Parameterized Approximation Schemes Using Graph Widths

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Visit Kyoto for ICALP ’15!




Overview

Topic of this talk:

Randomized Parameterized Approximation Algorithms

- **Approximation**: Ratio of $(1+\epsilon)$
- **Parameterized**: Parameter is tree/clique-width
- **Randomized**: Probabilistic rounding

Message: A generic technique for dealing with problems which are:

- W-hard: need time n^k to solve exactly
- APX-hard: cannot be $(1+\epsilon)$ approximated in poly time

Result: A natural $(\log n/\epsilon)^{O(k)}$ algorithm with ratio $(1+\epsilon)$

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Two concrete problems

- Max Cut parameterized by clique-width
  - Given: Graph $G(V, E)$ (along with a clique-width expression)
  - Wanted: A partition of V into L, R that maximizes edges cut.
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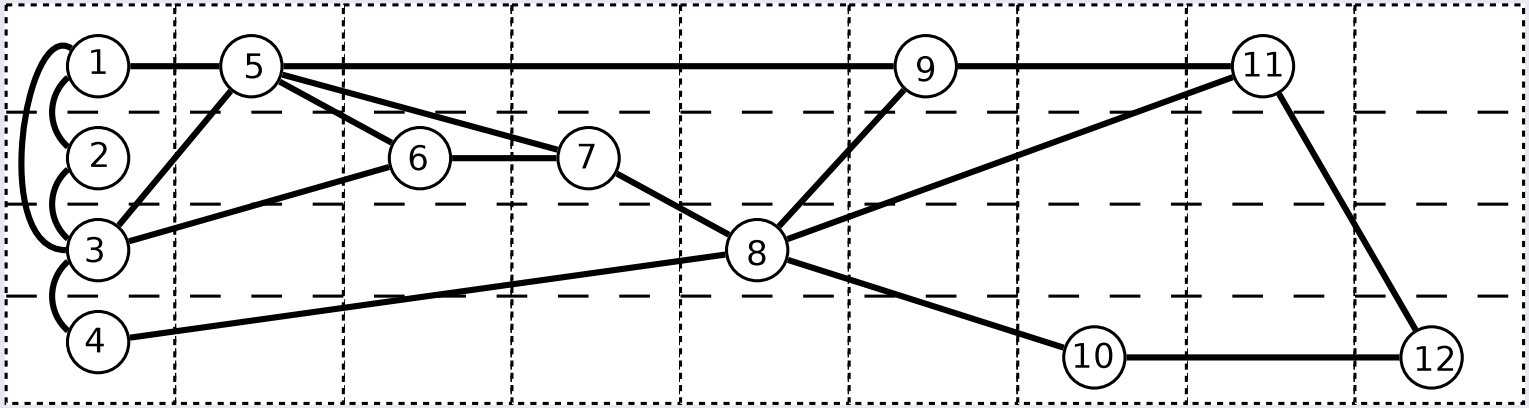
Treewidth - Pathwidth reminder

Good tree/path decompositions give a sequence of small separators



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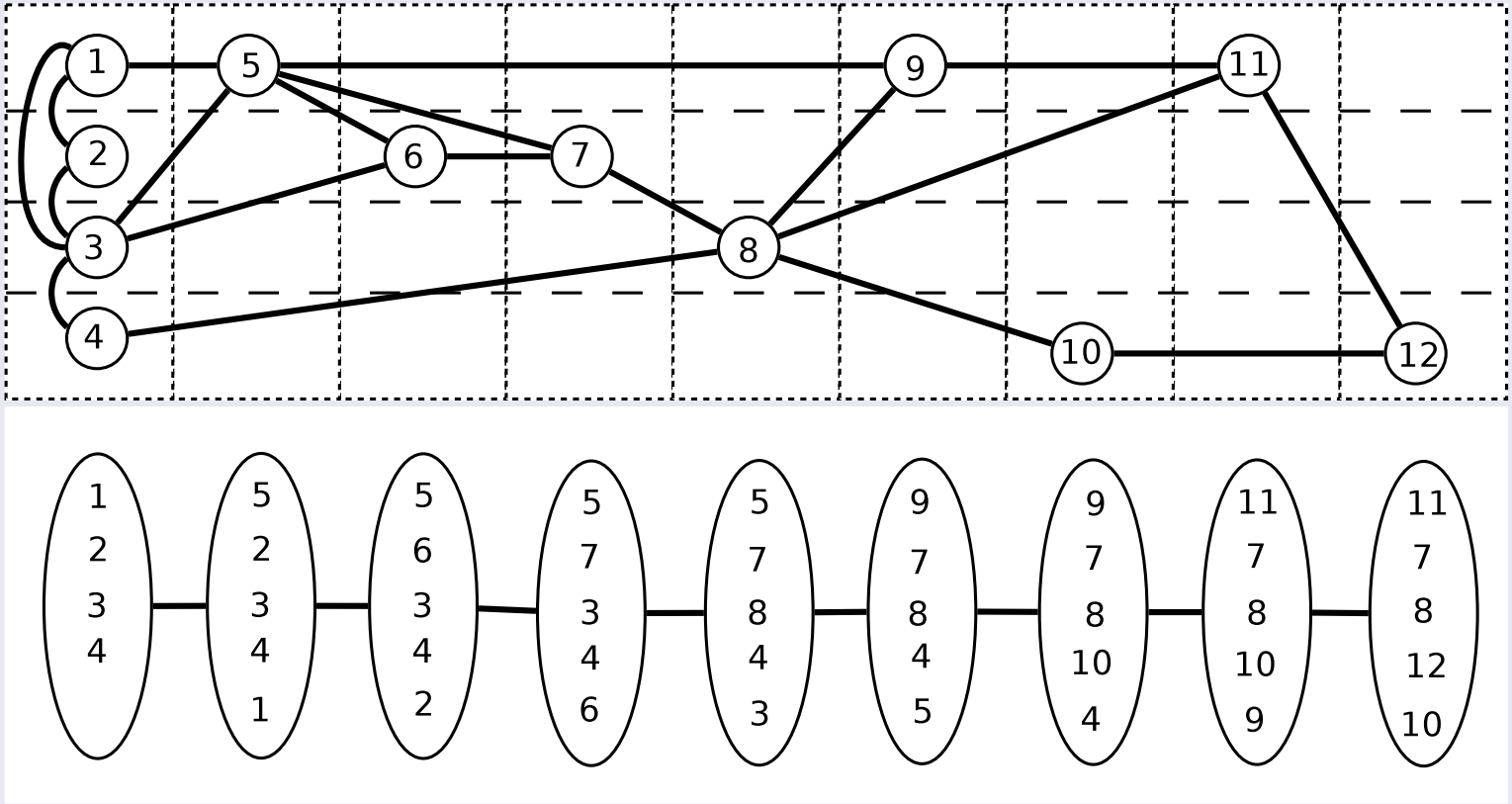

