Model Checking Lower Bounds for Simple Graphs

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Algorithmic Meta-Theorems

Positive results
- Problem X is **tractable**.

Negative results
- Problem X is **hard**.
Algorithmic Meta-Theorems

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Negative results

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• An algorithmic meta-theorem is a statement of the form: “All problems in a class C are **tractable**”
Algorithmic Meta-Theorems

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- Problem X is \textit{tractable}.

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- Problem X is \textit{hard}.

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- Meta-theorems are great! (more in a second)
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Main objective of today’s talk: barriers to meta-theorems:

“There exists a problem in class C that is hard”
Good news so far

- Most famous meta-theorem: Courcelle’s theorem
  All MSO-expressible properties are solvable in linear time on graphs of bounded treewidth.
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• Can we do better?
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- Can we do better?
  - More graphs?
  - Wider classes of problems?
  - Faster?
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Meta-theorems for clique-width, local treewidth,…
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- Can we do better?
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  This can be extended to optimization versions of MSO.
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- Can we do better?
  - More graphs? ✓
  - Wider classes of problems? ✓
  - Faster? ?
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  - More graphs? ✓
  - Wider classes of problems? ✓
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Faster than linear time?
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  Faster than linear time?

This is the main question we are concerned with today.
Courcelle’s theorem:

There exists an algorithm which, given an MSO formula $\phi$ and a graph $G$ with treewidth $w$ decides if $G \models \phi$ in time $f(w, \phi)|G|$. 
Some bad news

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  There exists an algorithm which, given an MSO formula $\phi$ and a graph $G$ with treewidth $w$ decides if $G \models \phi$ in time $f(w, \phi)|G|$.

- But the function $f$ is a tower of exponentials!
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- Unfortunately, this is not Courcelle’s fault.
  Thm: If $G \models \phi$ can be decided in $f(w, \phi)|G|^c$ for elementary $f$ then $P=NP$. [Frick & Grohe ’04]
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  Thm: If $G \models \phi$ can be decided in $f(w, \phi)|G|^c$ for elementary $f$ then P=NP. [Frick & Grohe ’04]

- In fact, Frick and Grohe’s lower bound applies to FO logic on trees!
There is still hope

This is bad! Can we somehow escape the Frick and Grohe lower bound?
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- For vertex cover, neighborhood diversity, \textit{max-leaf} [L. ’10]
- For twin cover [Ganian ’11]
- For shrub-depth [Ganian et al. ’12]
- For tree-depth [Gajarský and Hliňený ’12]
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Predominant idea: Removing isomorphic parts of the graph, when we have too many
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Predominant idea: Removing isomorphic parts of the graph, when we have too many

What’s next?
Let’s destroy all hope!

• In this talk the pendulum swings again.

• Main goal: prove hardness results even more devastating than Frick & Grohe.

• Motivation: If we know what we can’t do, we might find things we can do.
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- Main goal: prove hardness results even more devastating than Frick & Grohe.
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Today: Three new hardness results.

- Threshold graphs
- Paths
- Bounded-height trees
An appetizer:

Threshold Graphs
Theorem:

- MSO₁ expressible properties can be decided in linear time on graphs of bounded clique-width [Courcelle, Makowsky, Rotics ’00]
Theorem:

- MSO\textsubscript{1} expressible properties can be decided in linear time on graphs of bounded clique-width [Courcelle, Makowsky, Rotics ’00]

A graph has clique-width \(k\) if it can be constructed with the following operations using \(\leq k\) labels:

- Introduce a new vertex with label \(i \in [k]\).
- Connect all vertices with label \(i\) to all vertices with label \(j\).
- Rename all vertices with label \(i\) to label \(j\).
- Take the disjoint union of two clique-width \(k\) graphs.
Theorem:

- MSO$_1$ expressible properties can be decided in linear time on graphs of bounded clique-width [Courcelle, Makowsky, Rotics ’00]

An MSO$_1$ formula $\phi$ may contain:

- $\exists x, \forall x$ (quantifying over a graph’s vertices)
- $\exists X, \forall X$ (quantifying over a set of vertices)
- Relation $E(x, y)$ (edges), $x = y$
- Boolean connectives
Theorem:

- MSO$_1$ expressible properties can be decided in linear time on graphs of bounded clique-width [Courcelle, Makowsky, Rotics ’00]

- Trees have clique-width 3. Frick&Grohe \(\rightarrow\) non-elementary dependence.

- Graphs with clique-width 1 are easy for MSO$_1$.

