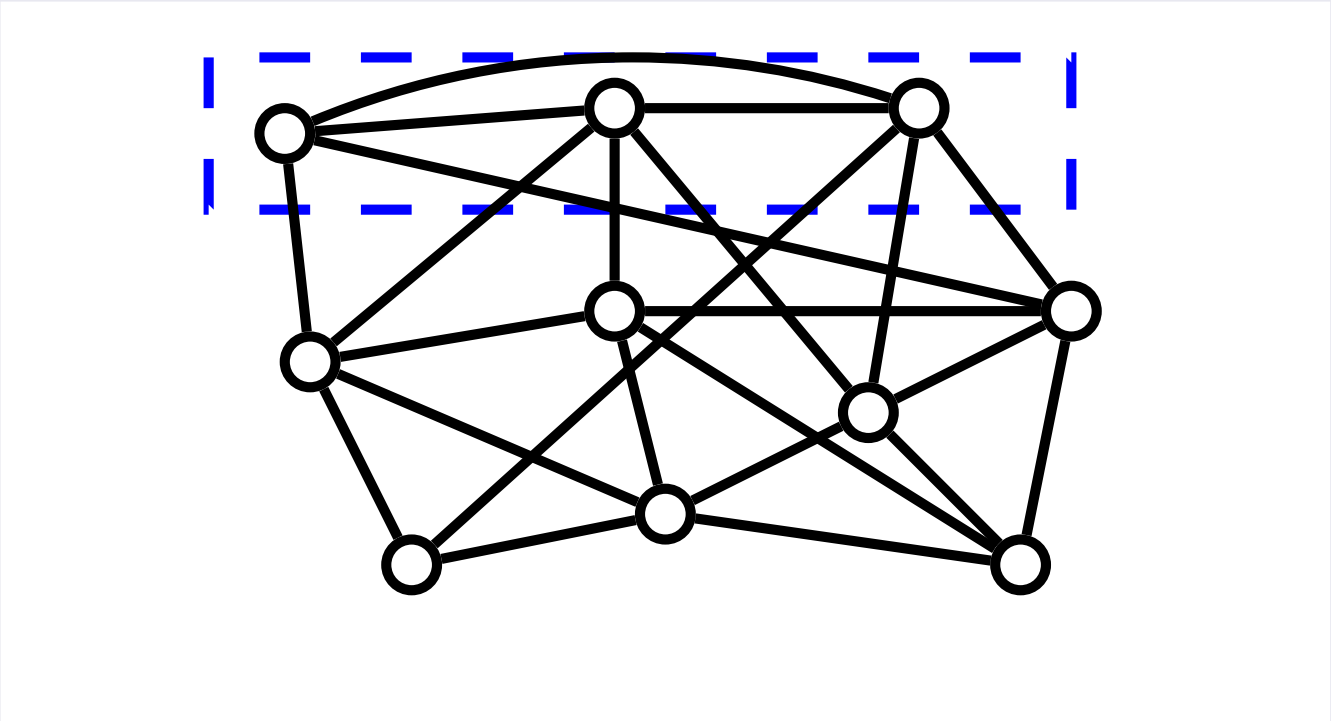
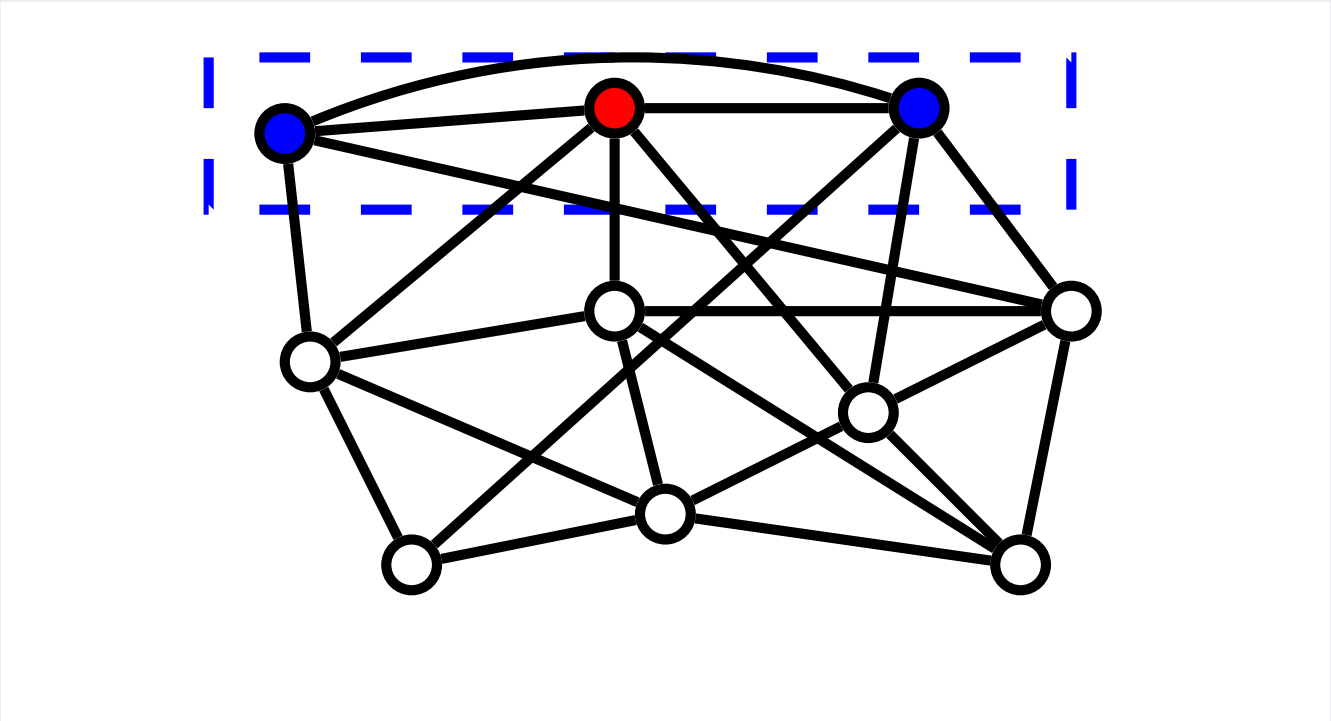


Randomly select a “sample” of its vertices

Basic scheme (Max Cut)



Guess their correct partition

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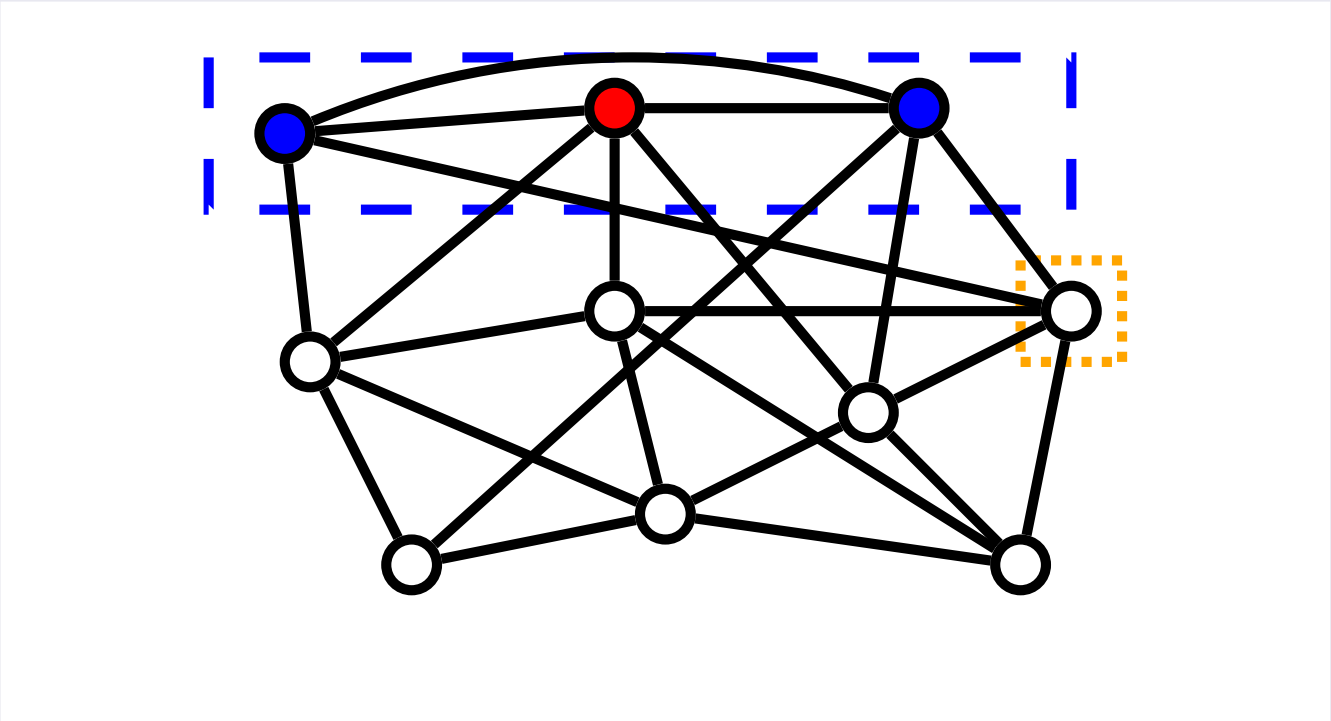
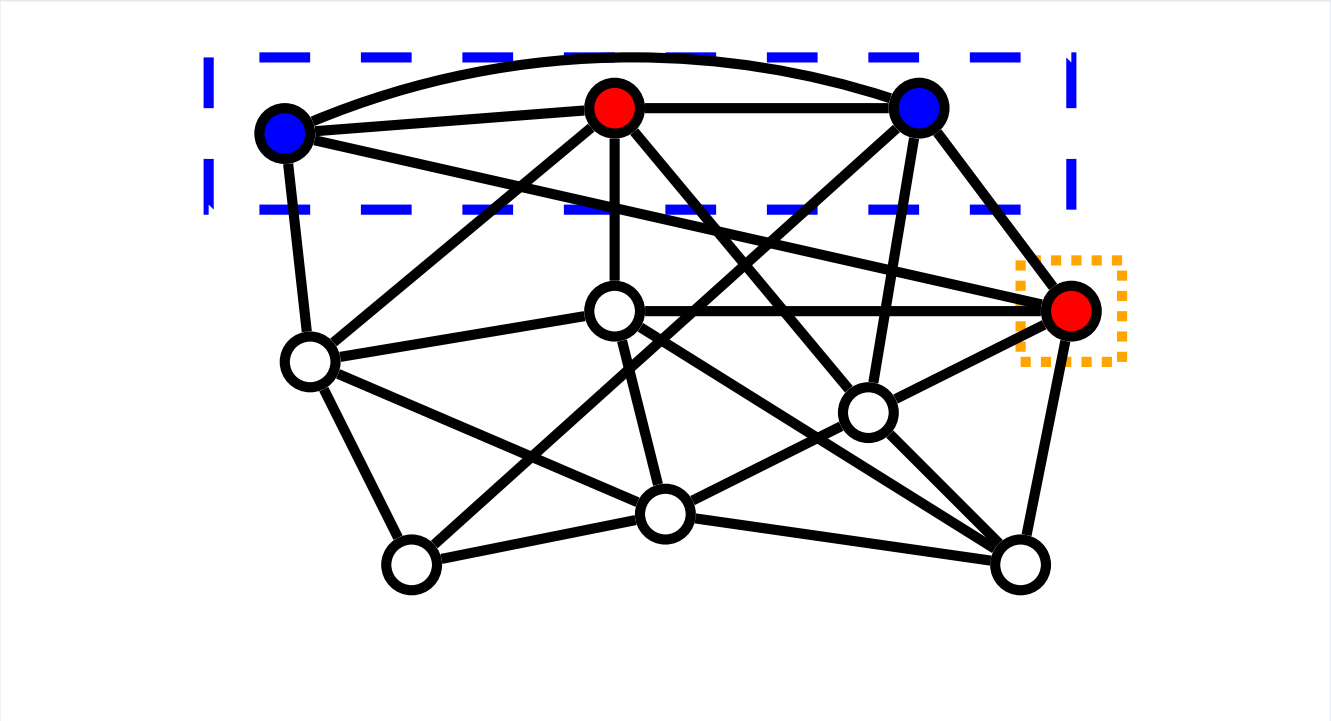


For every vertex outside the sample, examine its neighbors in the sample

Basic scheme (Max Cut)



Greedily set its value depending on this neighborhood

Basic scheme (Max Cut)

- The sample we select has size $O(\log n)$ (hidden constants depend on degree and ϵ)
- $\rightarrow$ running time $n^{O(1)}$ (will try all partitions of sample)

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Why this works (intuitively):

- Because graph is dense $\rightarrow$ every vertex outside sample S has many neighbors in S
- $\rightarrow$ examining $N(u) \cap S$ is (whp) a good representation of $N(u)$ in the optimal solution
- If a vertex in $V \setminus S$ has $>> 50\%$ of its neighbors on one side in the optimal solution, it will (whp) have $>> 50\%$ of its neighbors on that side in S

(de la Vega '96)

Sub-exponential Extension (Max Cut)

Summary of algorithm:

- Estimate the neighborhood partitions using a sample
  - How can we do this if a graph is sparse?
- The rest works as previously...

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- Suppose graph has average degree $\Delta = n^{\delta}$
- We sample $\frac{n \log n}{\Delta} = n^{1-\delta} \log n$ vertices
- Running time: $\approx 2^{n^{1-\delta}}$

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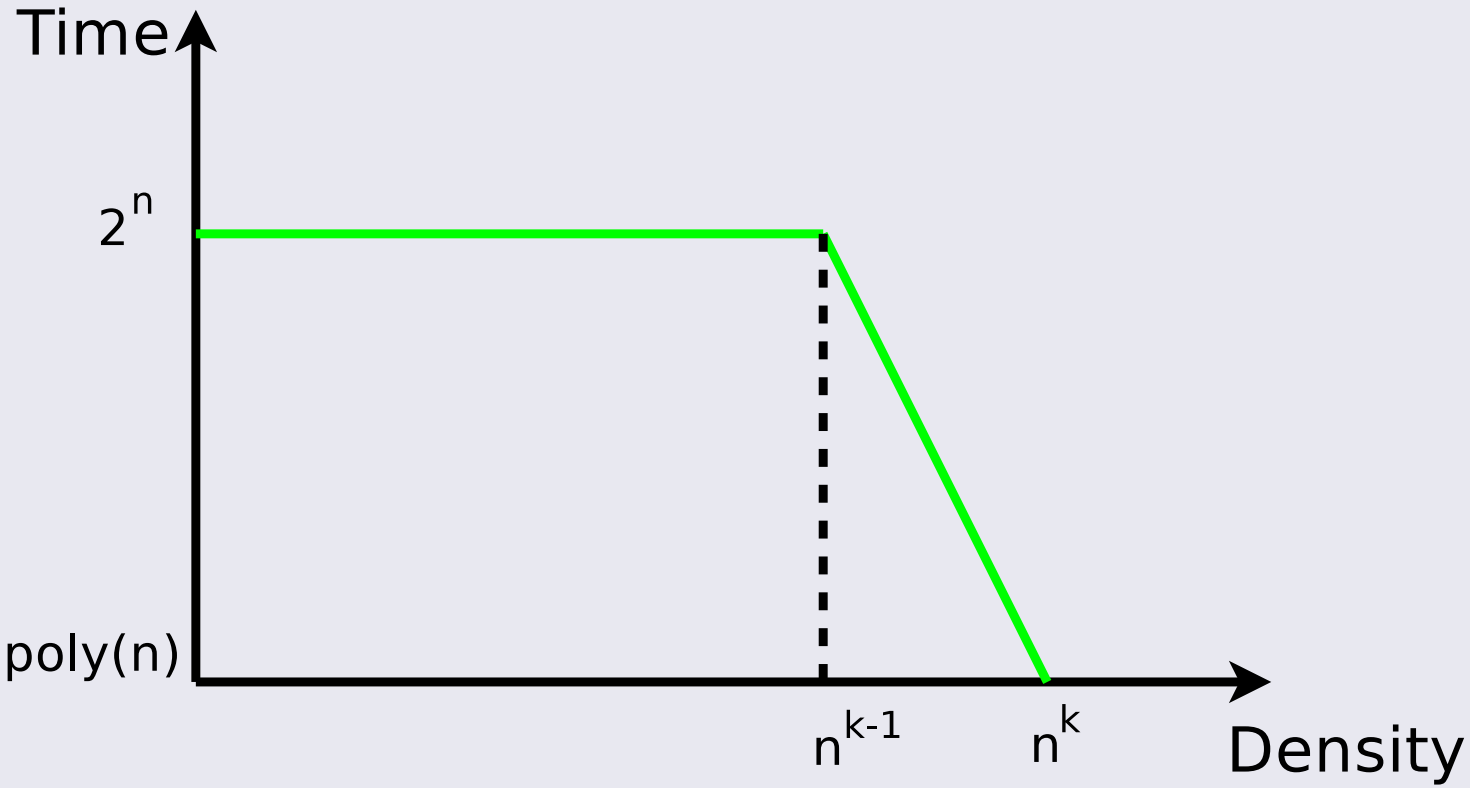
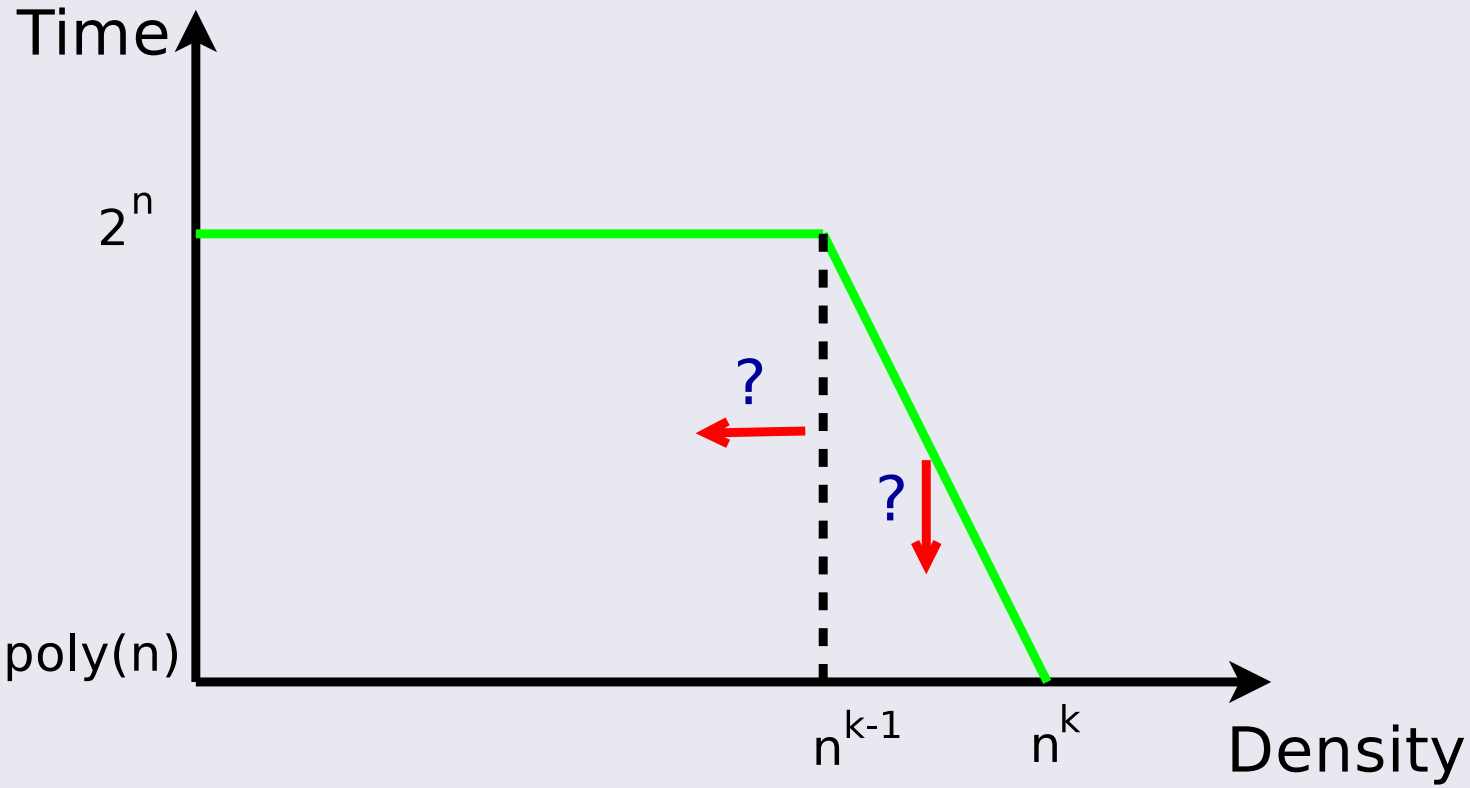
- Suppose graph has average degree $\Delta = n^\delta$
- We sample $\frac{n \log n}{\Delta} = n^{1-\delta} \log n$ vertices
- Running time: $\approx 2^{n^{1-\delta}}$
- Can be extended to Max-k-CSP!
- If $n^{k-1+\delta}$ clauses $\rightarrow 2^{n^{1-\delta}}$ time