Algorithmic view

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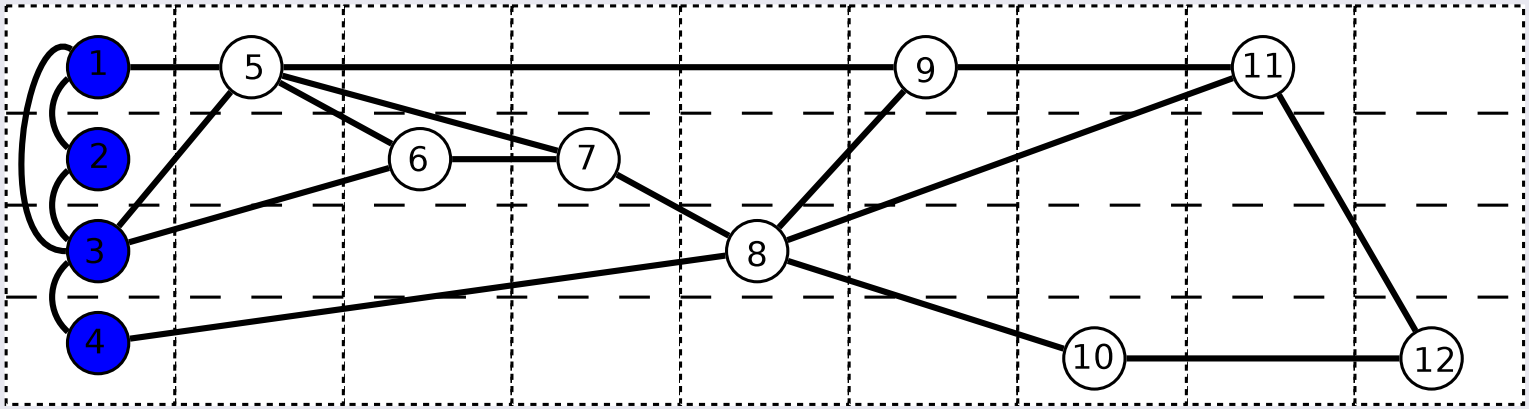

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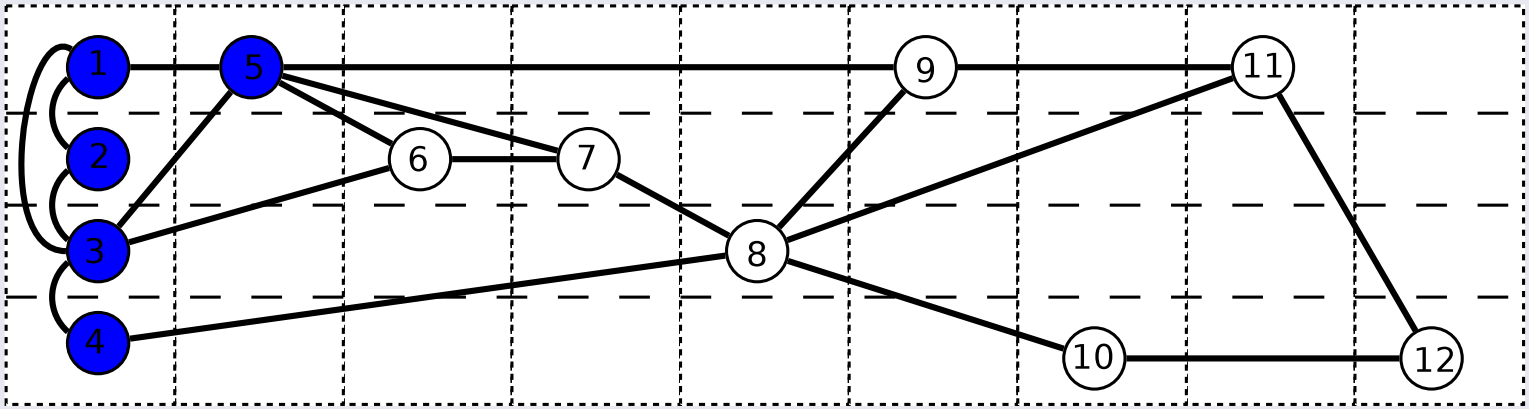

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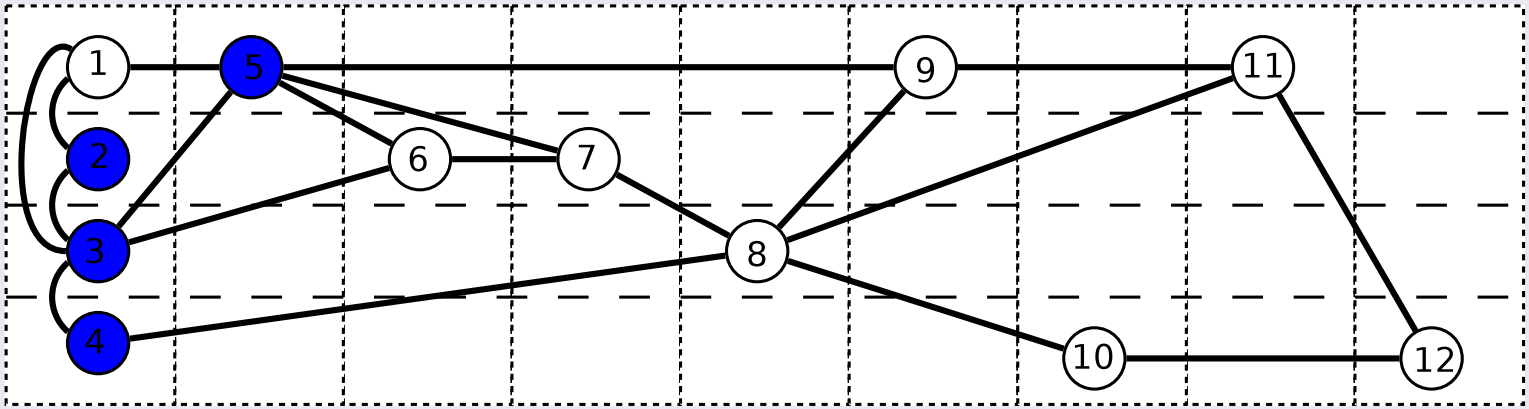