What about clique-width 2?
Threshold Graphs

A graph is a threshold graph if it can be constructed with the following operations:

- Add a new vertex and connect it to everything.
- Add a new vertex and connect it to nothing.
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Thm: Threshold graphs have clique-width 2.
We use the following result of Frick& Grohe:

- There is no elementary-dependence model-checking algorithm for FO logic on binary strings.
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Input:

- String $w$, FO formula $\phi$:
  - $\exists x, \forall x$ ($x$ will correspond to a character in the string)
  - Relation $\prec$ ($x \prec y$ if $x$ comes before $y$ in the string)
  - Relation $P_1(x)$ (the character $x$ is a 1)
  - Boolean connectives
Hardness for threshold graphs

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  - Boolean connectives

Example:

$$\forall x P_1(x) \rightarrow \exists y \neg P_1(y) \land x \prec y$$
Hardness for threshold graphs

Given a string \( w \) we construct a threshold graph \( G \)

- \( w : \)
- \( G : uuj \)
Given a string $w$ we construct a threshold graph $G$

- $w : 0$
- $G : uu_j u_j$
Given a string $w$ we construct a threshold graph $G$

- $w: \quad 0 \quad 1$
- $G: \quad uuj \quad uj \quad ujj$
Hardness for threshold graphs

Given a string $w$ we construct a threshold graph $G$

- $w : 0\ 1\ 1$
- $G : uujujujjujj$
Hardness for threshold graphs

Given a string $w$ we construct a threshold graph $G$

- $w : \quad 0 \quad 1 \quad 1 \quad 0 \ldots$
- $G : \quad uujuj \quad ujj \quad ujj \quad uj \ldots$
Given a string $w$ we construct a threshold graph $G$

- $w: \quad 0 \ 1 \ 1 \ 0 \ldots$
- $G: \quad uu \ uj \ ujj \ ujj \ uj \ldots$

Idea: union vertices represent the characters

\[
\text{union}(x) \ := \ \forall y \forall z \left( E(x, y) \land E(x, z) \land y \neq z \right) \rightarrow E(y, z)
\]
\[
\text{main}(x) \ := \ \text{union}(x) \land (\exists y \neg \text{union}(y) \land \neg E(x, y))
\]
Given a string \( w \) we construct a threshold graph \( G \)

- \( w : \) 0 1 1 0…
- \( G : uujujujjujjujjujj… \)

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\begin{align*}
\text{union}(x) & \ := \ \forall y \forall z (E(x, y) \land E(x, z) \land y \neq z) \rightarrow E(y, z) \\
\text{main}(x) & \ := \ \text{union}(x) \land (\exists y \neg \text{union}(y) \land \neg E(x, y))
\end{align*}
\]

This allows us to interpret \( \exists x \psi(x) \) (in the string) to \( \exists x (\text{main}(x) \land \psi^I(x)) \) (in the graph).
Interpretation continued:

- The \( \prec \) relation can be expressed as

\[
prec(x, y) := \exists z \neg \text{union}(z) \land E(x, z) \land \neg E(y, z)
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Interpretation continued:

- The $\prec$ relation can be expressed as

$$ prec(x, y) := \exists z \neg \text{union}(z) \land E(x, z) \land \neg E(y, z) $$

- The $P_1$ relation can also be expressed in FO logic...
Interpretation continued:

- The \( \prec \) relation can be expressed as

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- The \( P_1 \) relation can also be expressed in FO logic...

Thm: There is no elementary-dependence model-checking algorithm for FO logic on threshold graphs.
Recall some of the “good” graph classes we know

- Some are closed under complement (neighborhood diversity, shrub-depth)
- Some are closed under union (tree-depth)
Recall some of the “good” graph classes we know

- Some are closed under complement (neighborhood diversity, shrub-depth)
- Some are closed under union (tree-depth)
- None are closed under both operations...

Any class of graph closed under both operations must* contain threshold graphs.
Main course:

Paths
Why paths?

Main question:

- Is there an elementary-dependence algorithm for MSO$_1$ on paths?
Why paths?

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Equivalent question:
- Is there an elementary-dependence algorithm for MSO$_1$ on unary strings?
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- Is there an elementary-dependence algorithm for MSO\(_1\) on paths?

Equivalent question:

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Why?

- Do Frick and Grohe really need all trees?
- FO is easy on paths.
- MSO is hard on binary strings/colored paths.
Why paths?

Main question:

- Is there an elementary-dependence algorithm for MSO\textsubscript{1} on paths?

Equivalent question:

- Is there an elementary-dependence algorithm for MSO\textsubscript{1} on unary strings?

Why?