Summary – with a picture



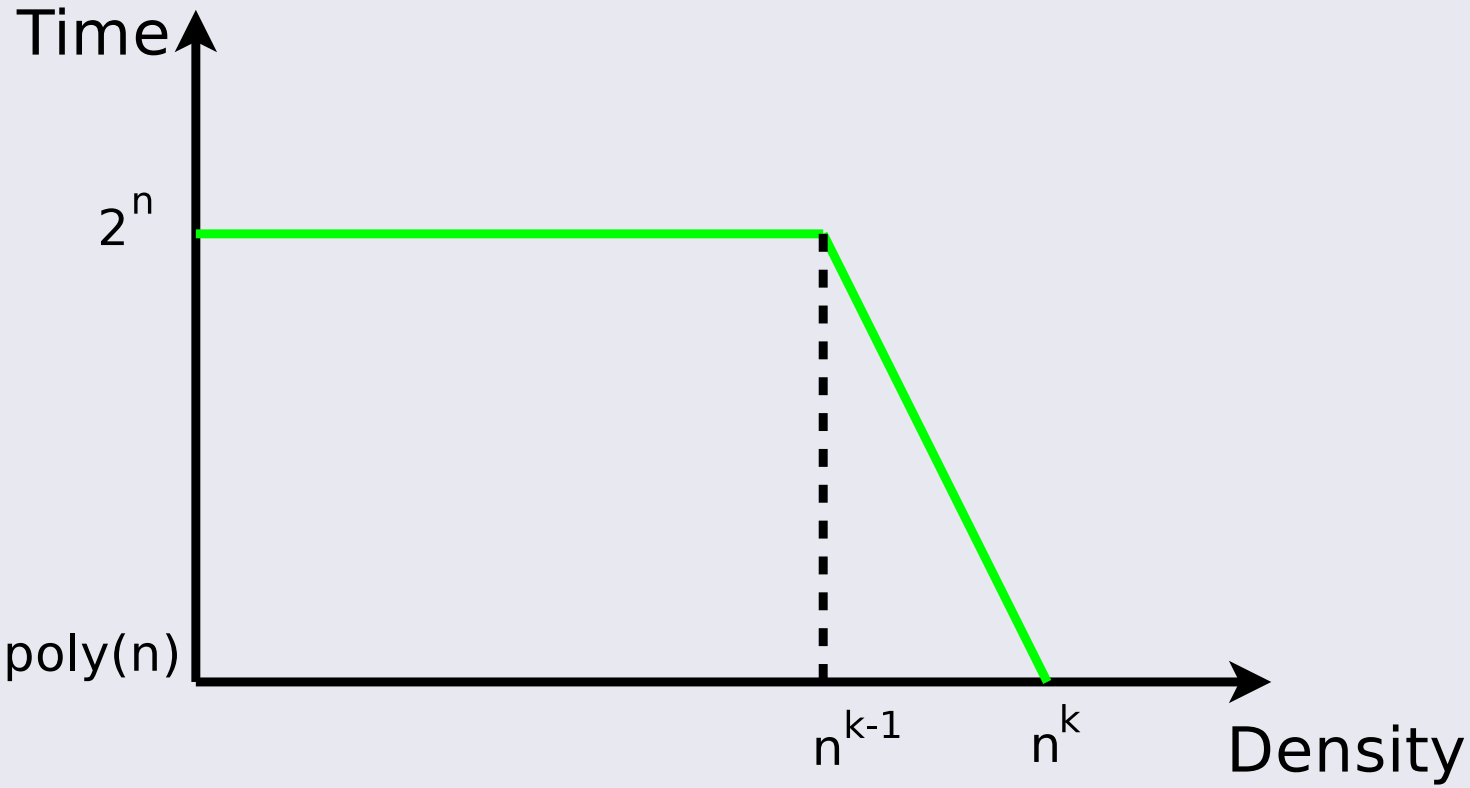
Complexity/Density trade-off for Max-k-CSP

Summary – with a picture



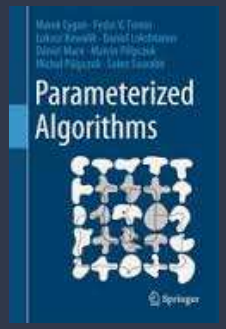
Possible Improvements? Faster? More general?

Summary – with a picture



Both are tight! (under ETH)

Parameterized Approximation



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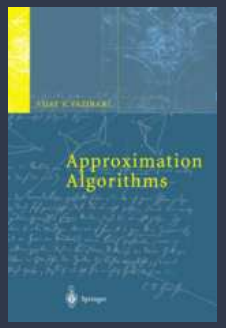

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General Approach

- Parameterized Complexity == Trading Time for Generality
- General question: What is the best trade-off curve?
  - n^k vc. 2^k

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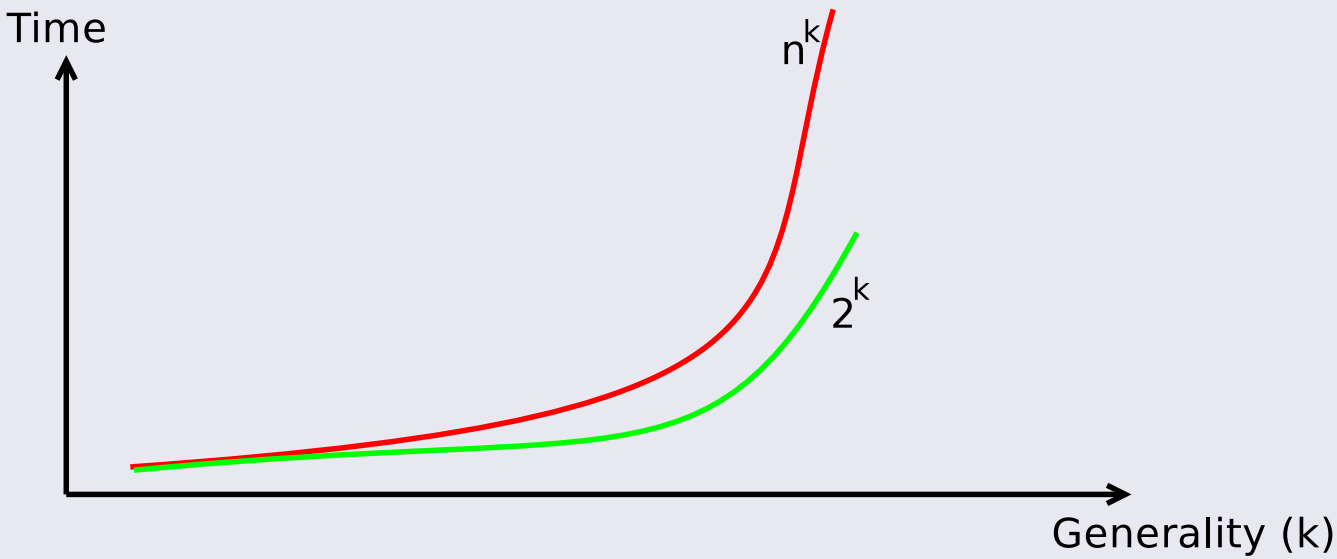


Does the trade-off curve become (much) better if we allow $(1+\epsilon)$ error?

Win/Win – Max-SAT

Joint work with:

Holger Dell

Eunjung Kim

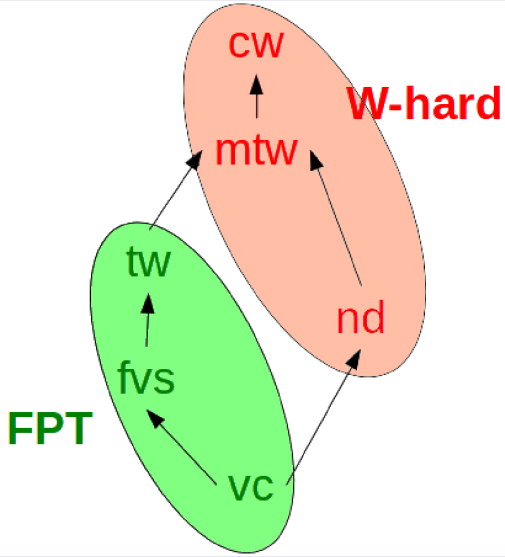
Valia Mitsou

Tobias Moemke

"Complexity and Approximability of Parameterized Max-CSPs", IPEC '15

Win/Win – Max-SAT

Problem: Max-SAT with structural graph parameters



- Solvable in 2^k for treewidth
- Needs n^k for clique-width
- (APX-hard in general)
- **Goal:** Faster algorithm for more general case (clique-width), for $(1+\epsilon)$-approximation error.

Win/Win – Max-SAT

Algorithmic Strategy: **Win/Win**

Win/Win – Max-SAT

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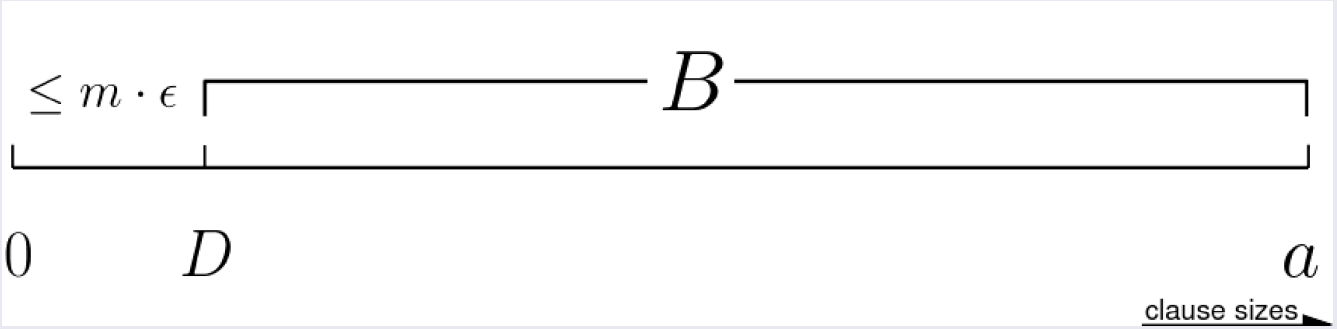
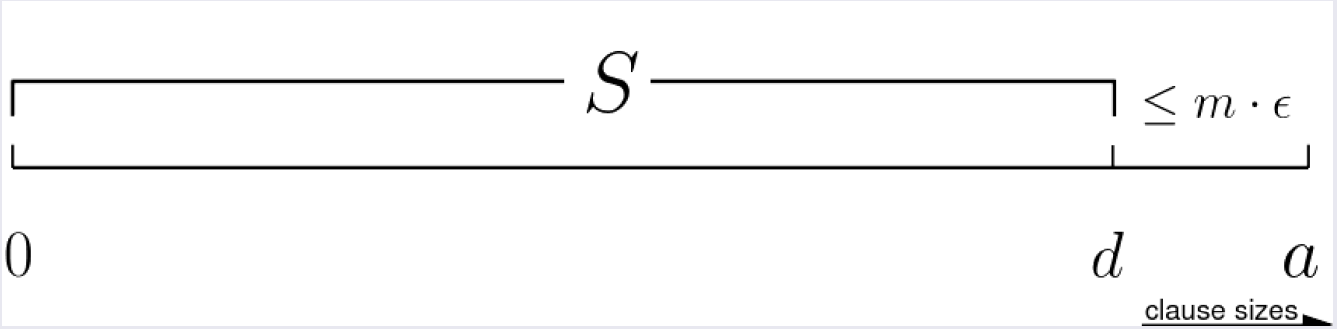


- If (almost) all clauses large $\rightarrow$ Easy!
  - Set everything randomly.

Algorithmic Strategy: **Win/Win**



- If (almost) all clauses large $\rightarrow$ Easy!
- If (almost) all clauses small $\rightarrow$ Easy!
  - Instance has small treewidth.

Win/Win – Max-SAT

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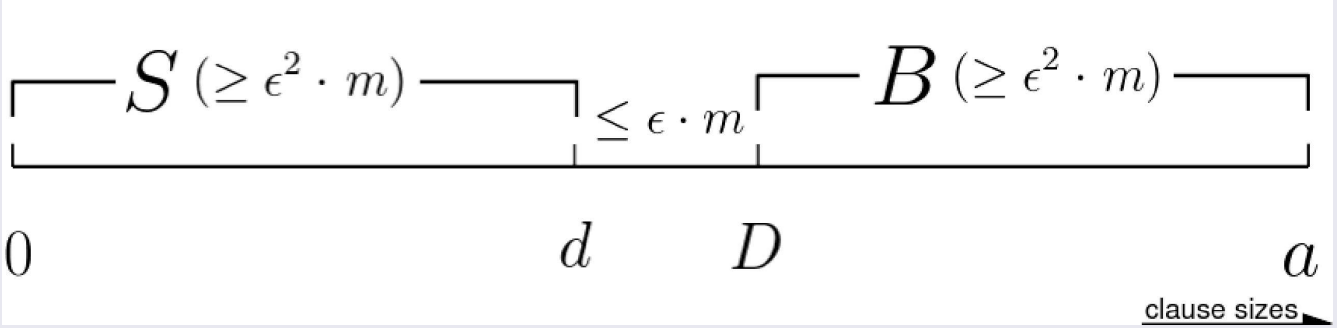
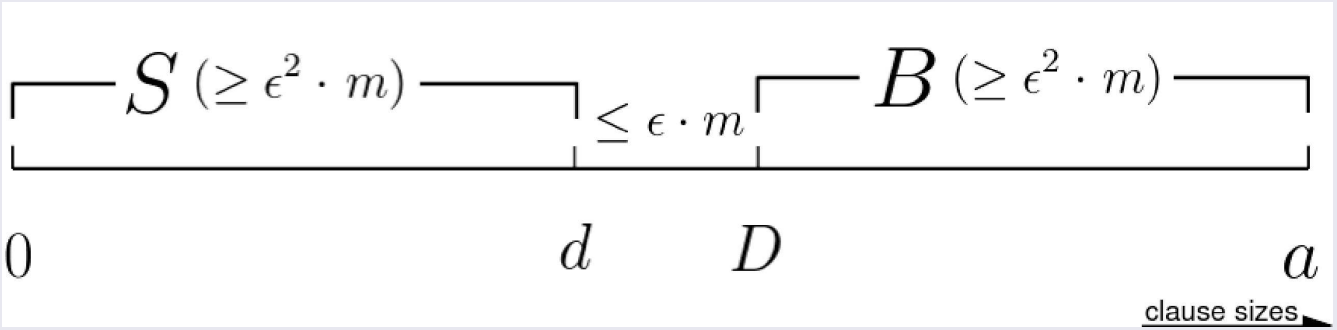


- If (almost) all clauses large $\rightarrow$ Easy!
- If (almost) all clauses small $\rightarrow$ Easy!
- If many large and many small $\rightarrow$ Reduce to previous cases
  - There exists variable appearing overwhelmingly in large clauses.
  - Set it randomly, repeat.

Win/Win – Max-SAT

Algorithmic Strategy: **Win/Win**



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- Bottleneck: FPT algorithm for treewidth (small clause case)
- $(1+\epsilon)$-approximation
- Running time: $2^{O(k)}$, where constant depends on ϵ
- **Contrast:** Exact algorithm needs n^k.

FPT Rounding – Ordered Weighted Assignment

Joint work with:

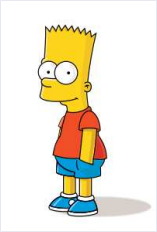
Julien Lesca

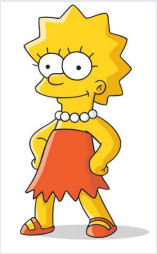
Florian Sikora

FPT Rounding – Ordered Weighted Assignment










- n agents
- n items
- Each agent has different utility for each item
- Maximize **Social Welfare** (not necessarily sum)

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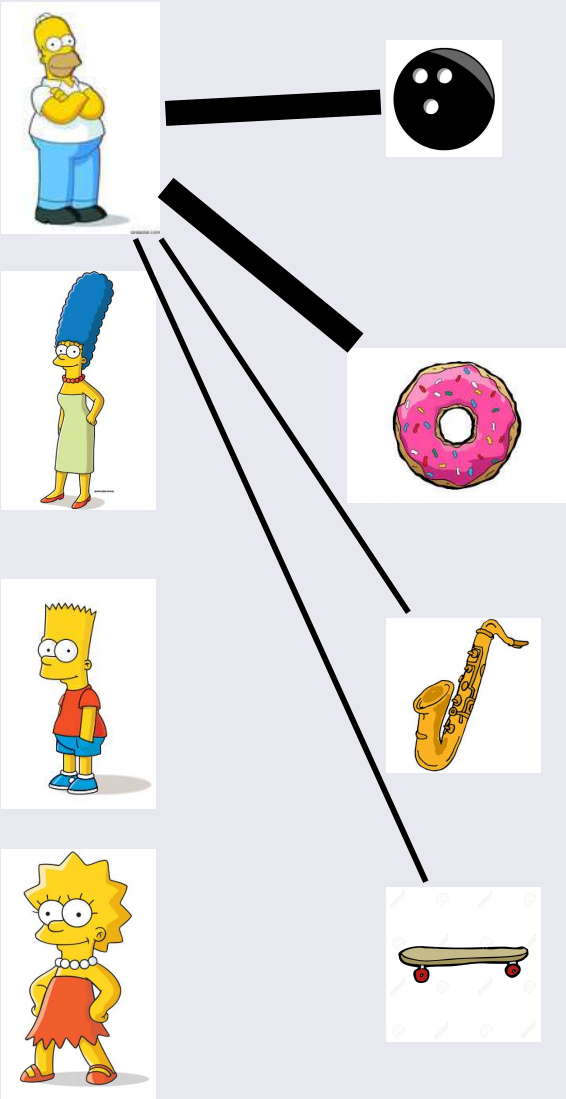