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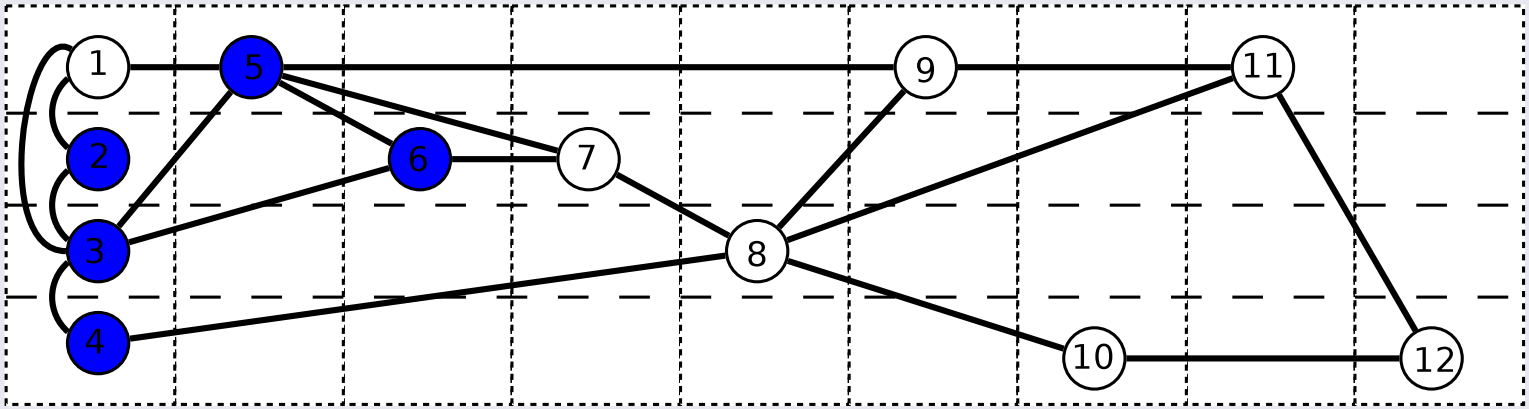

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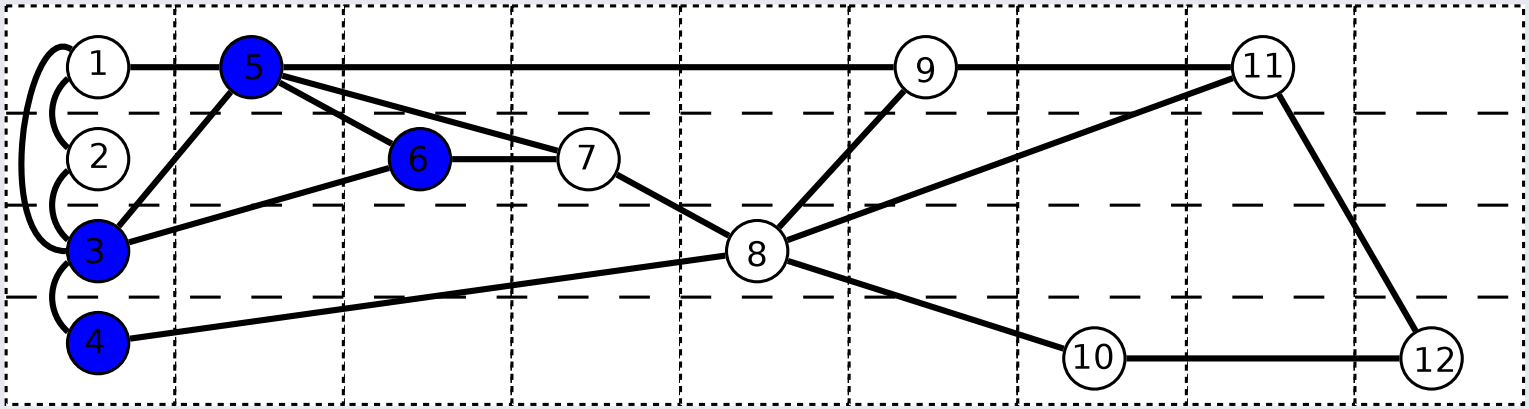

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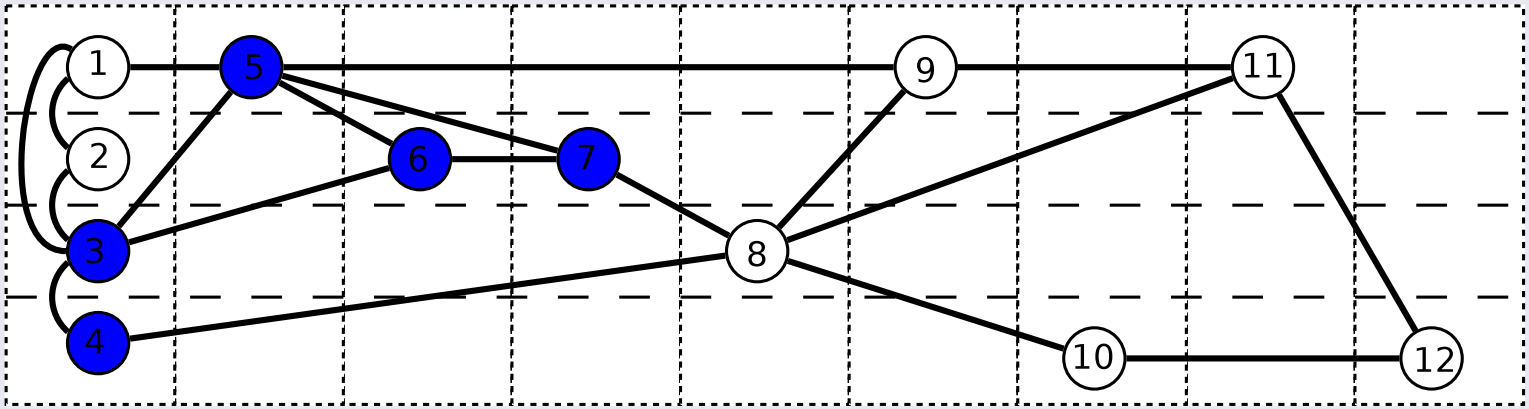

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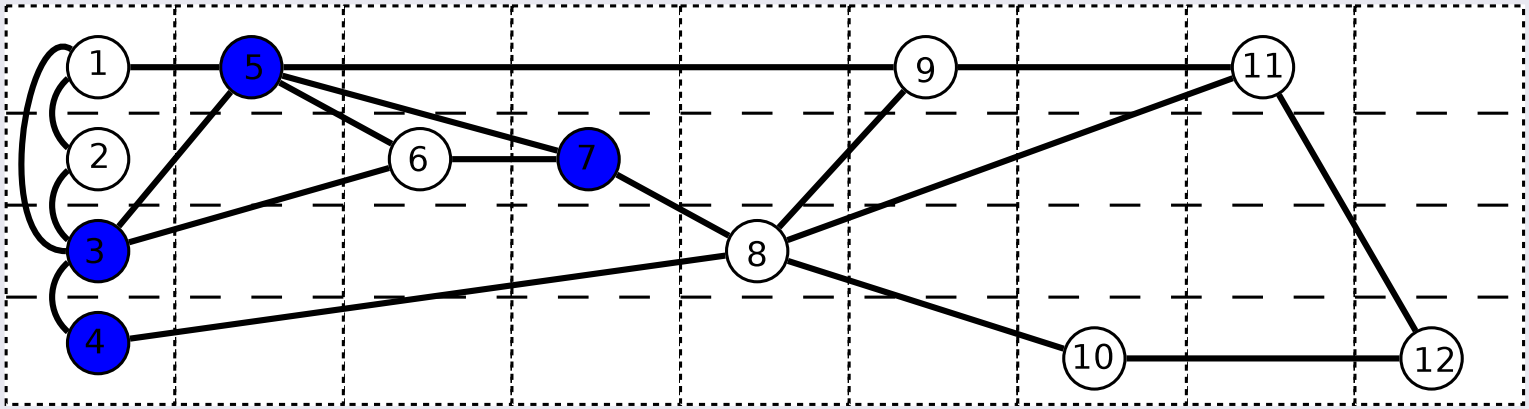