- Do Frick and Grohe really need all trees?
- FO is easy on paths.
- MSO is hard on binary strings/colored paths.
- MSO for max-leaf is open!
Why would this be easy?

- MSO on paths = Regular language over unary alphabet
- FO is easy
Why would this be easy?

- MSO on paths = Regular language over unary alphabet
- FO is easy
- Reduction seems impossible...

“Normal” reduction:

- Start with $n$-variable 3-SAT
- Construct graph $G$ with $|G| = n^c$
- Construct formula $\phi$ with $|\phi| = \log^* n$
- Prove YES instance $\leftrightarrow G \models \phi$

Problem: New instance would be encodable with $O(\log n)$ bits. We are making a sparse NP-hard language!
How the reduction can work

Key idea: do not use $P \neq NP$ but $EXP \neq NEXP$

- Motivation: reduction must construct exponential-size graph, so should be allowed exponential time.
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- Motivation: reduction must construct exponential-size graph, so should be allowed exponential time.

Plan:

- Start with an $NEXP$-complete problem and $n$ bits of input.
- Construct a path on $2^{n^c}$ vertices.
- Construct a formula $\phi$ with $|\phi| = \log^* n$.
- Prove YES instance $\leftrightarrow G \models \phi$.

Elementary parameter dependence gives $EXP=\text{NEXP}$. 
How the reduction can work

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Elementary parameter dependence gives $EXP = NEXP$.

- Formula will be somewhat larger, but still small enough.
The basic obstacle (as in Frick and Grohe) is counting efficiently.

Given two sets of elements $S_1, S_2$ with $|S_1| \neq |S_2|$, what is the smallest MSO formula that can verify this?
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Example: For independent sets, $q$ quantifiers work for size $2^q$.

Main goal:

- Increase counting power exponentially with each added quantifier.
- Frick and Grohe do this, but they are allowed to design their graphs. We are (essentially) not!
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Today: $q$ quantifiers count up to size $tow(\Omega(\log q))$ on unary strings.
Induction:

- We have a MSO formula $eq_L(P_1, P_2)$ which correctly compares sets up to size $L$.

- The formula is only true for equal sets (independent of size).

Use this to compare larger sets economically.

First idea: division
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First idea: division

Given an ordered set of elements to compare with another, we first select a subset of it.
Learning to count

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First idea: division

We can impose some structure: each “section” must have the same length ($\leq L$). We do this on both sets.
Learning to count

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First idea: division

Now we need to count the number of sections. Select one representative from each. Compare the two sets of representatives.
Learning to count

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First idea: division

This allows us to go from $L$ to $L^2$ with $O(1)$ additional quantifiers (if done carefully).
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First idea: division

This allows us to go from $L$ to $L^2$ with $O(1)$ additional quantifiers (if done carefully).

Counting power: $2^{2^q}$. Not good enough, but we’re moving.
Learning to count better

- Good: a single set gives many sections.
- Bad: hard to count how many sections we have. Using induction not good enough.

Idea: count in binary!
Learning to count better

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Select the same division into sections.
Good: a single set gives many sections.

Bad: hard to count how many sections we have. Using induction not good enough.

Idea: count in binary!

To count sections, select a subset that “writes” a binary number in each section.
Learning to count better

- Good: a single set gives many sections.
- Bad: hard to count how many sections we have. Using induction not good enough.

Idea: count in binary!

Demand that counting is correct for consecutive sections.

- Proof: hand-waving (but check the paper!)
Learning to count better

- Good: a single set gives many sections.
- Bad: hard to count how many sections we have. Using induction not good enough.

Idea: count in binary!

We went from $L$ to $L2^L$ using $eq_L O(1)$ times.
→ each level of exponentiation increases size by a constant factor.
→ can compare sets of size $n$ with $2^{\log^* n}$ quantifiers.
Are we done with the math?

Using $eq_L$ it’s easy to do comparisons, div, mod, . . .

- We will also need exponentiation. $exp_L(P_1, P_2)$ is true if $|P_2| = 2^{P_1}$. 
Using $eq_L$ it’s easy to do comparisons, div, mod, ...  

- We will also need exponentiation. $exp_L(P_1, P_2)$ is true if $|P_2| = 2|P_1|$. 

Idea: Find a set in $P_2$ with size $|P_1| + 1$. Ensure that consecutive distances are doubled. 

DONE!
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Idea: Find a set in $P_2$ with size $|P_1| + 1$. Ensure that consecutive distances are doubled.

DONE!