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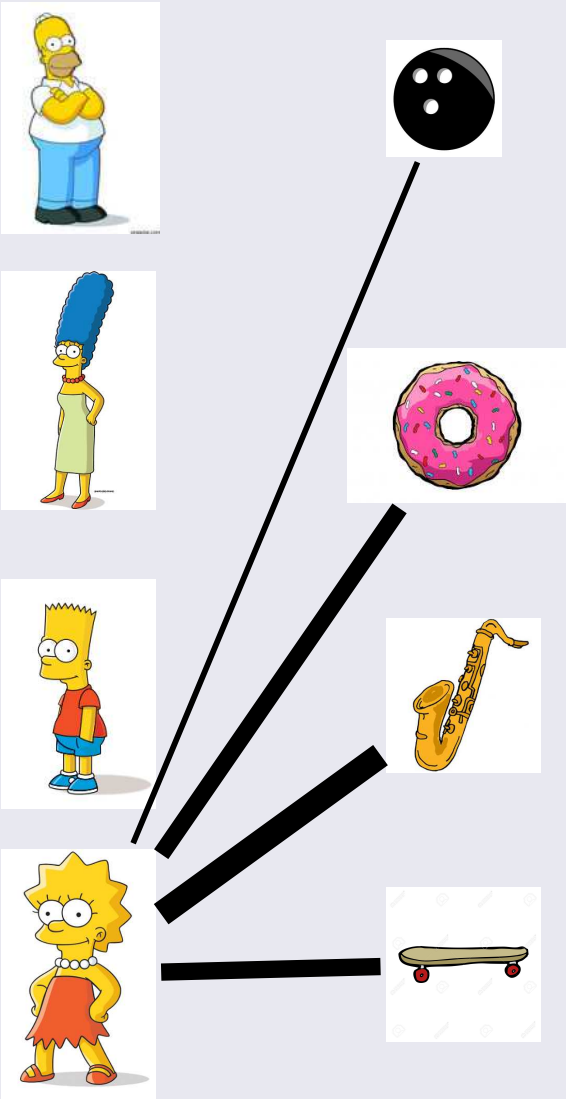

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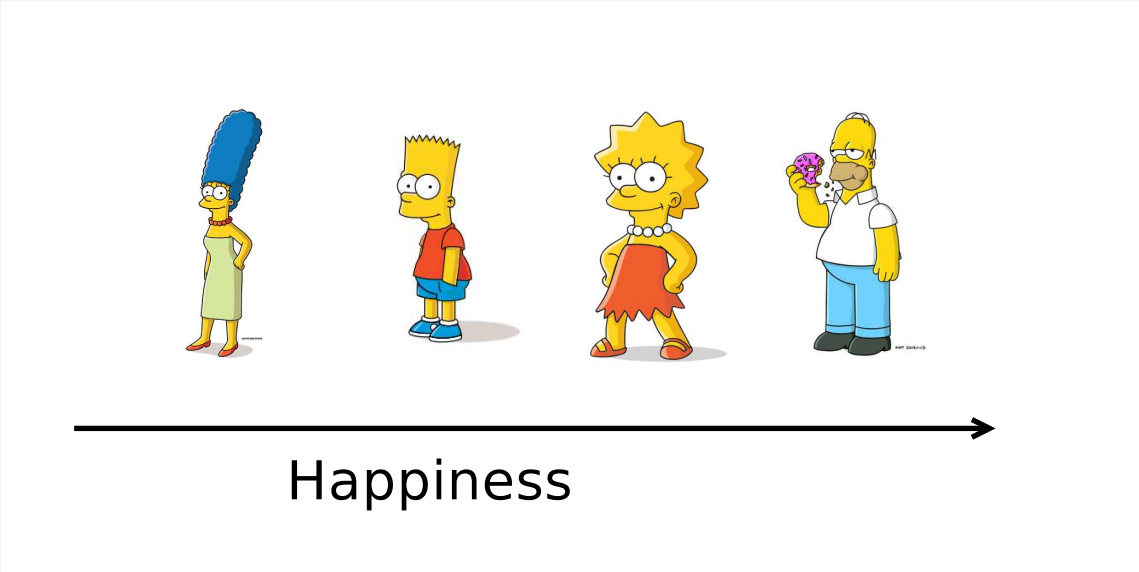

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  - Maximum Matching ($\in$ P)

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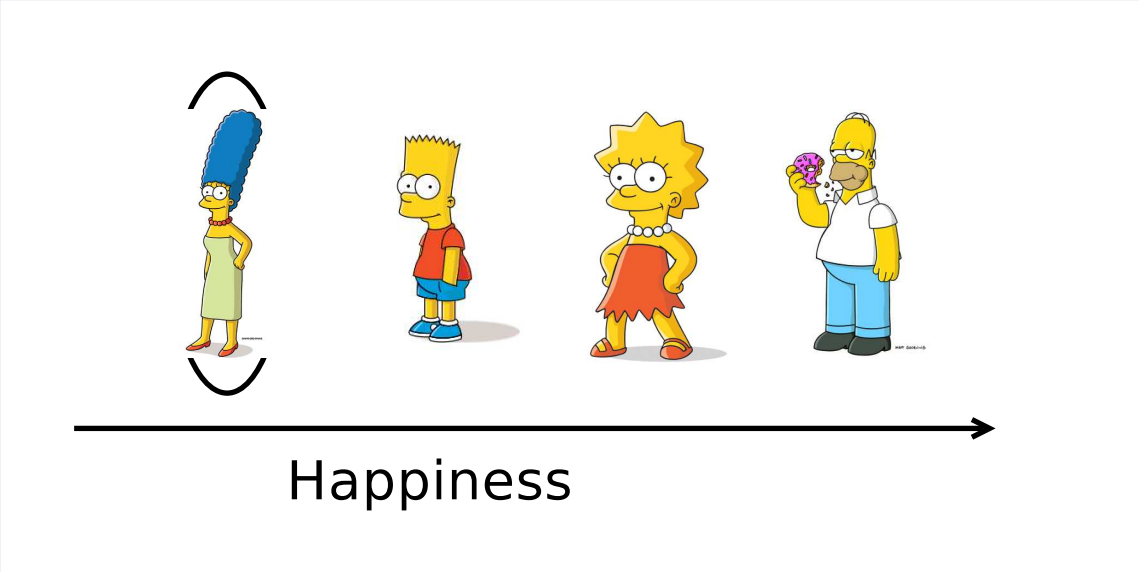

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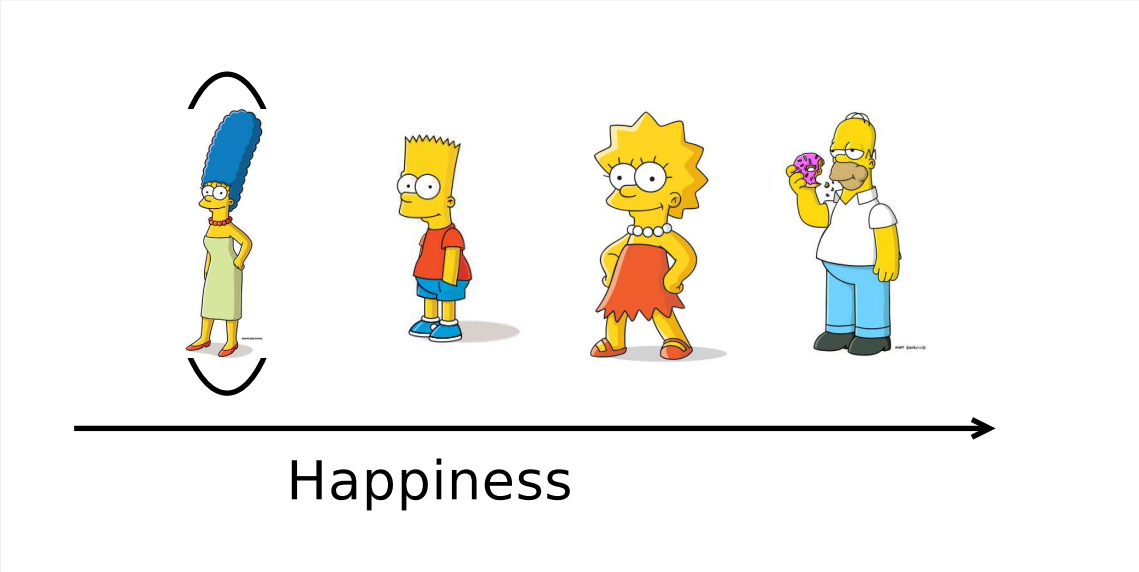

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Something between the two extremes?

OWA continued

Objective: Maximize the total utility of **the bottom 50%** of agents

- Strikes a balance between two extremes

OWA continued

Objective: Maximize the total utility of **the bottom 50%** of agents

- Strikes a balance between two extremes
- Solvable in P
  - Guess max utility in bottom 50%
  - Reduce higher utilities
  - Maximum matching

OWA continued

Objective: Maximize the total utility according to **social priorities**

Position	0-20%	20-40%	40-60%	60-80%	80-100%
Weight	14	12	7	5	1

- Algorithm for k groups
  - Guess max utility in each group
  - Adjust utilities
  - Maximum matching
- Time complexity: n^{2k}
- Optimal! (no $n^{o(k)}$)

OWA Rounding

Basic idea: “Round down” large utilities to decrease search space

- Suppose all utilities $n^{O(1)}$ (wlog)
- Replace each value u_{ij} with

$$(1+\epsilon)^{\lfloor \log_{1+\epsilon} u_{ij} \rfloor}$$

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- Running time: $O((\log n)^k \cdot \mathrm{poly}(n))$
- $(1+\epsilon)$-approximation!

- (Reminder:) Exact algorithm needs n^{2k}

Power Vertex Cover – Rounding

Joint work with:

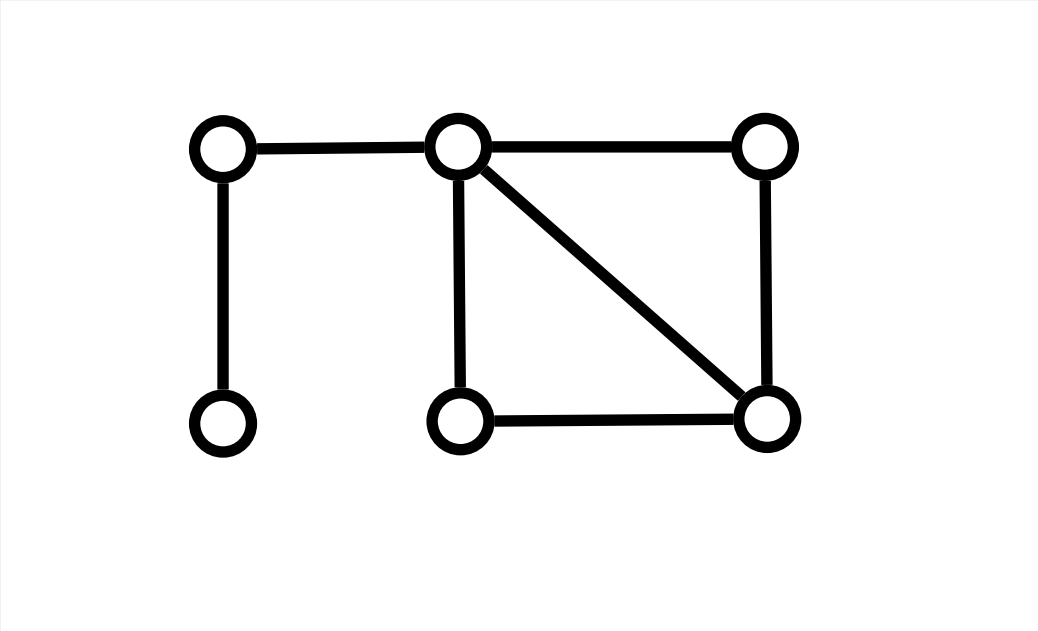
Eric Angel

Evripidis Bampis

Bruno Escoffier

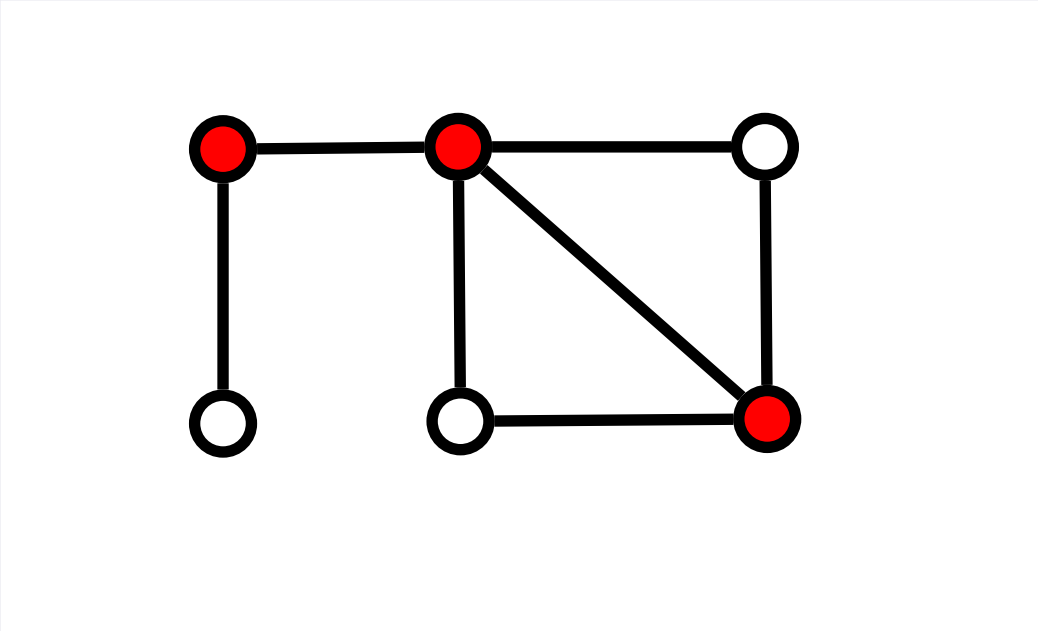
"Parameterized Power Vertex Cover", WG '16

Power Vertex Cover – Rounding



Vertex Cover: Select vertices that touch all edges

Power Vertex Cover – Rounding



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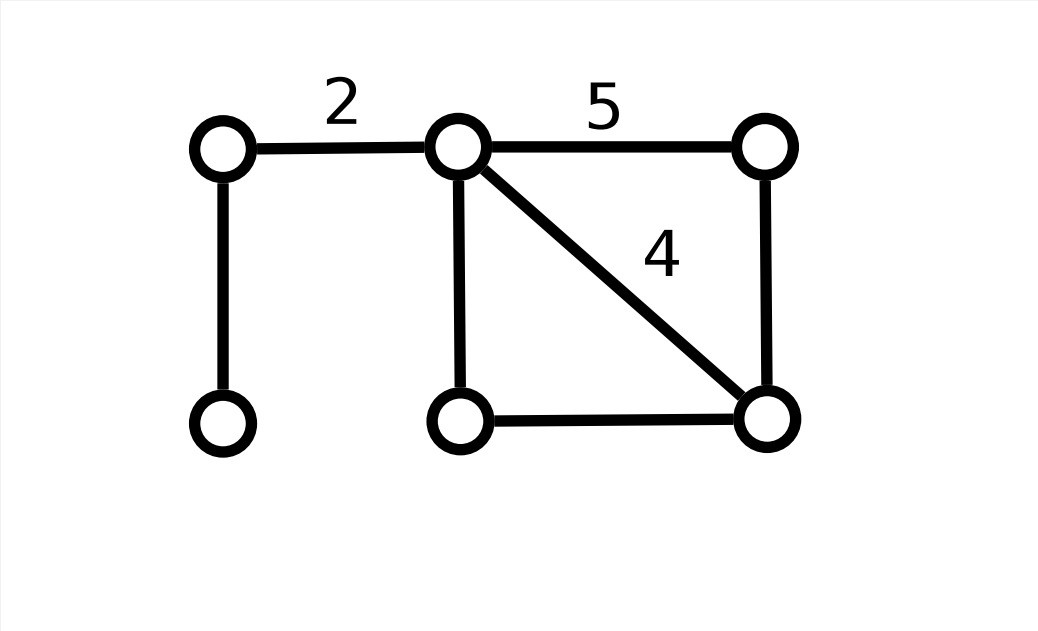

Power: Some edges **demand more power** to be covered

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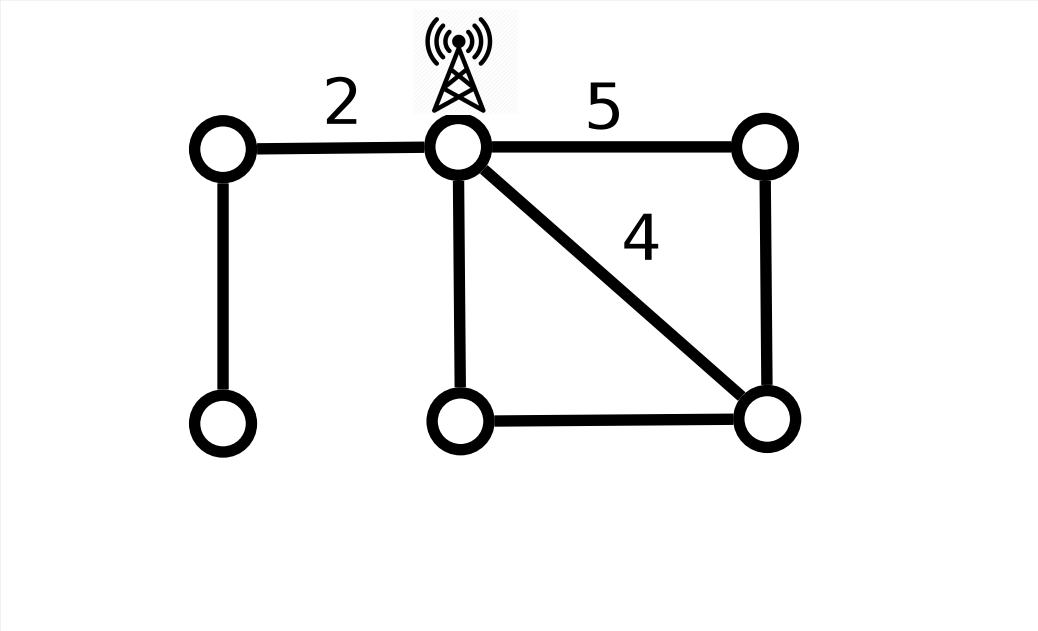

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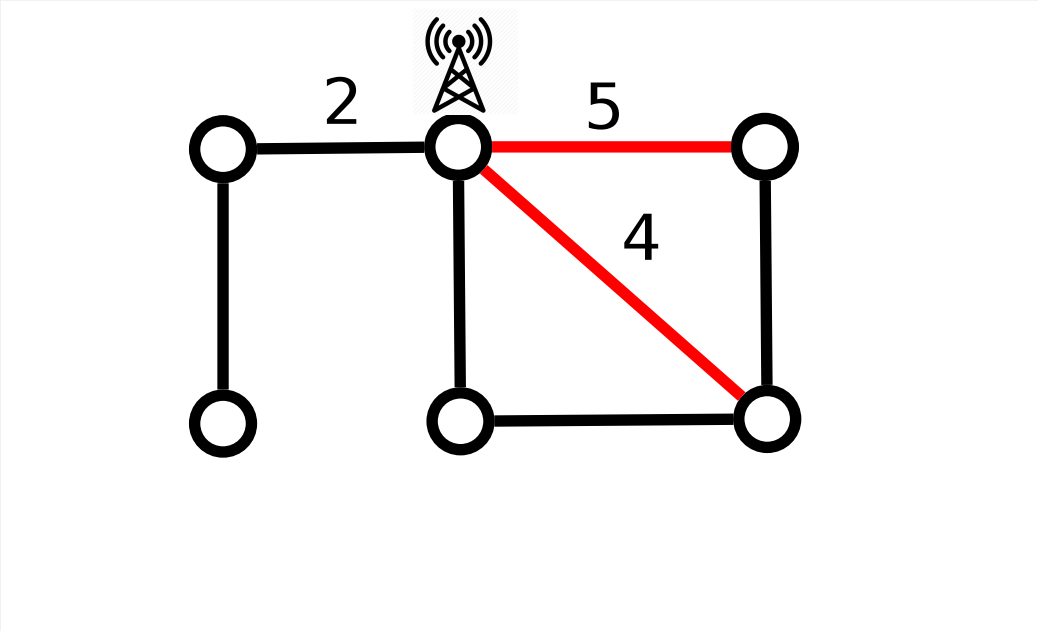