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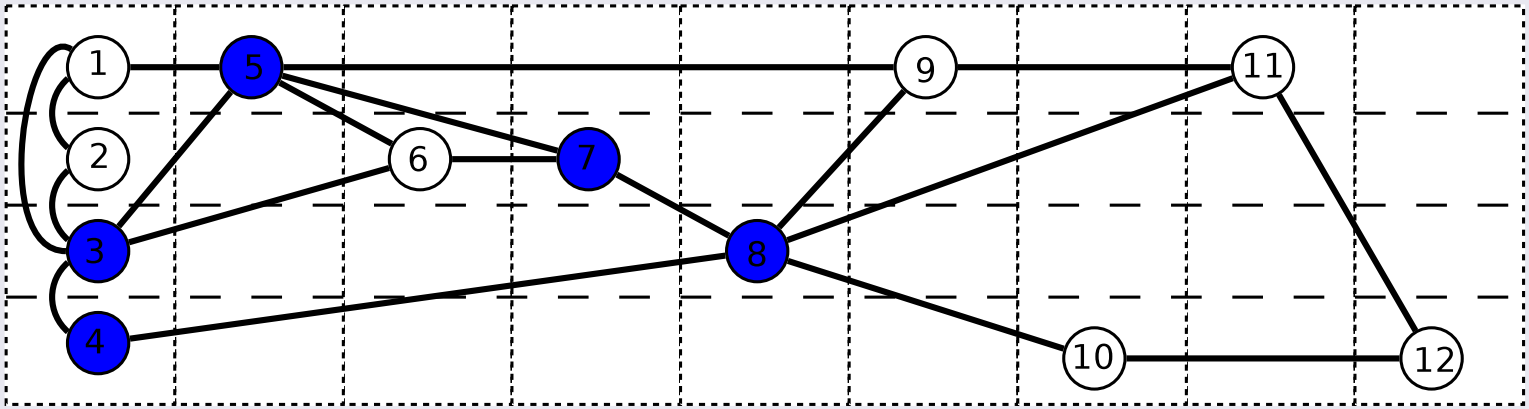

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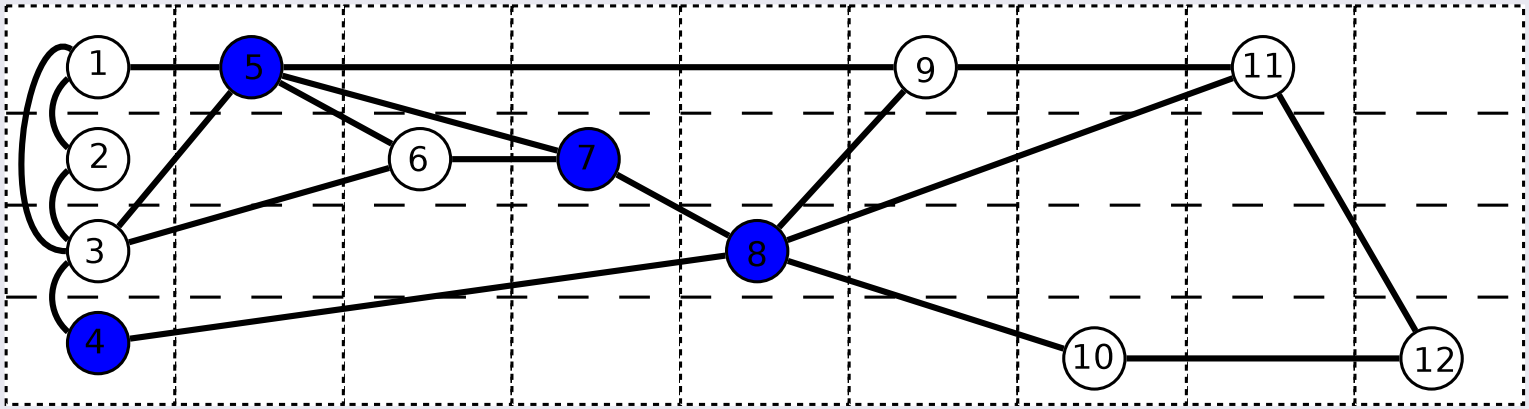

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For Dominating Set only need to remember information about boundary

- Selected (Blue)
- Not Selected – Already Covered (Green)
- Not Covered (Red)
- Total Cost

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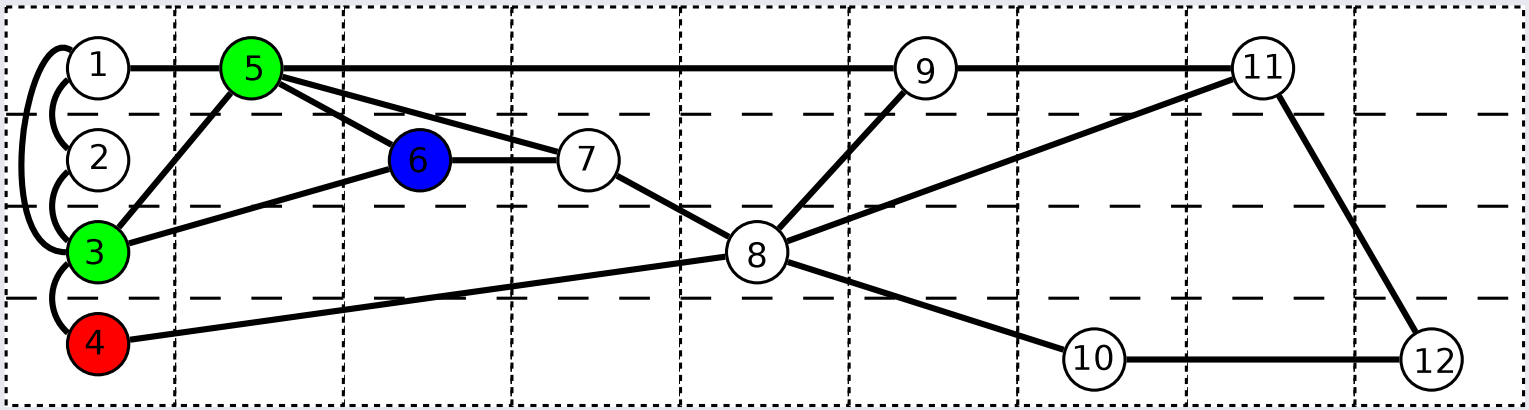

Separator: $\{3, 4, 5, 6\}$ includes tuple (3,4,5,6;?)