The hard part is over!
Putting things together

- Reduction from NEXP Turing machine acceptance with $n$ input bits.
- Machine runs in $T = 2^{nc}$ time. Input (read as binary number) is $I \leq 2^n$.
- Construct a path of length $T^2(2I + 1)$.
- Construct a $\phi$ that simulates the machine on the path.
Putting things together

- Reduction from NEXP Turing machine acceptance with \( n \) input bits.
- Machine runs in \( T = 2^{n^c} \) time. Input (read as binary number) is \( I \leq 2^n \).
- Construct a path of length \( T^2(2I + 1) \).
- Construct a \( \phi \) that simulates the machine on the path.

The last one is the tricky part. But we now have the right tools.

- Locate a set of length \( T^2 \). Divide it into sections of size \( T \). These will represent snapshots of the machine’s tape.
- Locate a set of length \( I \). Use \( exp \) to “read” input bits from it.
- Guess the contents of the tape.
- Check that the computation is correct and accepting.
Consequences

Unless \( \text{EXP}=\text{NEXP} \):

- Max-leaf is hard
Consequences

Unless $\text{EXP} = \text{NEXP}$:

- Max-leaf is hard
- Graph classes closed under edge sub-divisions are hard
Unless EXP=\text{NEXP}:

- Max-leaf is hard
- Graph classes closed under edge sub-divisions are hard
- Graph classes closed under induced subgraphs with unbounded (dense)* diameter are hard
Consequences

Unless \( \text{EXP} = \text{NEXP} \):

- Max-leaf is hard
- Graph classes closed under edge sub-divisions are hard
- Graph classes closed under induced subgraphs with unbounded (dense)* diameter are hard
- \( \text{MSO}_2 \) for cliques is very hard! (not in XP)

The last one was already known. But “easier” proof using that \( \text{eqL} \) has constant size on cliques with \( \text{MSO}_2 \).
Dessert:

Trees of bounded height
Why trees of bounded height?

This class of graphs is important for two recent meta-theorems:

- Shrub-depth in “When trees grow low: Shrubs and fast $\text{MSO}_1$” [Ganian et al. MFCS ’12]
- Tree-depth in “Faster deciding $\text{MSO}$ properties of trees of fixed height, and some consequences” [Gajarský and Hliňený FSTTCS ’12]

In both cases the main tool is the following:

MSO model-checking for $q$ quantifiers on trees of height $h$ colored with $t$ colors can be done in $\exp^{(h+1)}(O(q(t + q)))$ time.
Goal: prove that $h + 1$ levels of exponentiation are exactly necessary.

- Start from an $n$-variable 3-SAT instance.
- Construct a tree of height $h$. Use $t = \log^{(h)}(n)$ colors.
- Construct a formula with $q = O(h)$ quantifiers.
- Prove equivalence between instances.
Goal: prove that $h + 1$ levels of exponentiation are exactly necessary.

- Start from an $n$-variable 3-SAT instance.
- Construct a tree of height $h$. Use $t = \log^{(h)}(n)$ colors.
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Argument: an algorithm running in $\exp^{(h+1)}(o(t))$ would run in $2^{o(n)}$ here, disproving ETH.
Let’s count some more!

Fix $h$. The main problem is again to count efficiently.

- We have $\log^h(n)$ colors available. These can represent numbers up to $\log^{h-1}(n)$ with a single vertex (and comparisons are propositional!).
Let’s count some more!

Fix $h$. The main problem is again to count efficiently.

- We have $\log^h(n)$ colors available. These can represent numbers up to $\log^{h-1}(n)$ with a single vertex (and comparisons are propositional!).

- Assuming we can do numbers up to $L$ with trees of height $i$. We do numbers up to $2^L$ with trees of height $i + 1$ (Frick & Grohe).
The rest is easy

- Construct a tree of height $h - 1$ for each variable, encoding its index.
- Construct a tree of height $h - 1$ for each clauses, encoding the indices of its three literals.
- Add a root.
- Express satisfiability with a constant quantifier-depth formula.

Essential idea: we are using the proof of Frick and Grohe for $h$ levels.
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- Add a root.
- Express satisfiability with a constant quantifier-depth formula.

Essential idea: we are using the proof of Frick and Grohe for $h$ levels.

Thm: There is no $\exp^{(h+1)(o(t))}$ algorithm for MSO logic on $t$-colored trees of height $h$ unless the ETH is false.
Conclusions - Open problems

- Three natural barriers to future improvements.
- Paths are probably the toughest to work around.

Future work

- (Uncolored) tree-depth?
- Height of tower for paths?
Conclusions - Open problems

- Three natural barriers to future improvements.
- Paths are probably the toughest to work around.

Future work

- (Uncolored) tree-depth?
- Height of tower for paths?
- Other logics?!?
Thank you!