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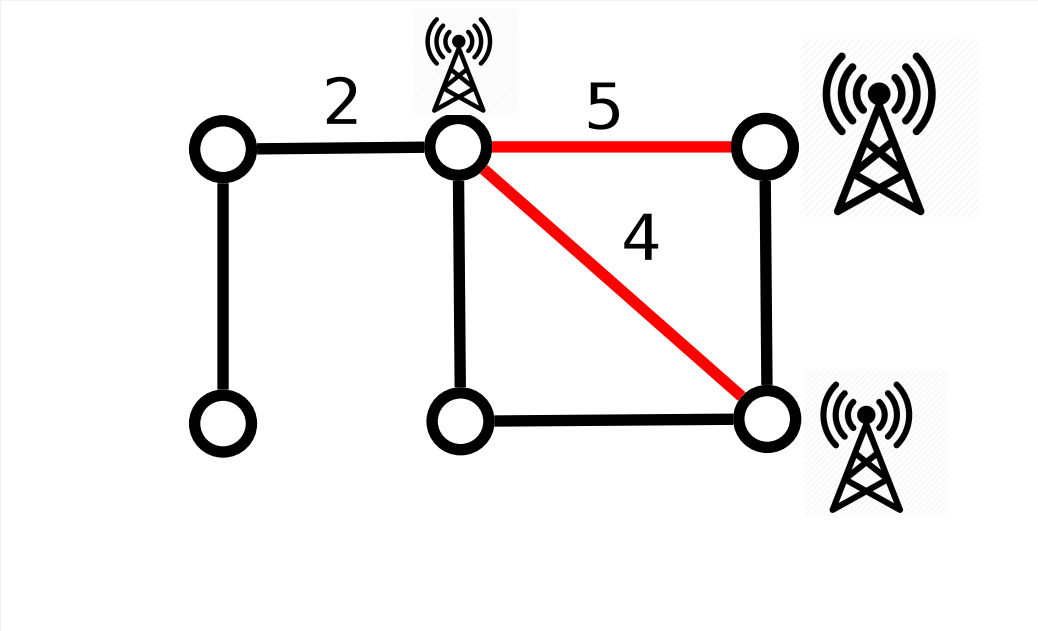
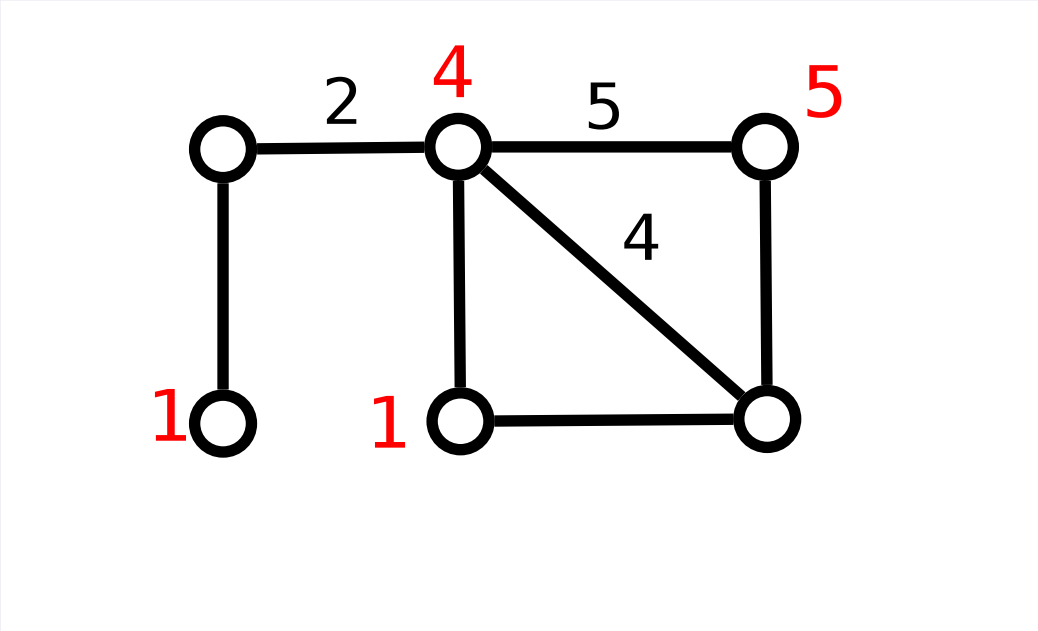


Power Vertex Cover: Must decide which vertices get power

. . . and how much

Power Vertex Cover – Rounding



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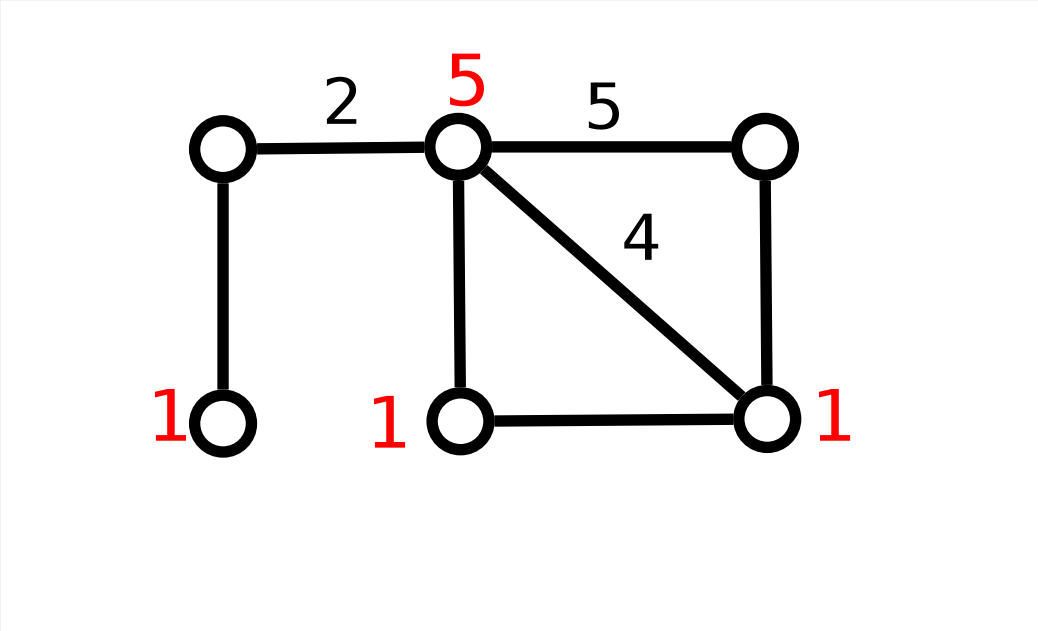

Formal Definition:

$$\min \sum p(v)$$

$$\max\{p(u), p(v)\} \geq d((u,v)) \;\; \forall (u,v) \in E$$

Motivation

- Applications to communication networks

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- Interesting Generalization of Vertex Cover
  - Note: added **non-linear** constraint
    $\max\{p(u), p(v)\} \geq d((u, v)) \;\; \forall (u, v) \in E$
    - Compare: $p(u) + p(v) \geq d((u, v))$
  - Is this problem really different/harder from Vertex Cover?
    - Admits 2 approximation
    - In P for bipartite graphs [Angel et al. ISAAC '15]

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  - What about Parameterized algorithms?
    - Vertex Cover is **flagship** problem
    - Compare: Weighted VC, Capacitated VC, Connected VC, ...

Power Vertex Cover – Hardness

- We show hardness for treewidth (n^t lower bound)
- Consider the class $\mathrm{Trivial} + t$ vertices, result still holds.

Power Vertex Cover – Hardness

Theorem: There is no $n^{o(t)}$ algorithm for PVC (under ETH)

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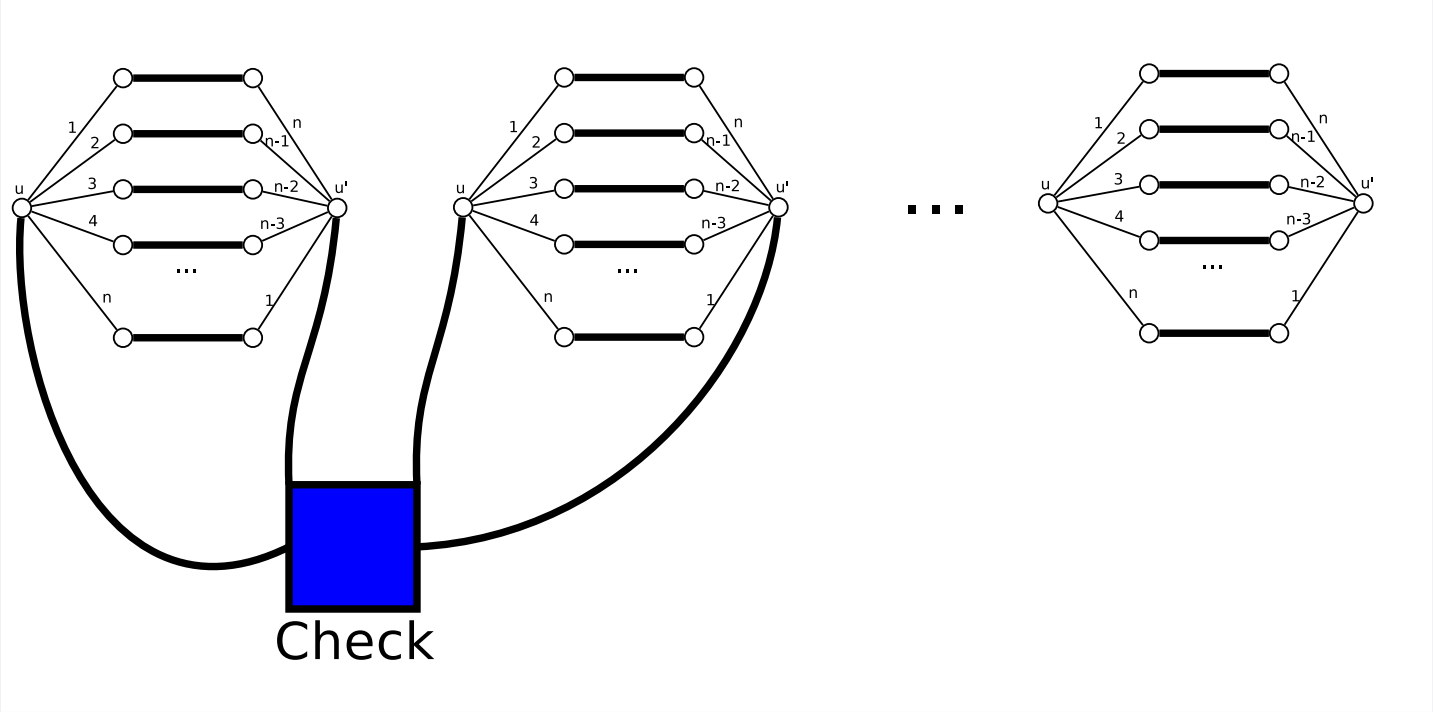

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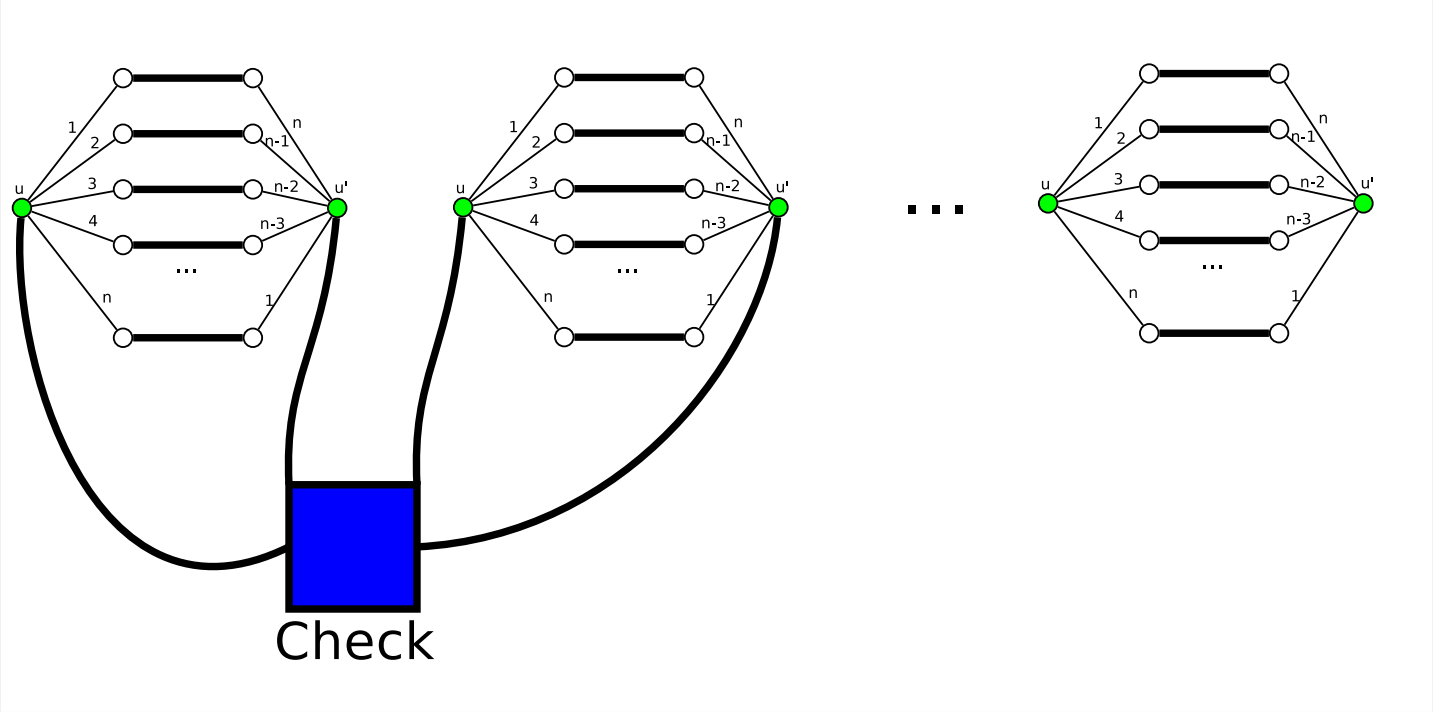

The only thing that matters is the power level of just k vertices!

Treewidth doesn't work!

Treewidth Algorithms

Treewidth doesn't work!

Actually it's not so bad...

Treewidth Algorithms

Easy **Exact** Algorithms

- $(\Delta + 1)^{tw} n$ time
- $(M + 1)^{tw} n$ time (M=maximum weight)

Main observation: Each vertex has limited number of reasonable power values.

(These running times are optimal)

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(These running times are optimal)
Can we do better?

FPT **Approximation** Scheme

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- Main idea: **Rounding**
  - Instead of power value p for each vertex store $\lfloor \log_{1+\epsilon}(p) \rfloor$
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  - At most a $(1+\epsilon)$ factor from correct value
  - If $M = n^{O(1)}$ running time $(\log n/\epsilon)^{tw}$
  - (If not, easy: think Knapsack)

Treewidth Algorithms

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Bottom line: Fast FPT algorithm for W-hard problem, only $(1+\epsilon)$ error!
(This is part of a more general technique [L. ICALP '14])

Huge but Soft Lower Bounds

Temporal Spanning Trees

Joint work with:

Dimitris Fotakis

Vangelis Paschos

Temporal Spanning Trees

Input: T copies of the same graph, with different edge weights
Output: T spanning trees
Objective: Min weight cost + buying cost

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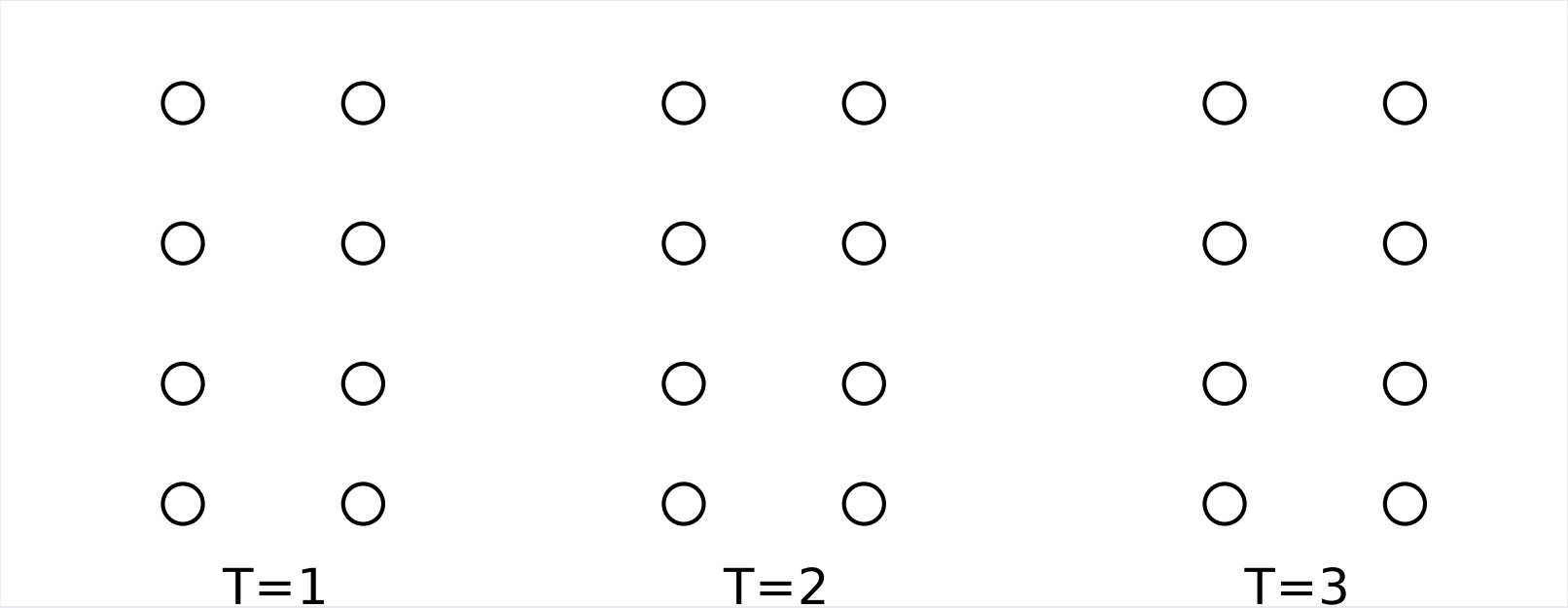