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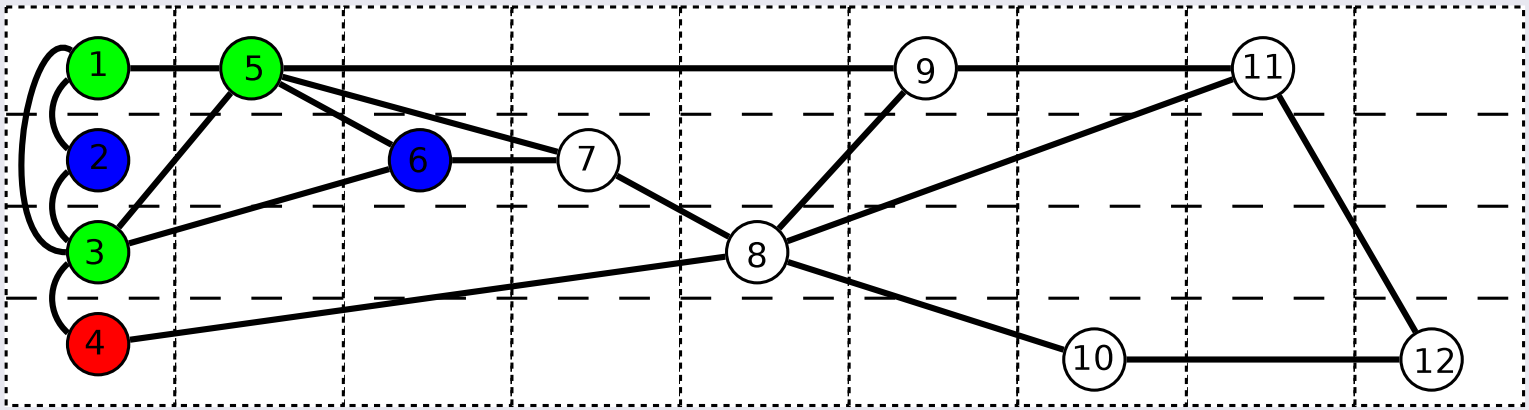

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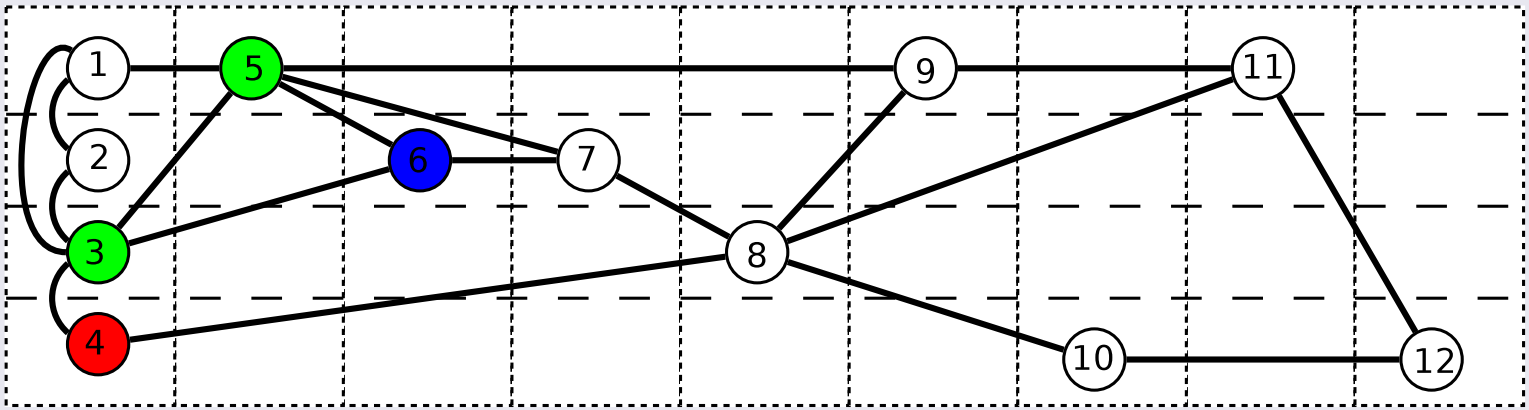

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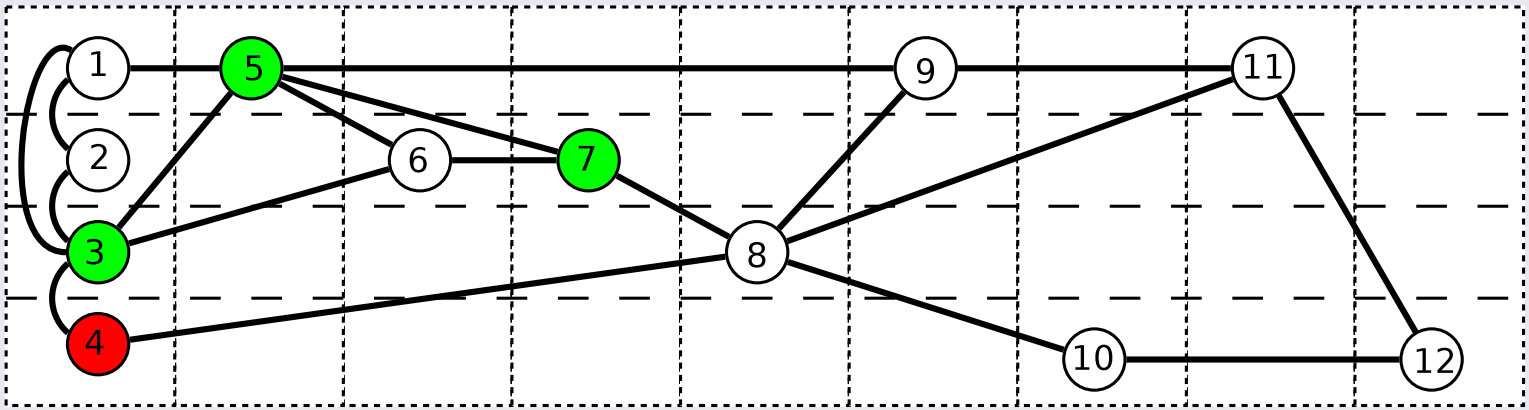