Example: $T = 3$, buy$= 1$

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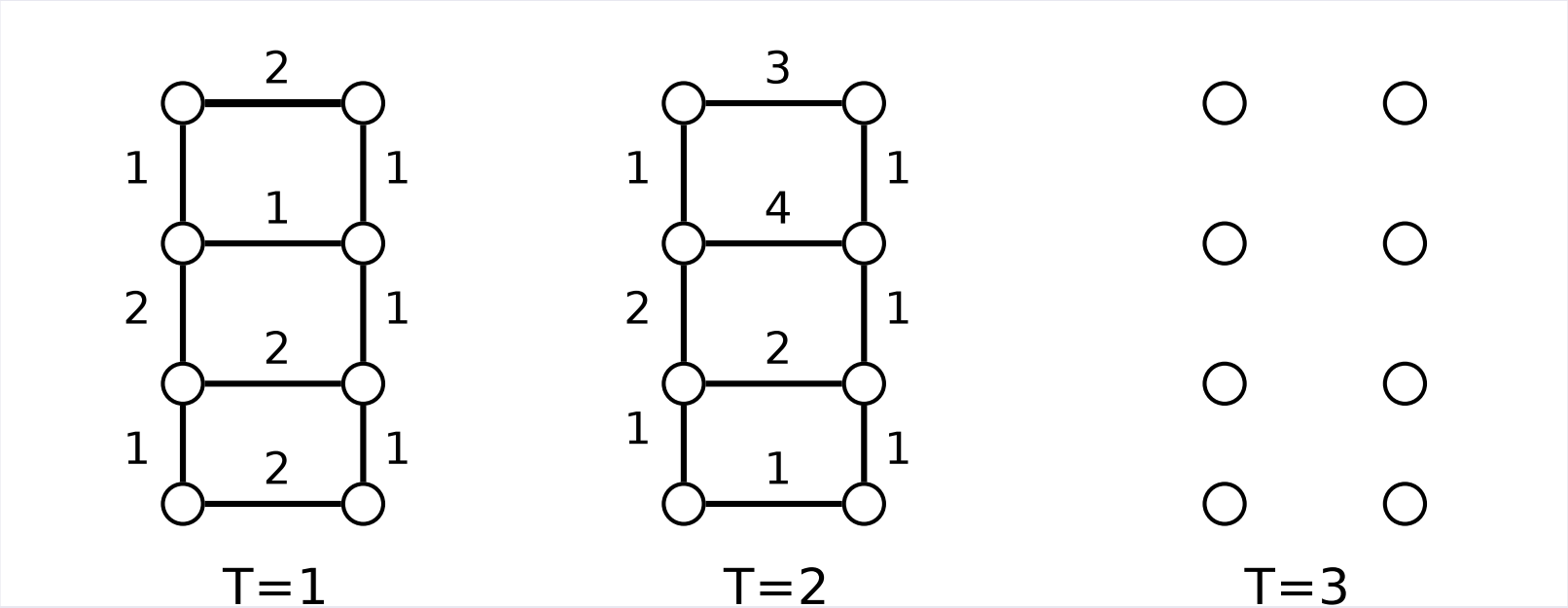

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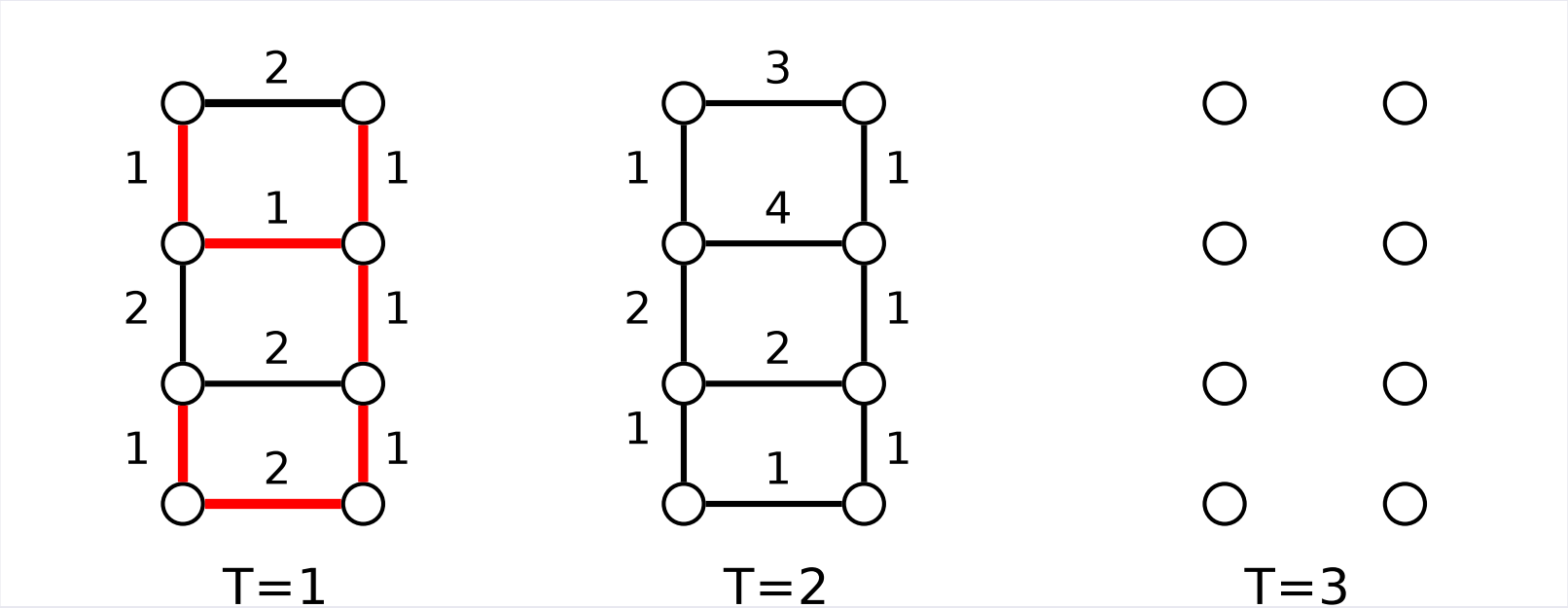

Example: $T = 3$, buy$= 1$
Cost = 8

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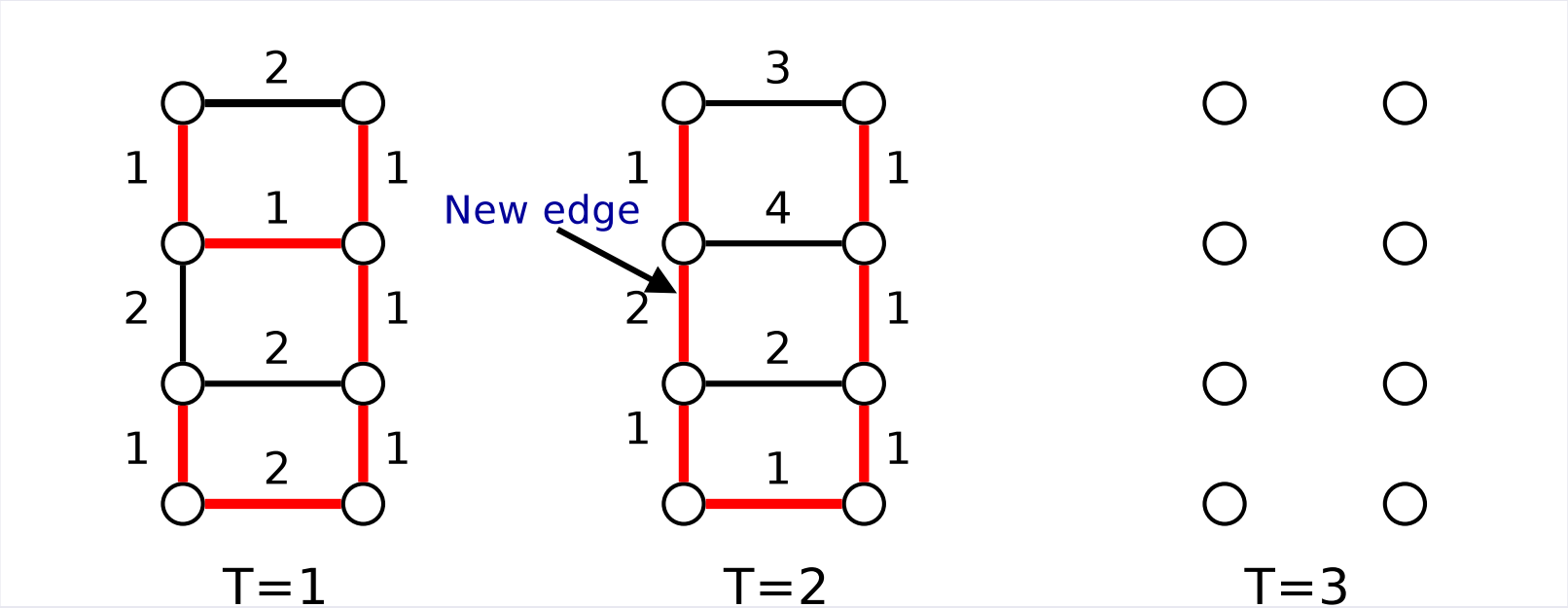

Example: $T = 3$, buy$= 1$
Cost = 8+8+1

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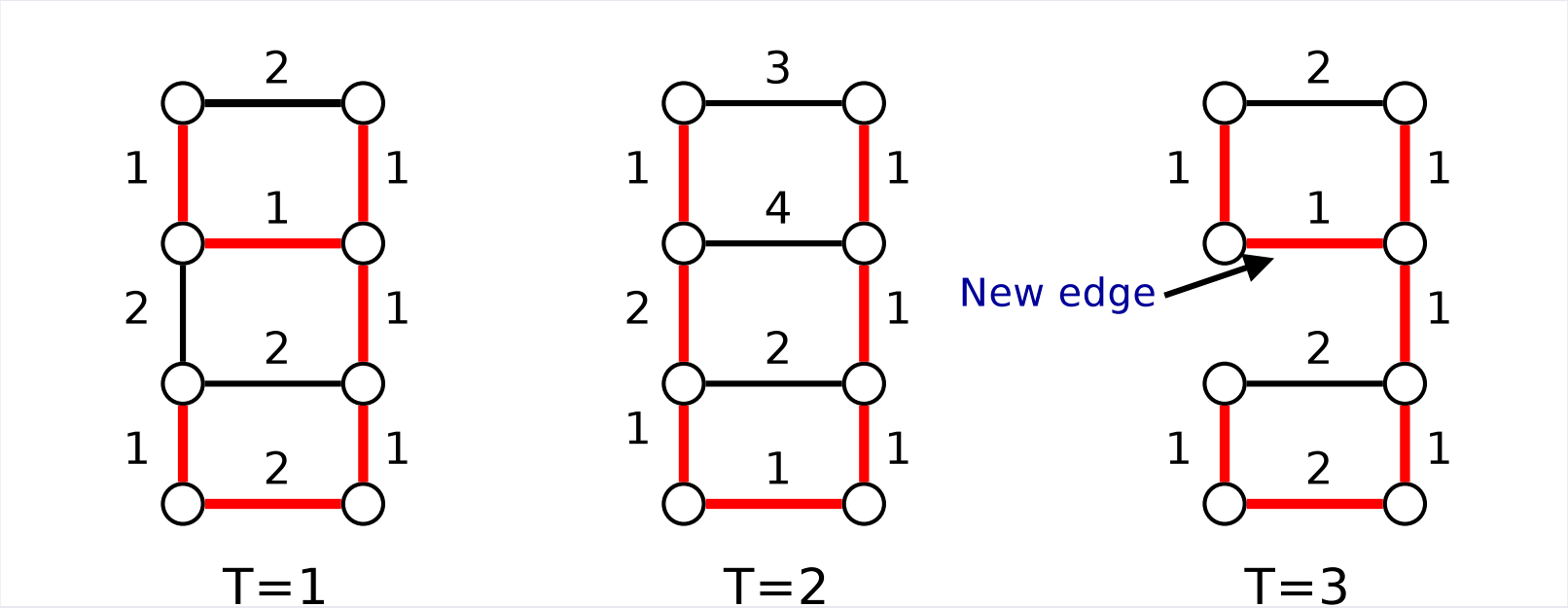

Example: $T = 3$, buy$= 1$
Cost = 8+8+1+8+1=26

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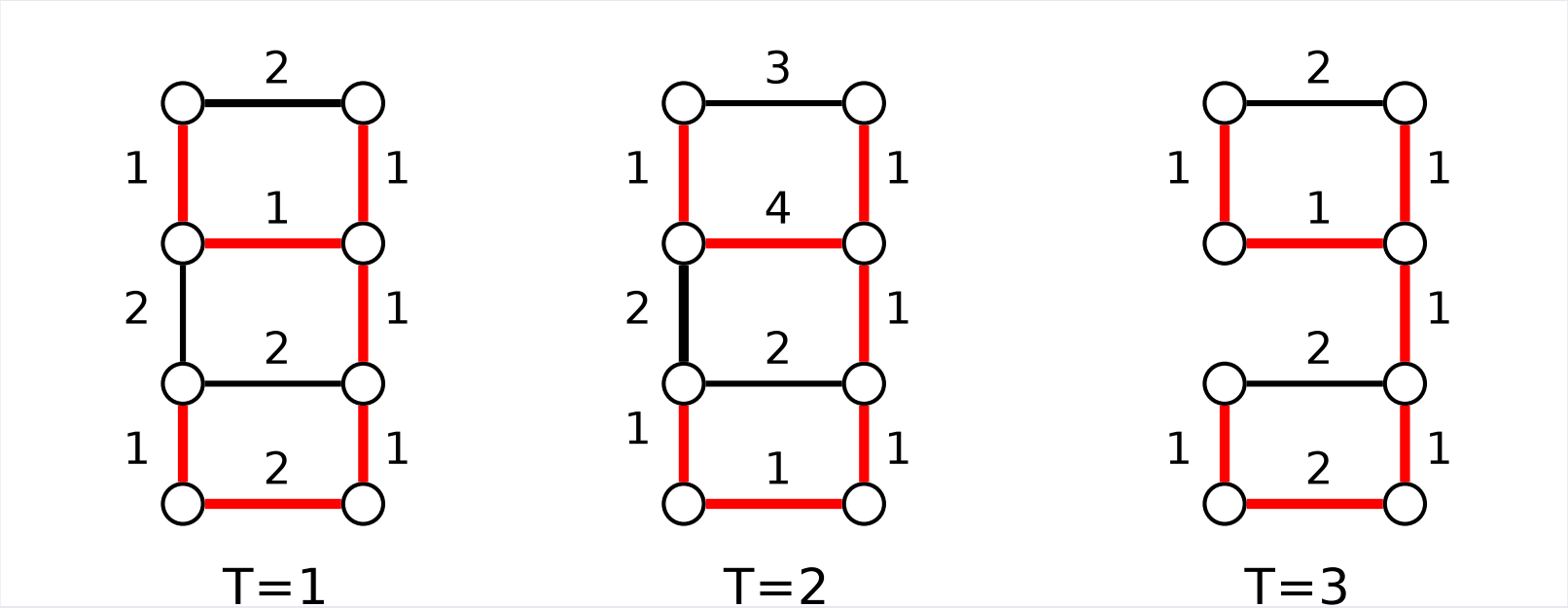

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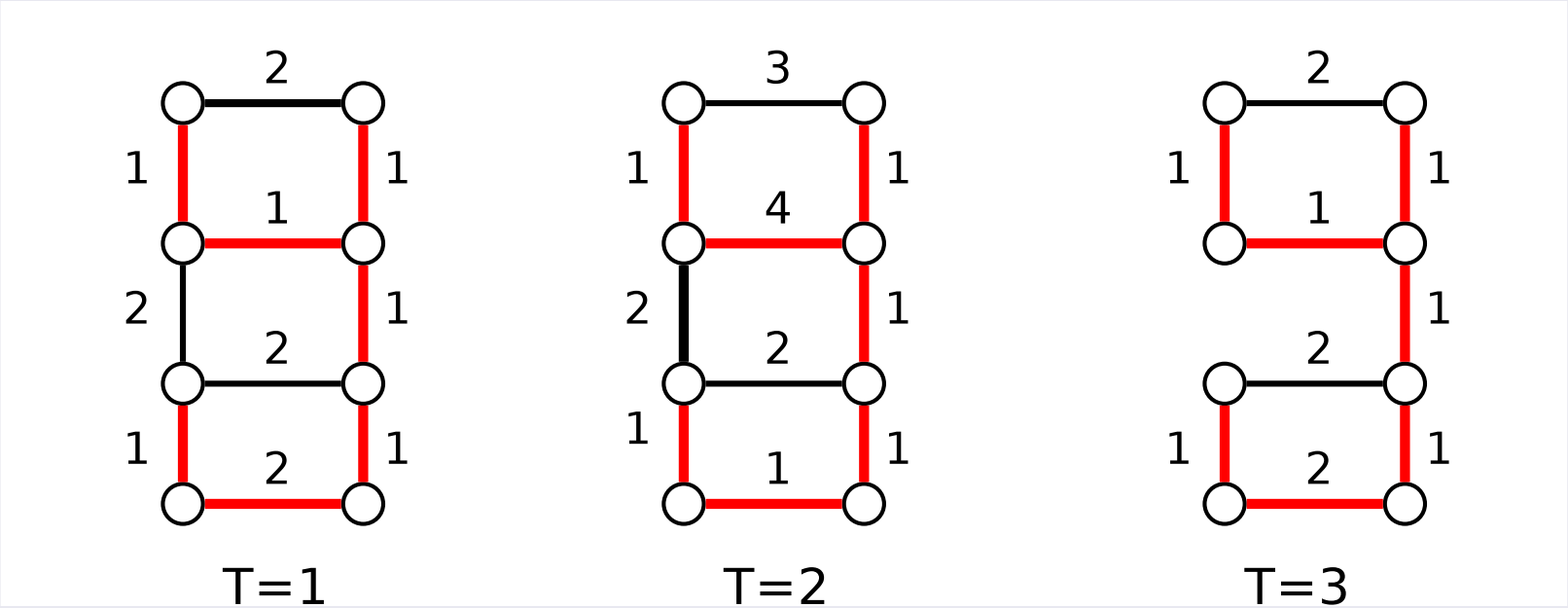

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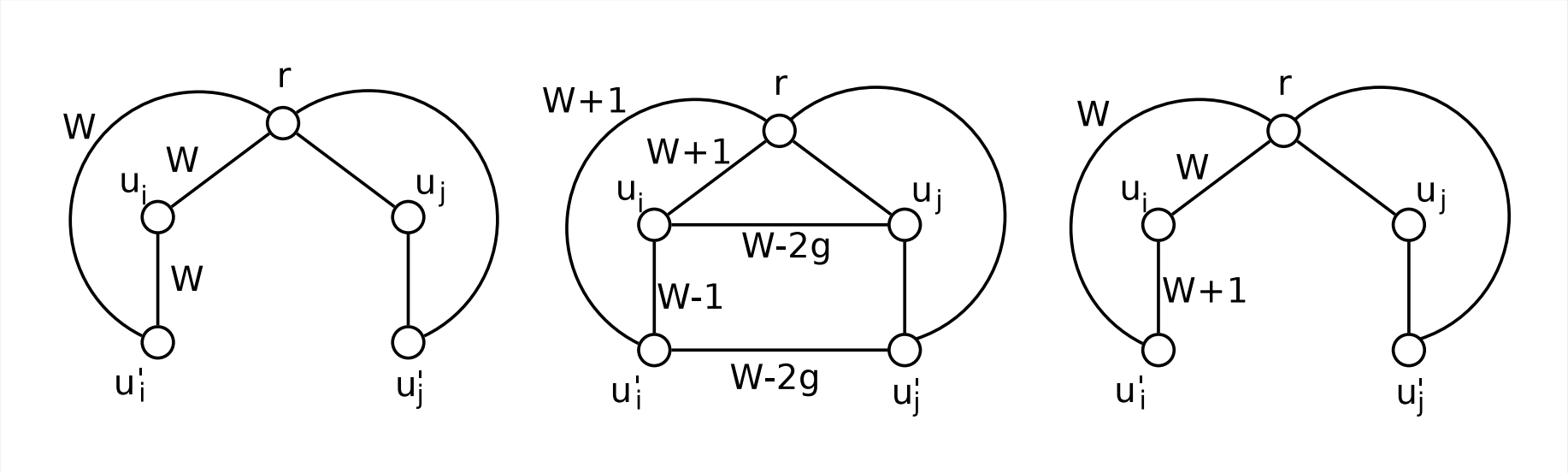
Cost = 8+10+0+8+0=26

- In P when buy=0
- In P when buy=∞

Temporal Spanning Tree – APX-hardness

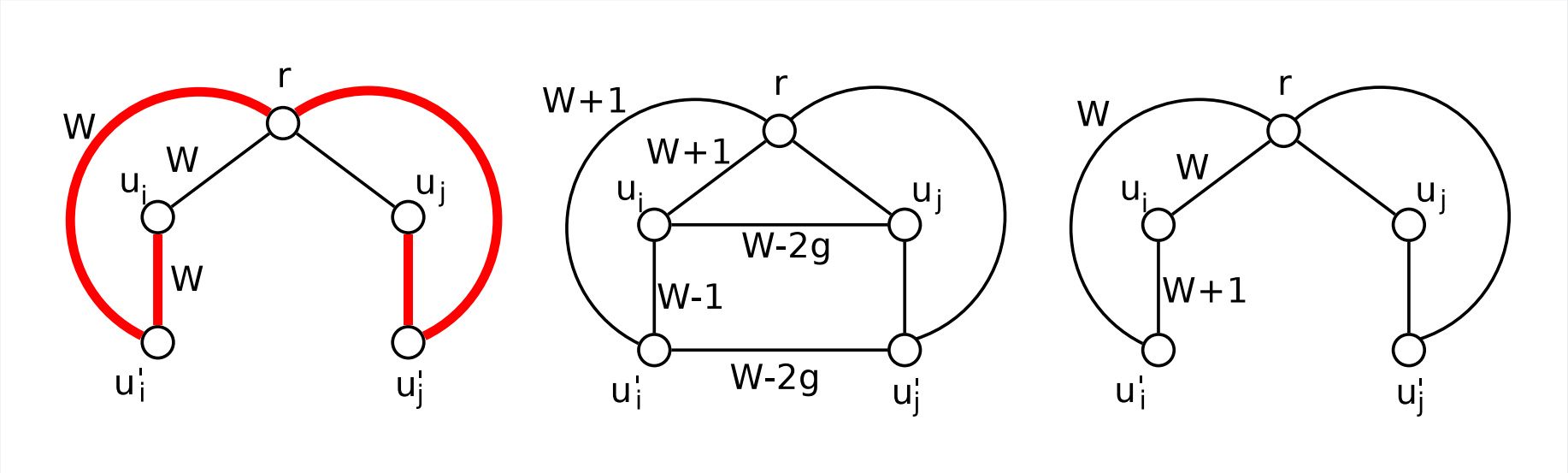


APX-hardness

Gadget that implements NAND constraint

Choices represented by “internal” edges used

Temporal Spanning Tree – APX-hardness



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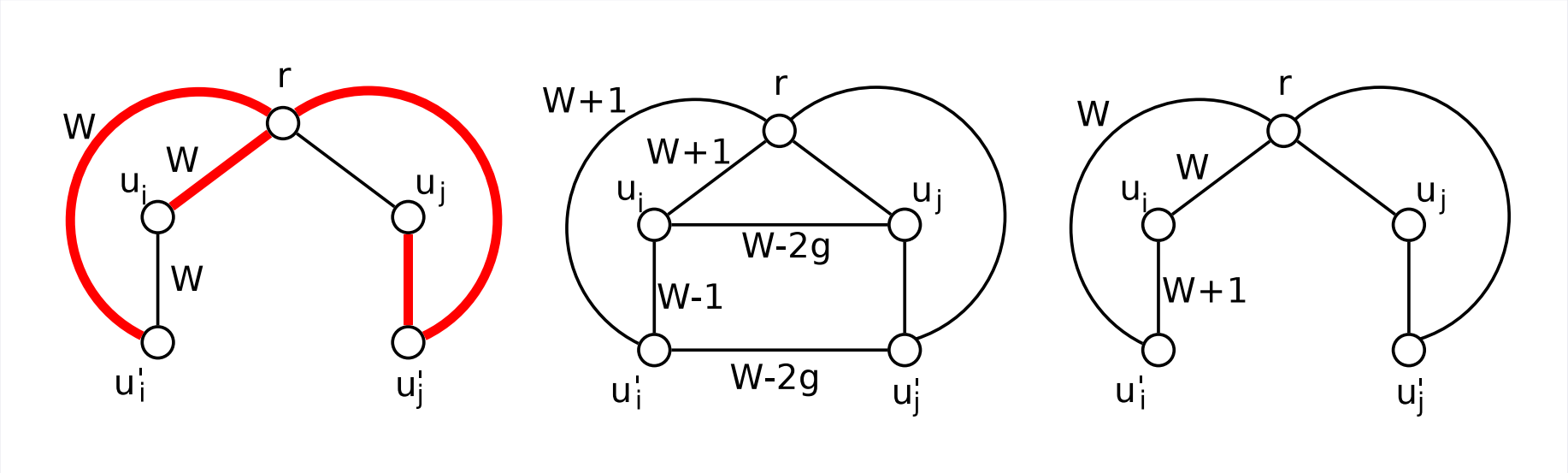
Both used (BAD)

Temporal Spanning Tree – APX-hardness



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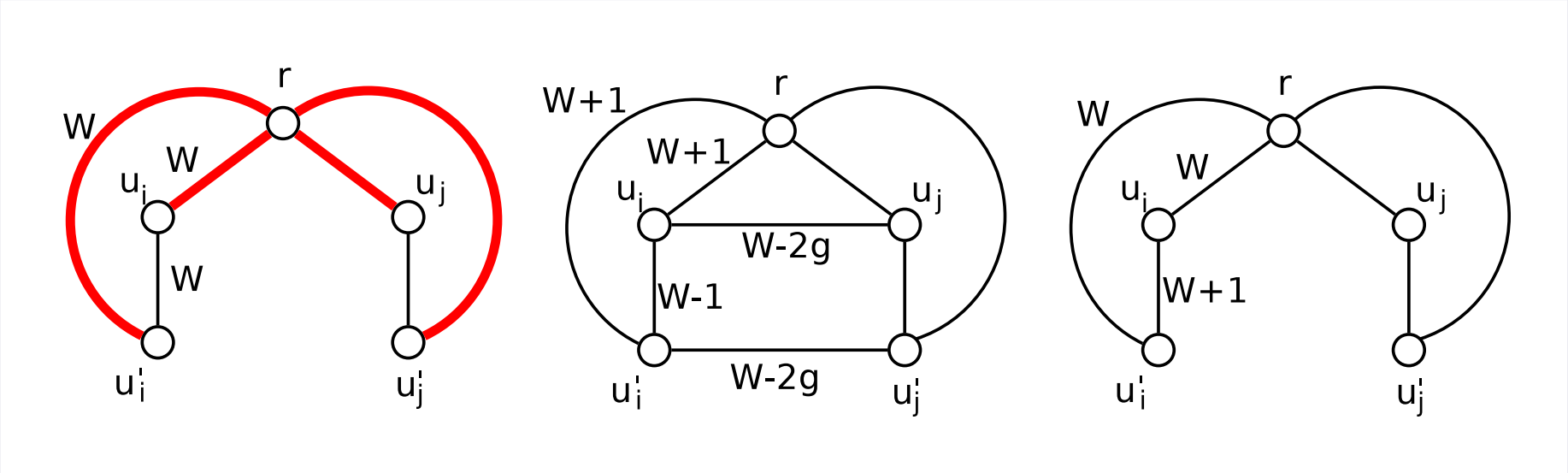
One used (Good)

Temporal Spanning Tree – APX-hardness



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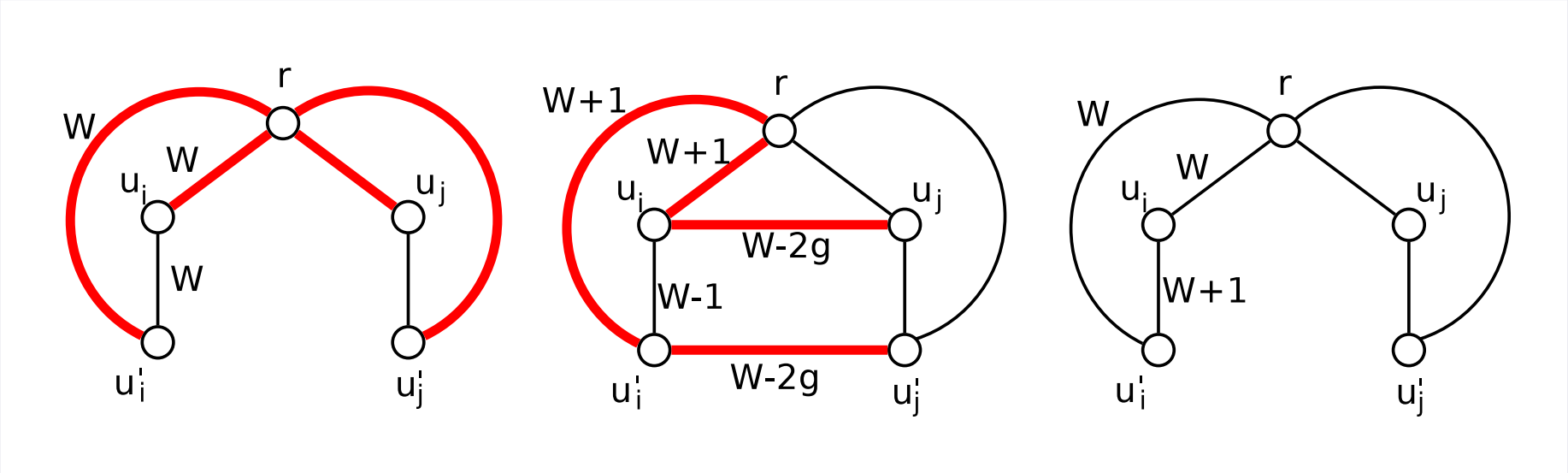

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Goal: Replace edges from r with temporary cheap edges

Do this as much as possible

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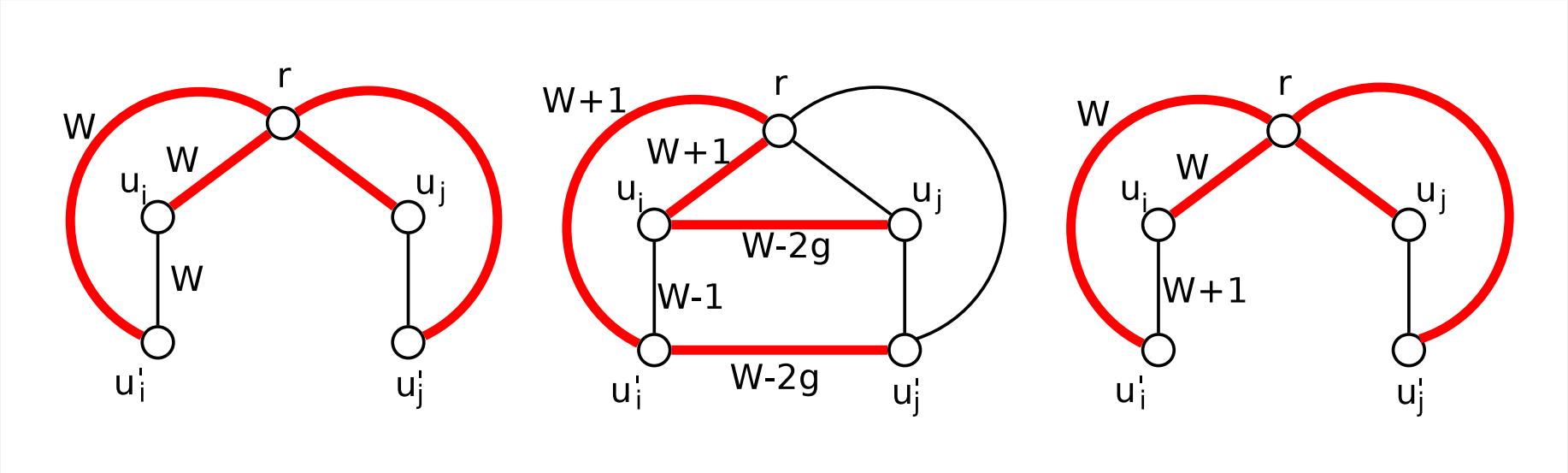

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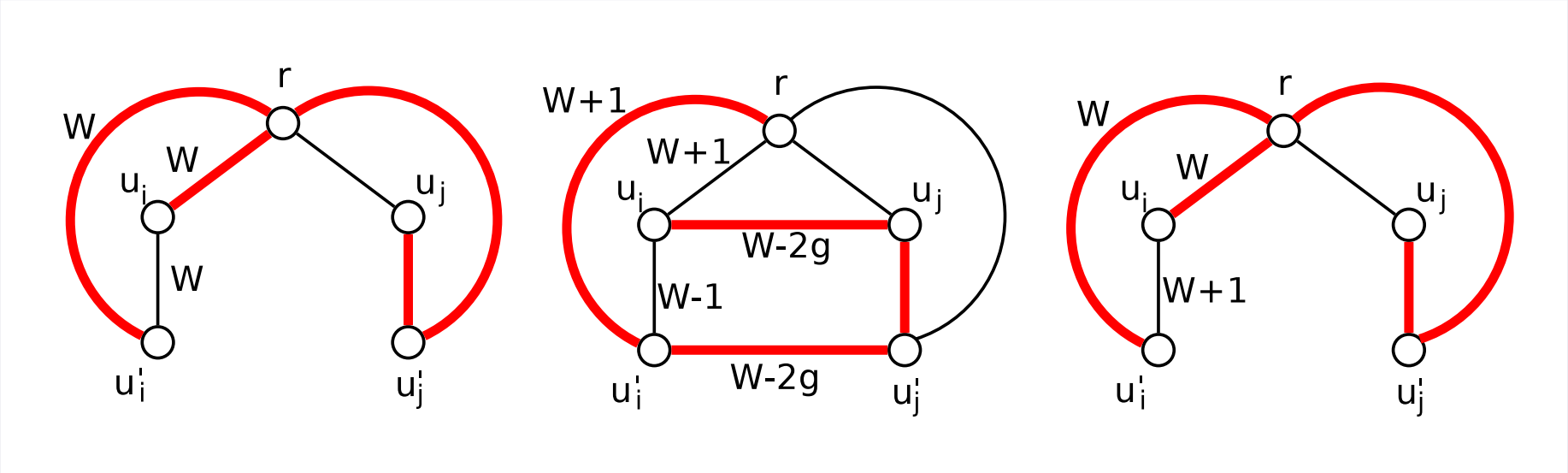
OK for no internal edges

Temporal Spanning Tree – APX-hardness



APX-hardness

Gadget that implements NAND constraint

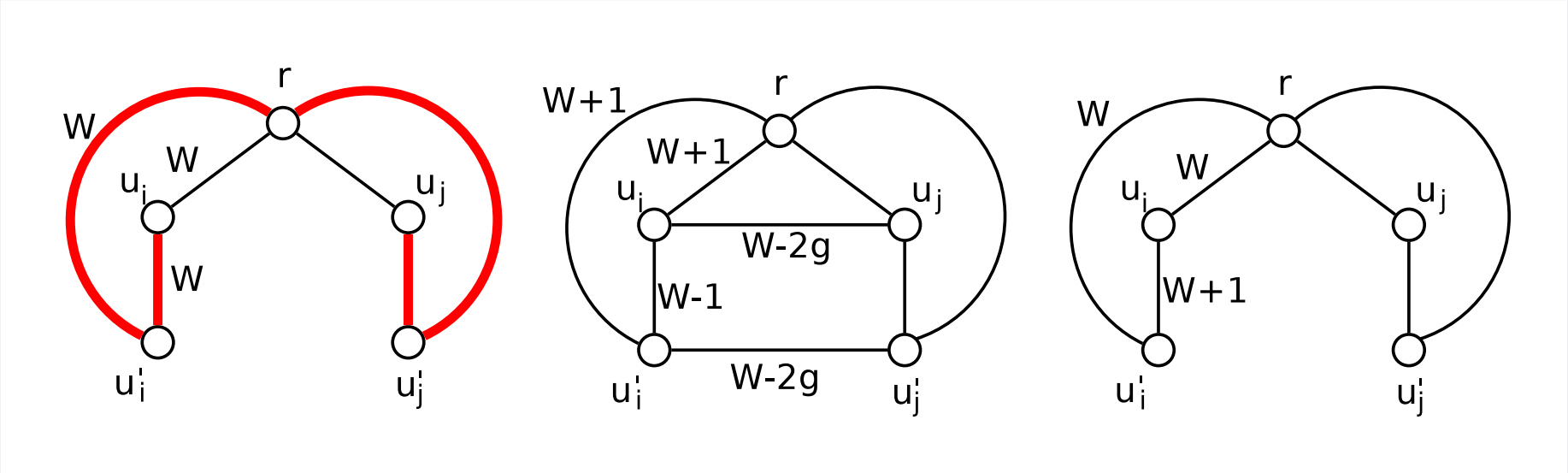
Choices represented by “internal” edges used

Goal: Replace edges from r with temporary cheap edges

Do this as much as possible

OK for one internal edge

Temporal Spanning Tree – APX-hardness



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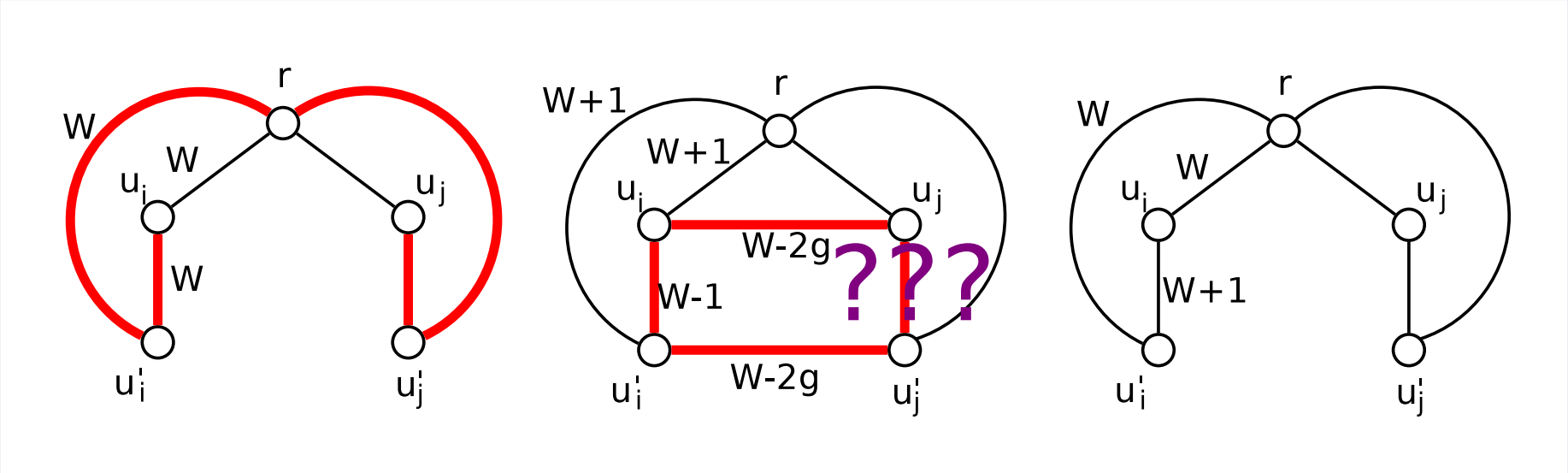

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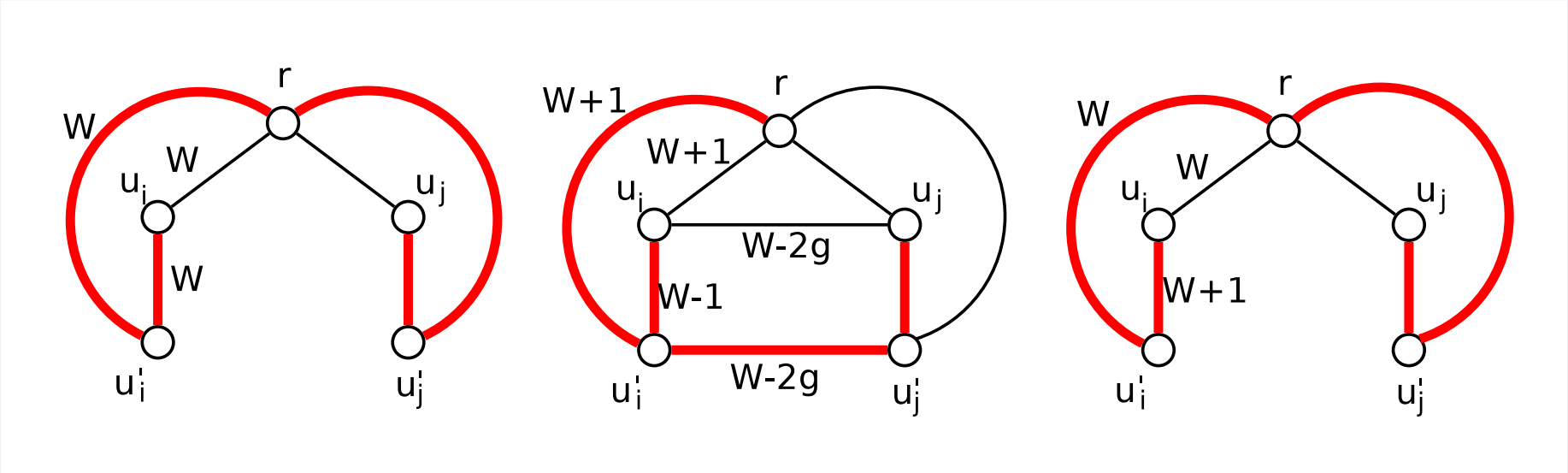

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Temporal Spanning Tree – APX-hardness

APX-hardness

- Reduction from Independent Set
  - One “internal edge” for each vertex
  - One “testing phase” (NAND) for each edge
- Can make $T = O(1)$ by testing in parallel
- Reduction is linear

Theorem: There is no Approximation Scheme for TST in time $2^{n^{1-\delta}}$ (under ETH) even for constant rounds.