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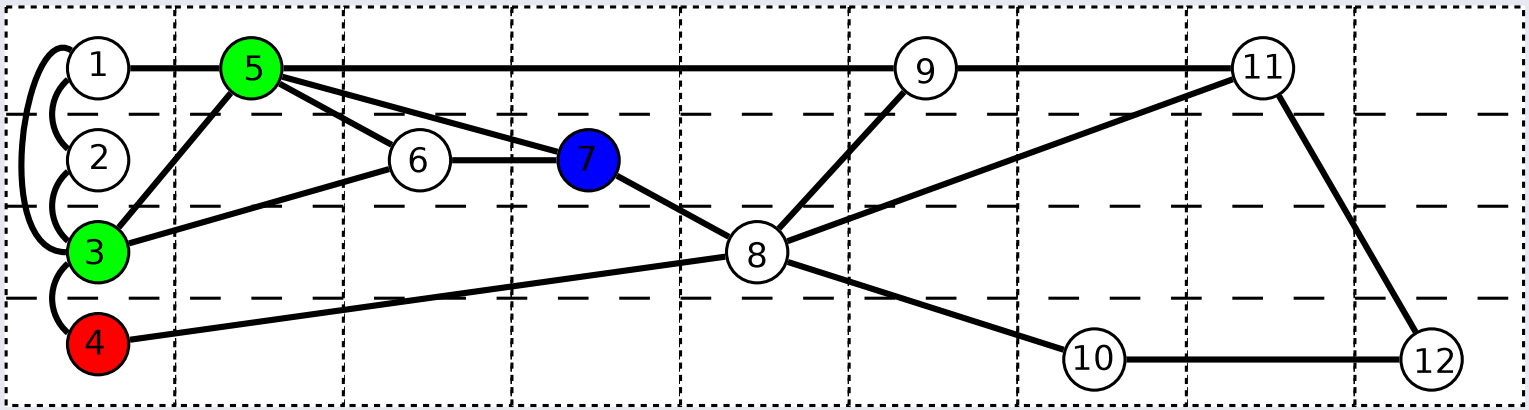

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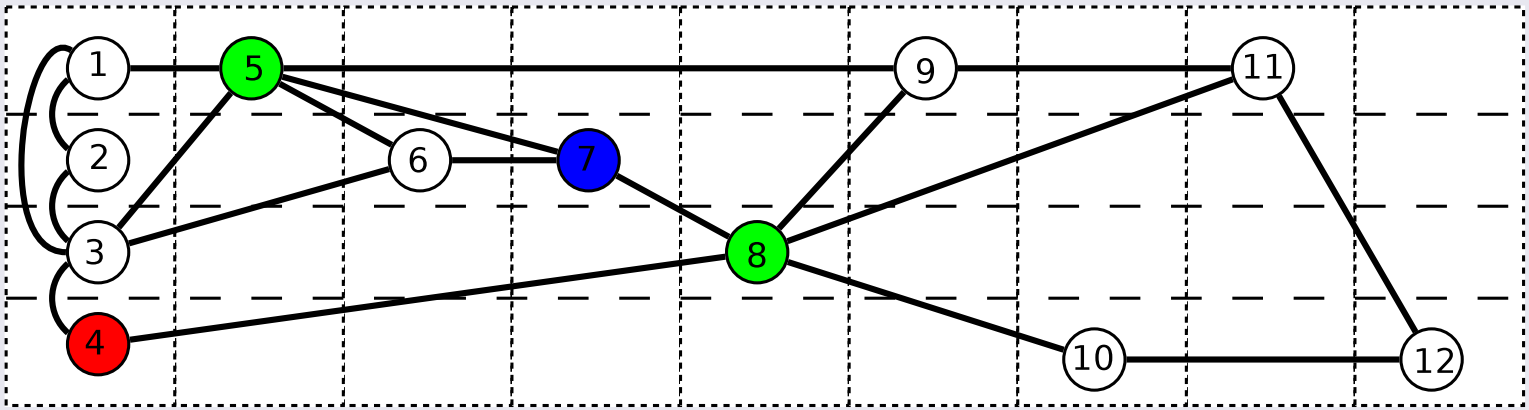

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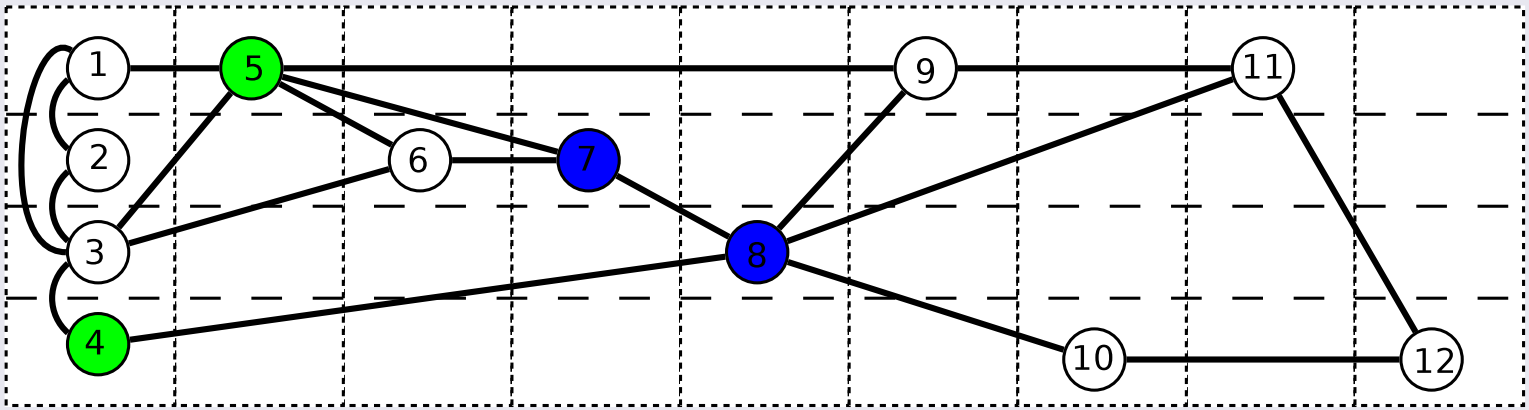

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- For Dominating Set DP tables have size 3^k.
- For Capacitated Dominating Set must remember capacity info for selected vertices
  - Table Size: C^k
- Note: May remember Capacity left **OR** Capacity used. Same thing?