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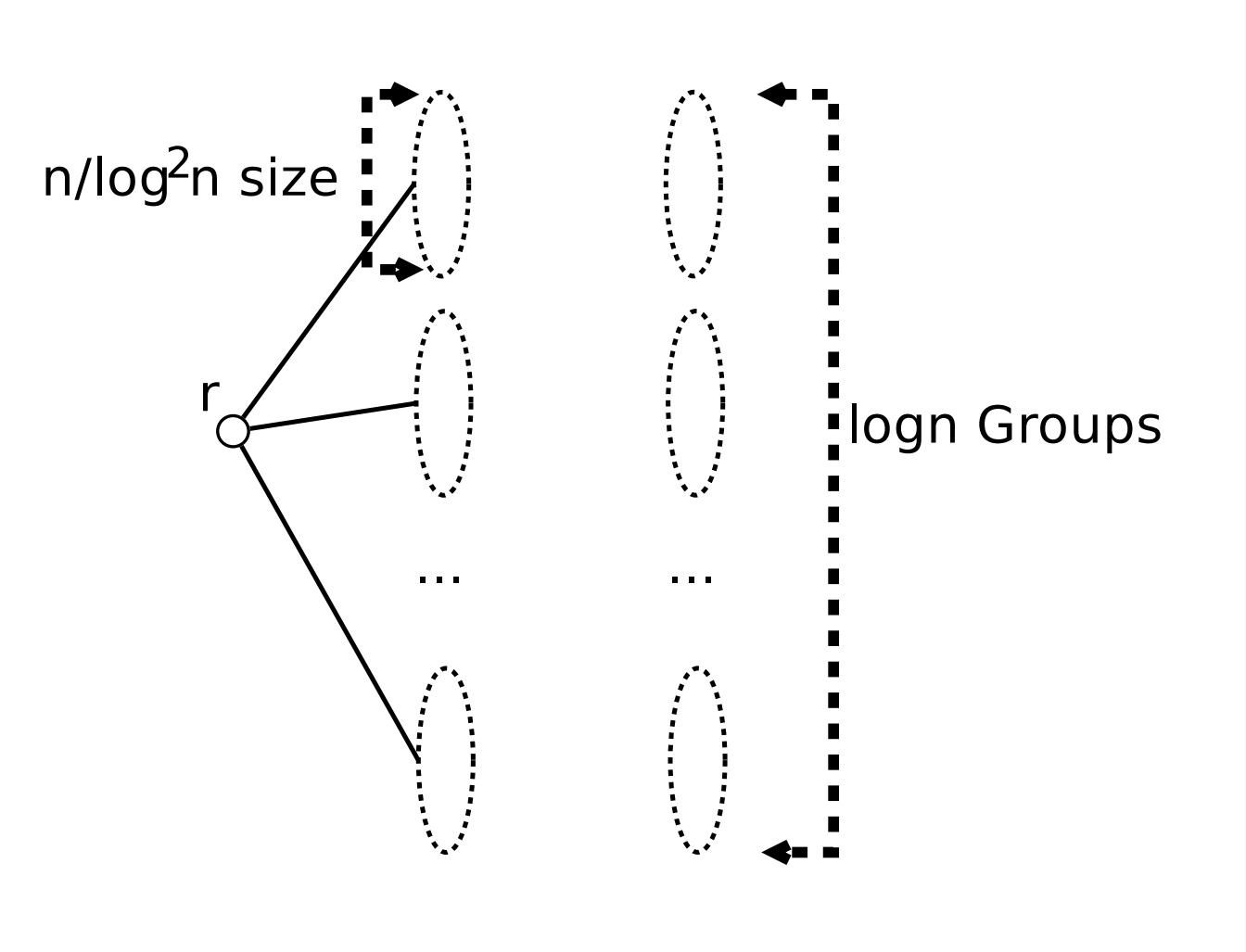
- For $T = 2$?
- Constant factor approximation algorithm?

Temporal Spanning Tree – Super-Exponential Lower Bound

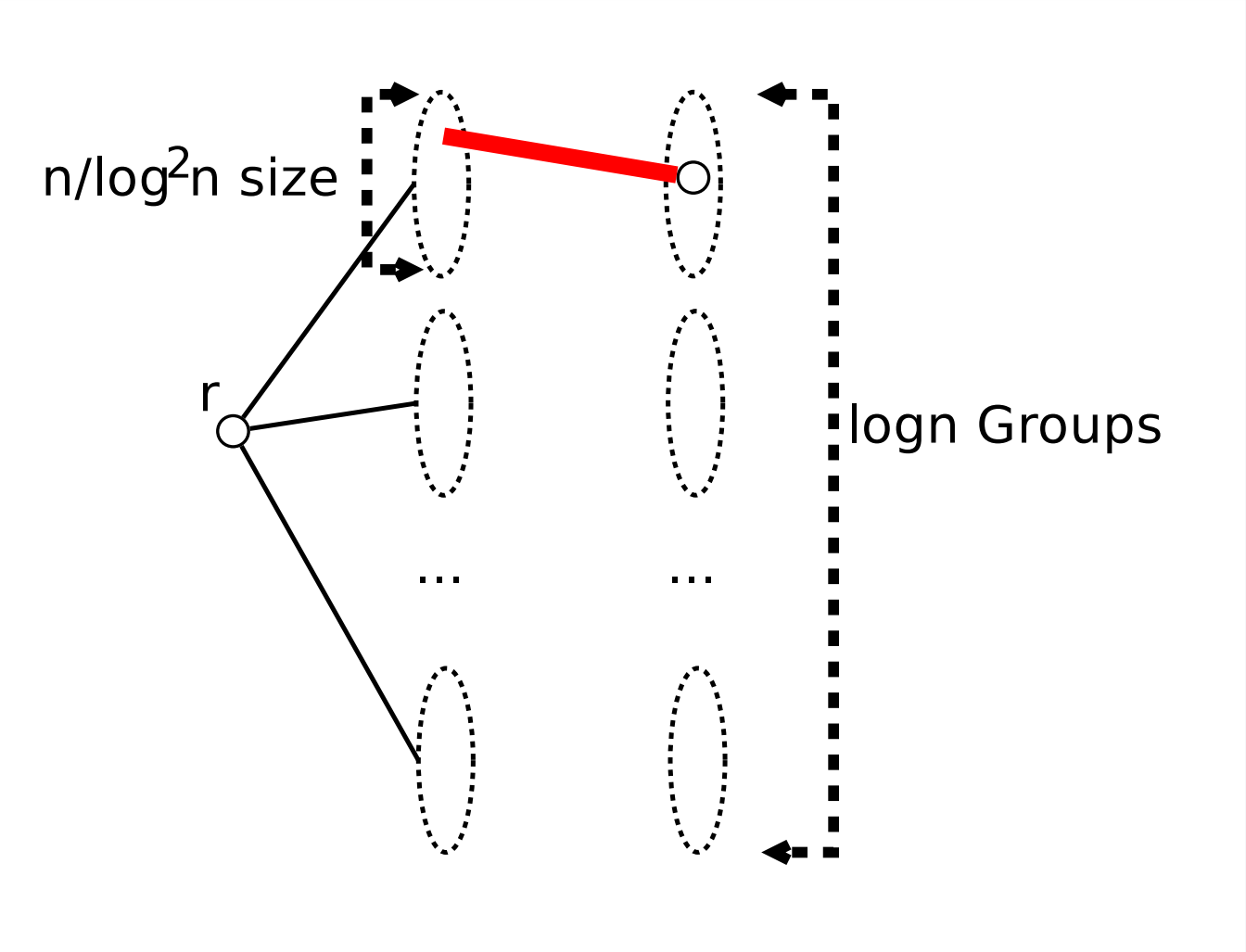
No talk is complete without a picture of a reduction that nobody understands (Daniel Marx)

Temporal Spanning Tree – Super-Exponential Lower Bound



- Represent an n-var 3-SAT formula
- Edges used on right encode $\log n$ bits
- NAND gadget catches incosistent assignments
- Total order: $O(n/\log n)$

Temporal Spanning Tree – Super-Exponential Lower Bound



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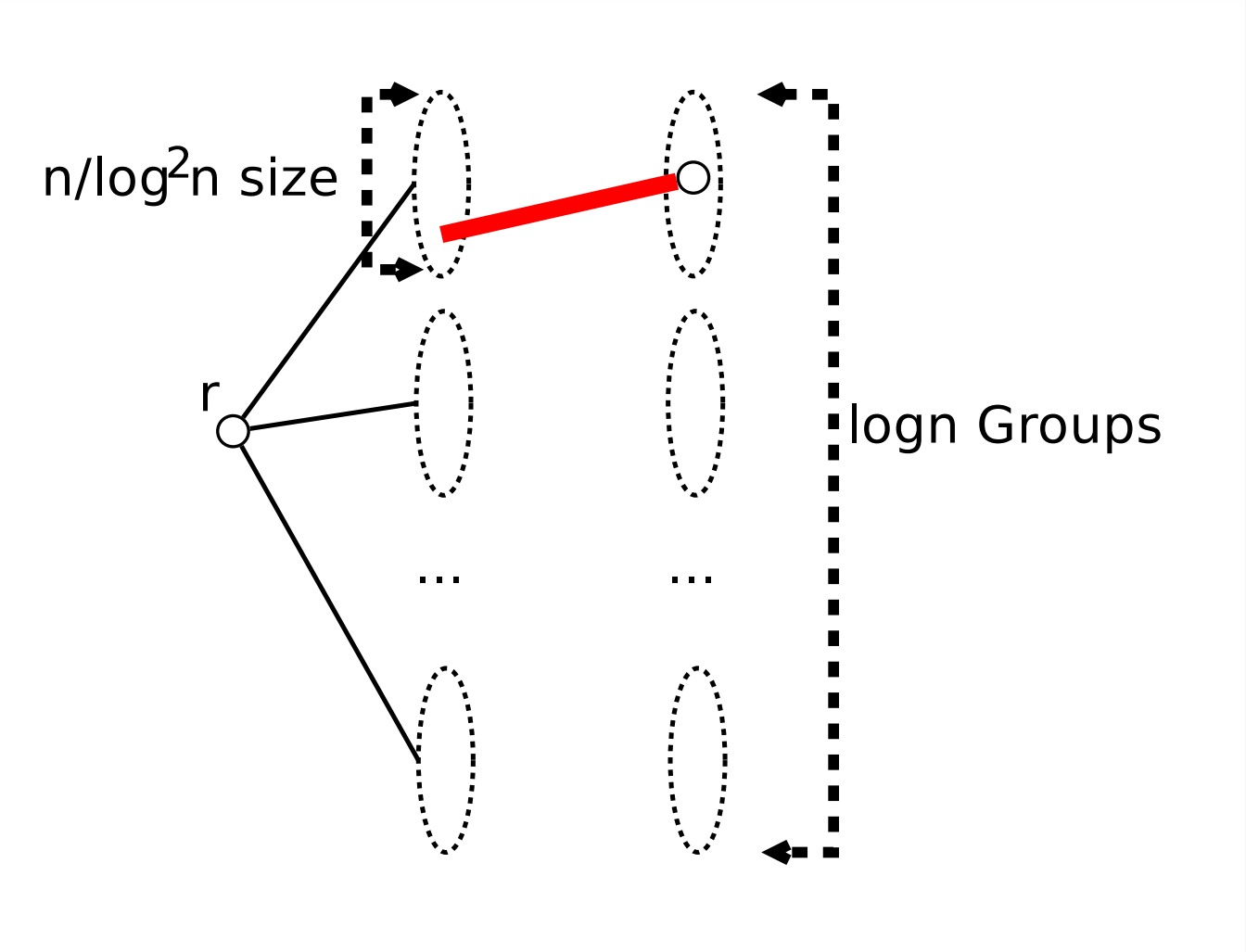

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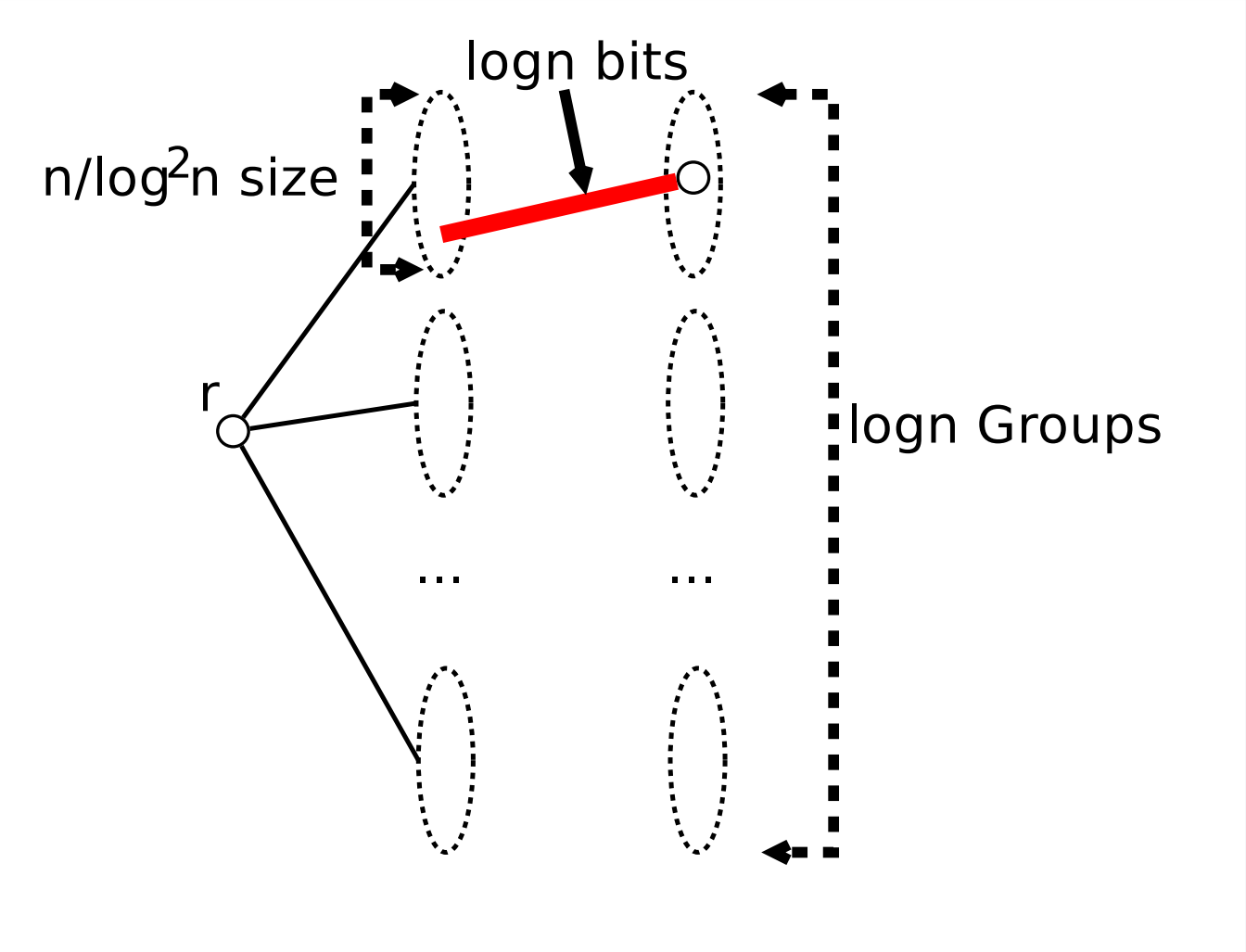

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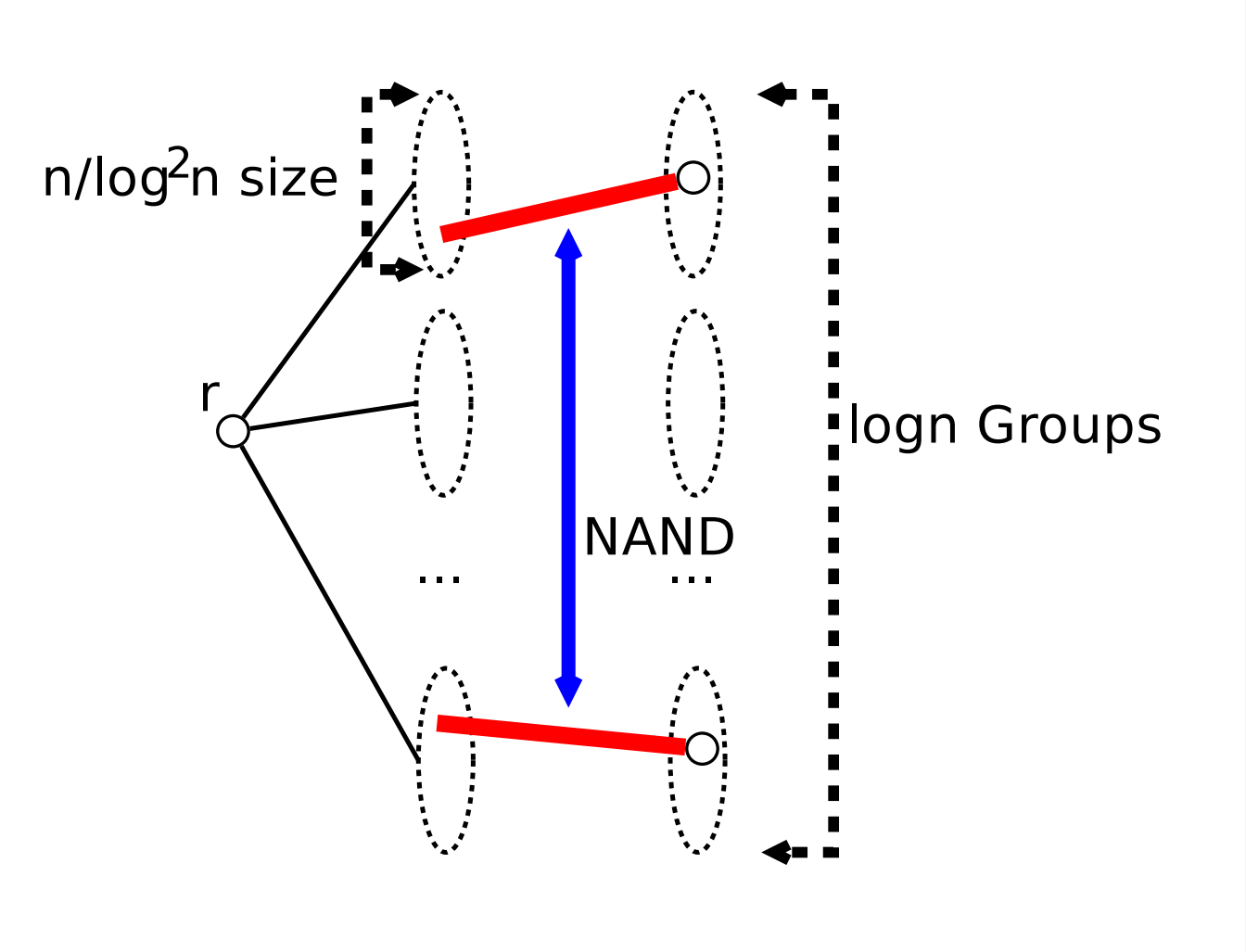

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Temporal Spanning Tree – Super-Exponential Lower Bound

- Reduction 3-SAT $\rightarrow$ TST
  - Formula with n vars
  - New graph has $n/\log n$ vertices
  - If $n^{o(n)}$ alg for TST...
  - ...then $2^{o(n)}$ alg for 3-SAT
- TST needs super-exponential time (n^n) to be solved! (under ETH)
- $n^n T$ agorithm is easy...(DP, list all spanning trees)

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What does this have to do with anything?