Why n^k for Max Cut? (1/2)

A labelled graph G has clique-width at most k if

- G is K_1 with some label in $\{1, \ldots, k\}$
- Union: $G = G_1 \cup G_2$, with cw k
- Join: $G = \mathrm{Join}(i, j, G')$, $i, j \in \{1, \ldots, k\}$ and G' has cw k
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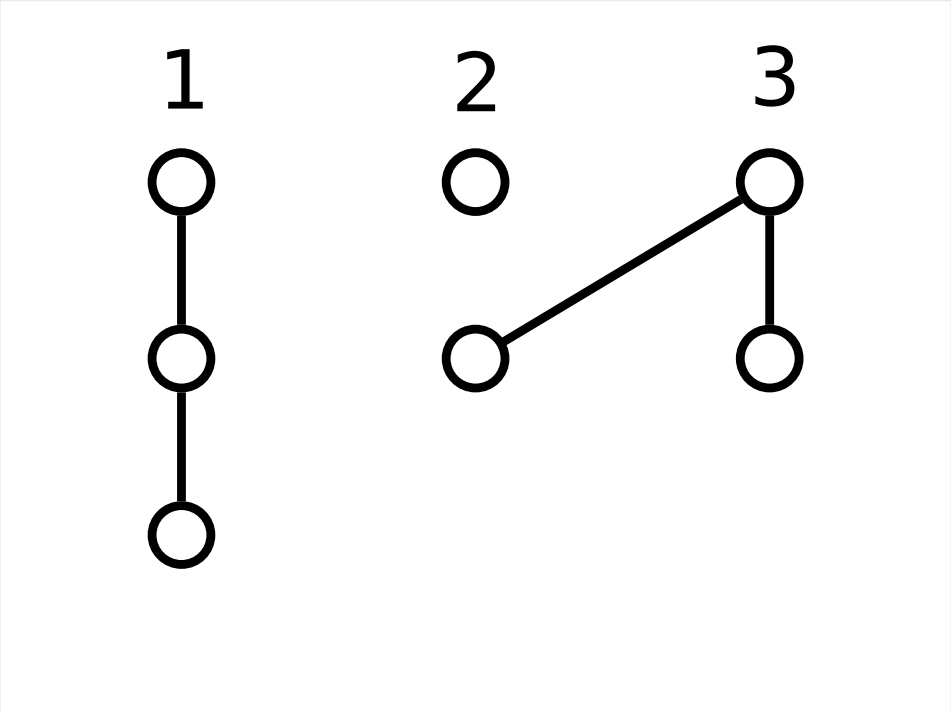

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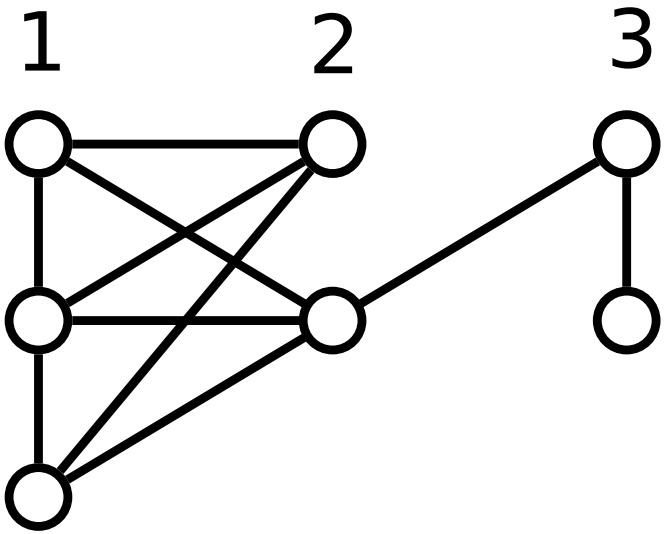

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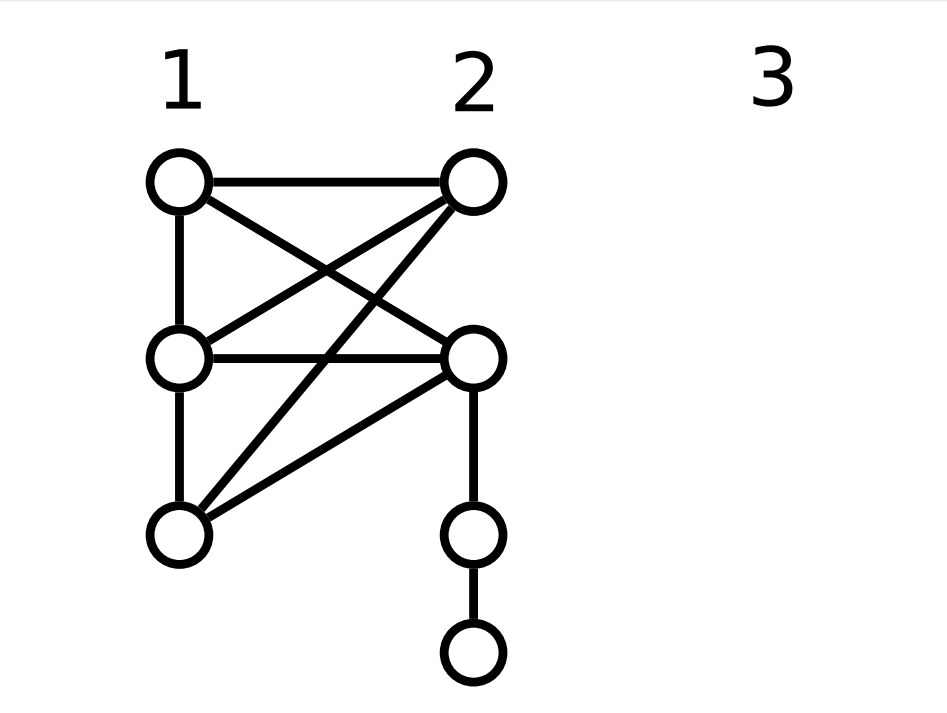

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- A clique-width expression for G is a "proof" that G can be built using these operations and k labels.
- Finding an optimal expression is generally hard...
- We "hope" that such an expression is supplied.
- We view it as a binary tree and perform dynamic programming.

Why n^k for Max Cut? (2/2)

Natural dynamic program for Max Cut

- For each node store a collection of tuples $(l_1, l_2, \ldots, l_k; C)$
- Meaning: There exists a solution that places **exactly** l_i vertices with label i in L and cuts C edges.