Temporal Spanning Tree – Few Rounds

- TST needs n^n time to solve exactly.
- Can we do c^n if $T = O(1)$?

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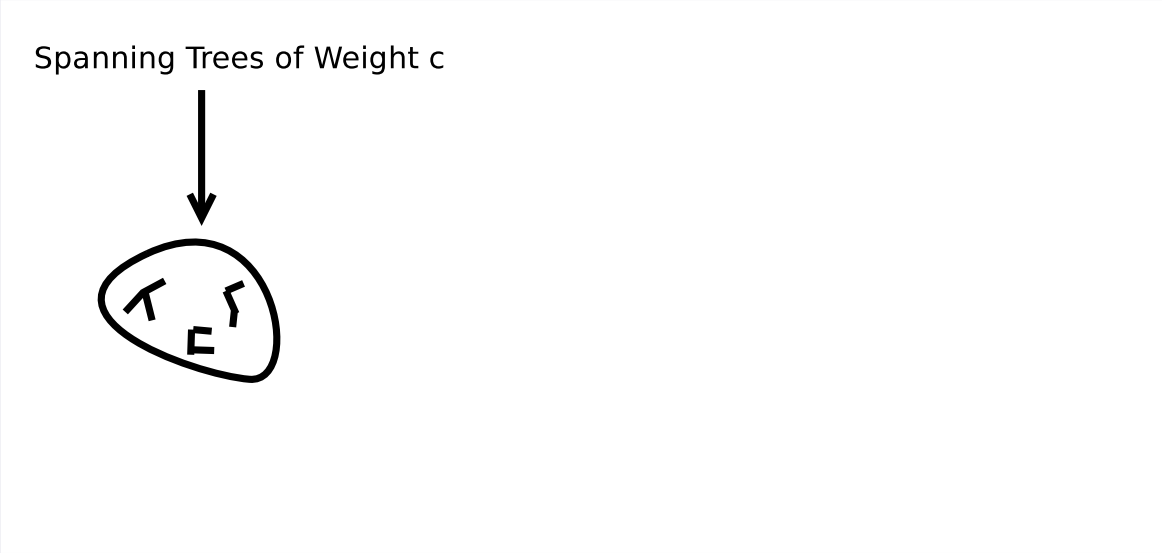

Consider the set of STs of weight c

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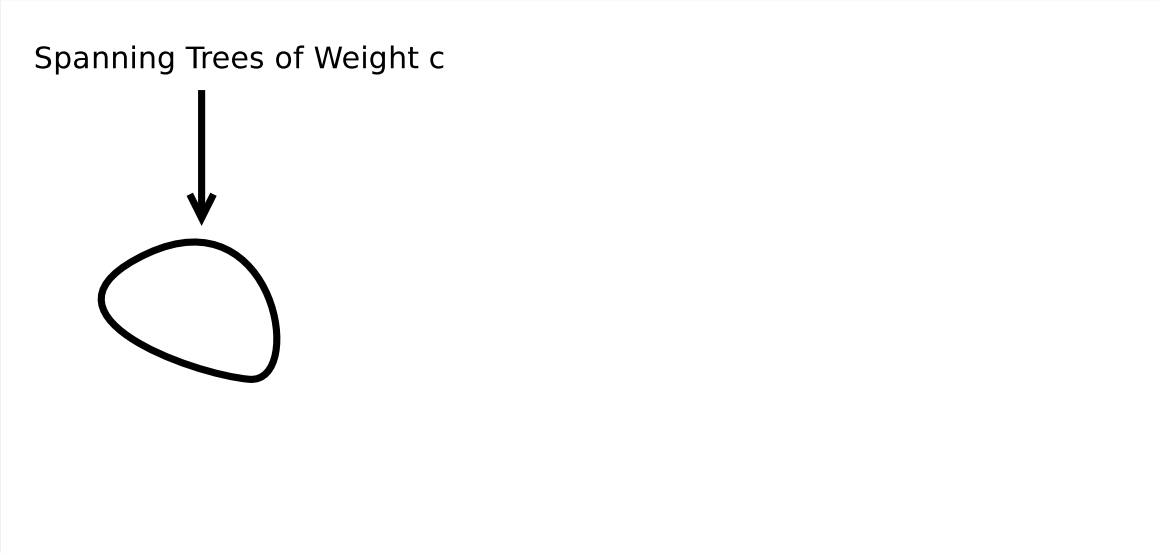

Consider the set of STs of weight c
Is this set empty?

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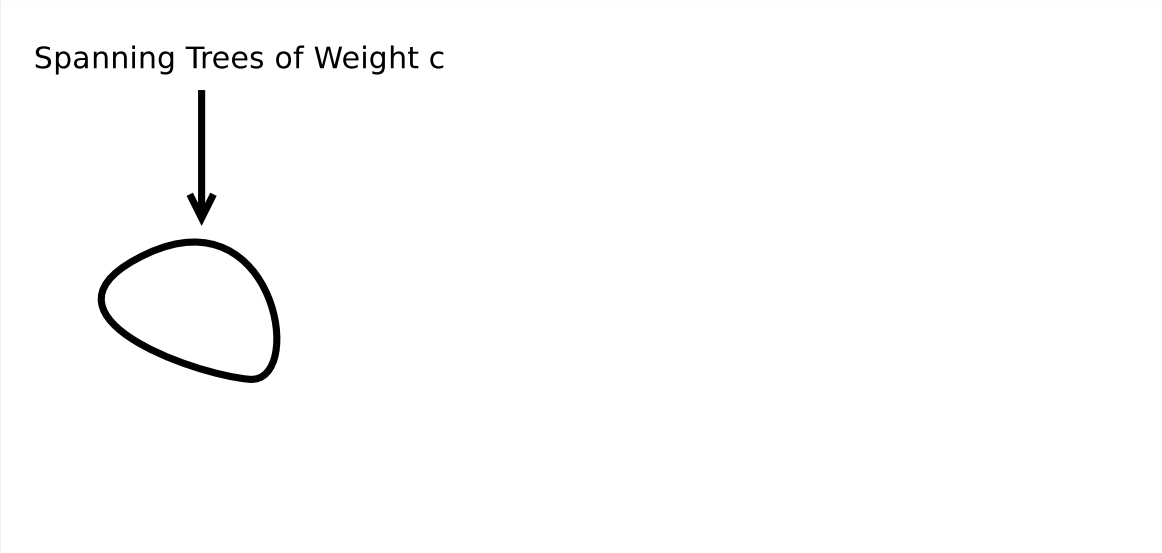

Consider the set of STs of weight c
Is this set empty?
We will decide by determining its parity!!!

Temporal Spanning Tree – Few Rounds

- TST needs n^n time to solve exactly.
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The Cut&Count method
Two problems:

- Determining the parity is potentially harder
- What if the # of solutions is even?

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The Cut&Count method
Two problems:

- Determining the parity is potentially harder
- What if the # of solutions is even?

- Bad solutions cancel out! $\rightarrow$ we can do this in c^n.
- Isolation Lemma $\rightarrow$ minimum solution is unique whp.

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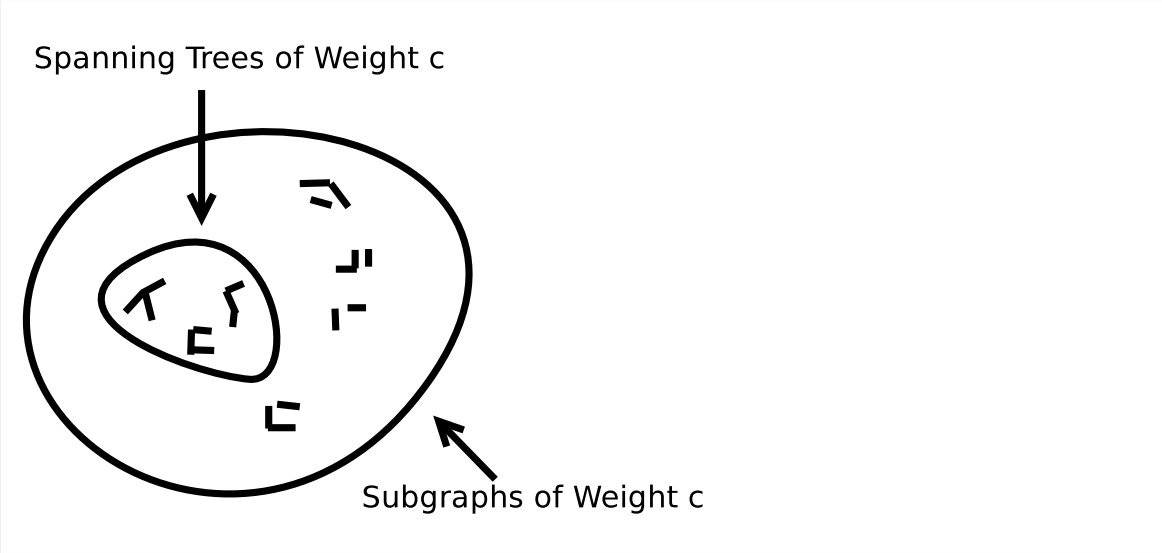
The Cut&Count method



- We can determine the size of bigger set in 2^n time (DP)

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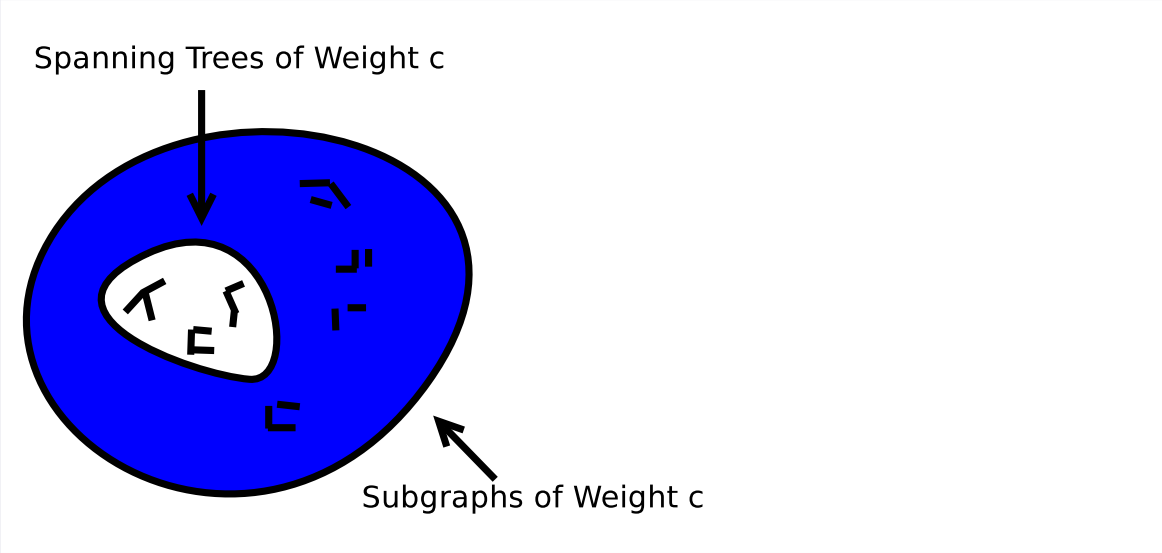

- We can determine the size of bigger set in 2^n time (DP)
- If blue area has even size $\rightarrow$ Done!

Temporal Spanning Tree – Few Rounds

- TST needs n^n time to solve exactly.
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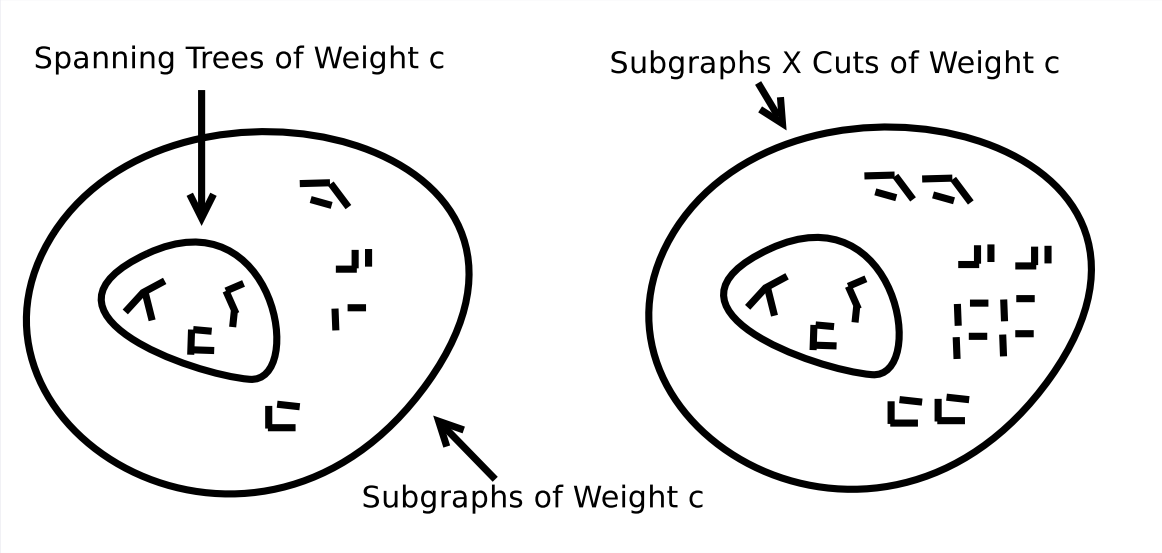
The Cut&Count method



- We can determine the size of bigger set in 2^n time (DP)
- Construct a new set:
  - Every subgraph appears one time for each cut of the graph it respects.
  - Good subgraphs only respect one cut! (the trivial one)

Temporal Spanning Tree – An Approximation Scheme

- TST is APX-hard for time $2^{n^{1-\delta}}$.
- TST needs n^n time to solve exactly.
- Can we $(1+\epsilon)$-approximate in single-exponential time? (c^n)

Temporal Spanning Tree – An Approximation Scheme

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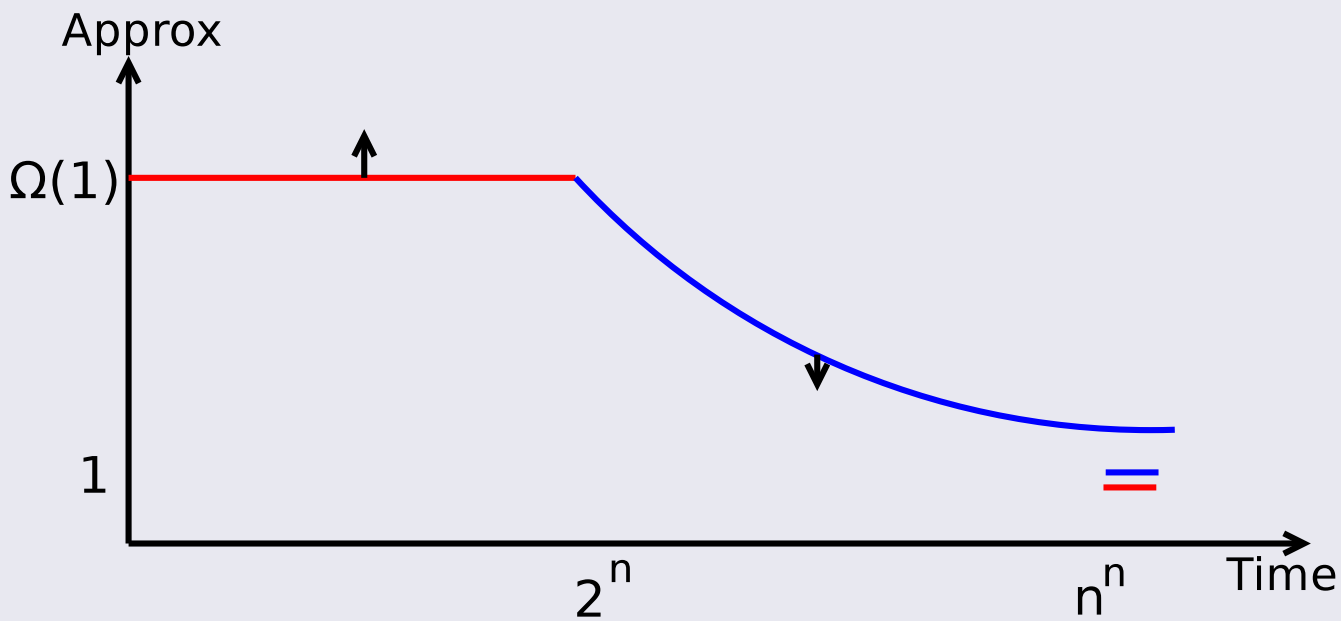
- Can we $(1+\epsilon)$-approximate in single-exponential time? (c^n)

- If $T = O(1)$ Yes! (previous slide)
- Otherwise, divide T into blocks of size $1/\epsilon$
- Solve each optimally
  - Added buying cost is at most $\epsilon \mathrm{OPT}$
- $(1+\epsilon)$-approximation in time $2^{O(n/\epsilon)}$.

- Is this trade-off optimal??

Temporal Spanning Tree – Summary



- At least constant inapproximability in $2^{o(n)}$ time.
- Any $(1+\epsilon)$ approximation in $2^{n/\epsilon}$ time.
- n^n time (tight) to solve exactly.

Temporal Spanning Tree – Summary

- At least constant inapproximability in $2^{o(n)}$ time.
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Approximation helps for fighting huge but soft lower bounds

Conclusions

- Time doesn't buy approximation
  - Except for very hard problems...
- Time buys generality
  - Time vs. Generality vs. Approximation trade-offs?
  - A 3-D theory of approximation??
- Use approximation to fight ETH/SETH lower bounds?
  - Double-exponential bounds? (see Valia's talk tomorrow)
  - SETH bounds? E.g. $(2-\epsilon)^{tw}$ for VC?

Thank you!