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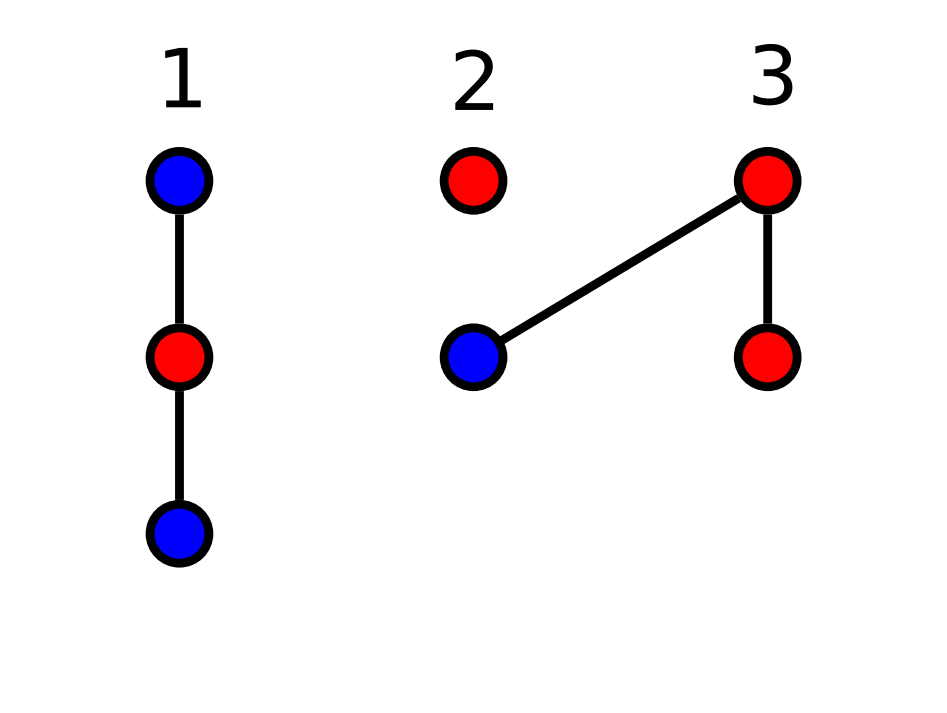

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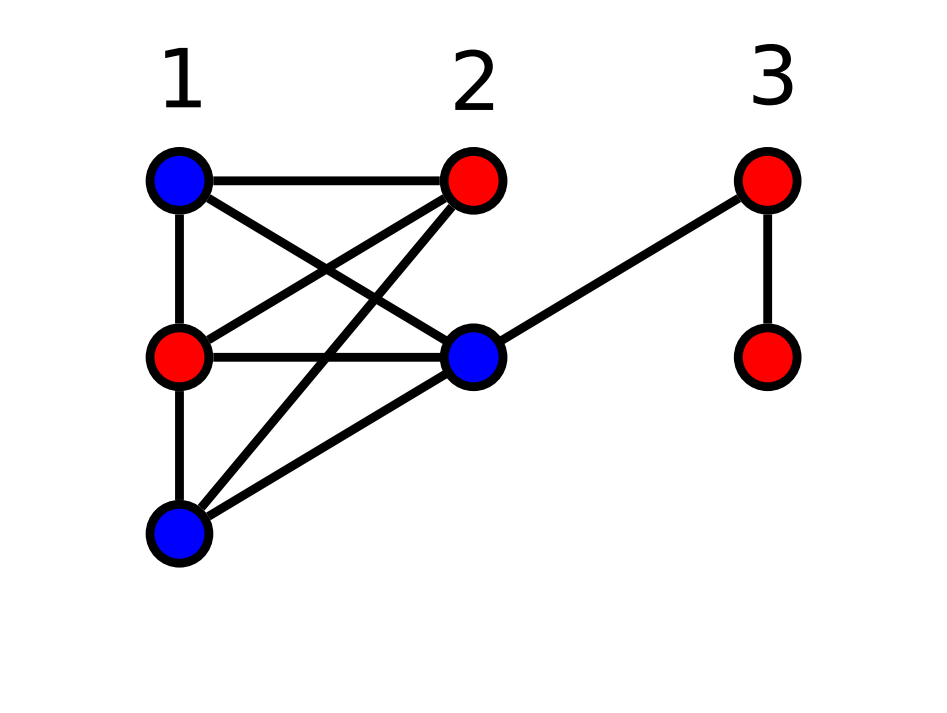

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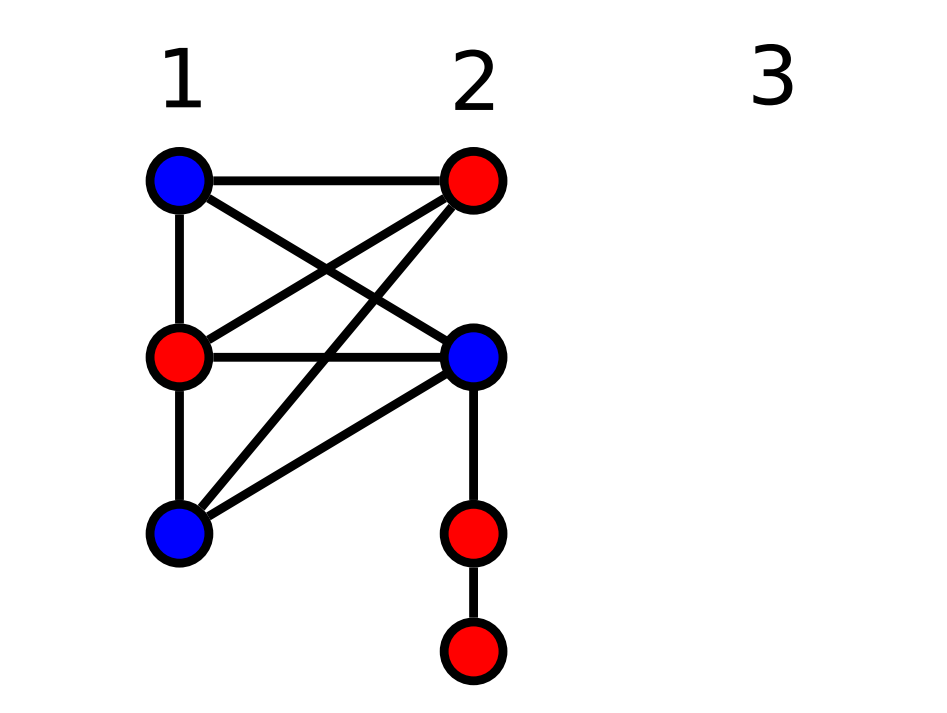

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Today's idea: keep **rounded** values for the l_i entries.
This can make the table smaller.

What is rounding?

Example rounding scheme:

- Normal table has values $l_i \in \{0, 1, 2, 3, \ldots, n\}$.
- We can store values $l_i \in \{0, 1, 2, 4, 8, 16, \ldots, n\}$.
  - Informal meaning: there exists a partition that places **roughly** l_i vertices with label i in L
- Running time $\approx$ table size $\approx (\log n)^k$
- But approximation ratio ≥ 2

What is rounding?

Example rounding scheme:

- Normal table has values $l_i \in \{0, 1, 2, 3, \ldots, n\}$.
- Fix some (small) parameter $\delta > 0$
- We will store values $l_i \in \{0, (1+\delta), (1+\delta)^2, (1+\delta)^3, \ldots\}$
  - Informal meaning: there exists a partition that places **roughly** l_i vertices with label i in L
- Running time $\approx$ table size
- For small δ we have $\log_{(1+\delta)} n = O(\frac{\log n}{\ln(1+\delta)}) = O(\frac{\log n}{\delta})$
- Table size $\to (\log n/\delta)^k$

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- This will be hard!

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Errors can propagate and pile up!

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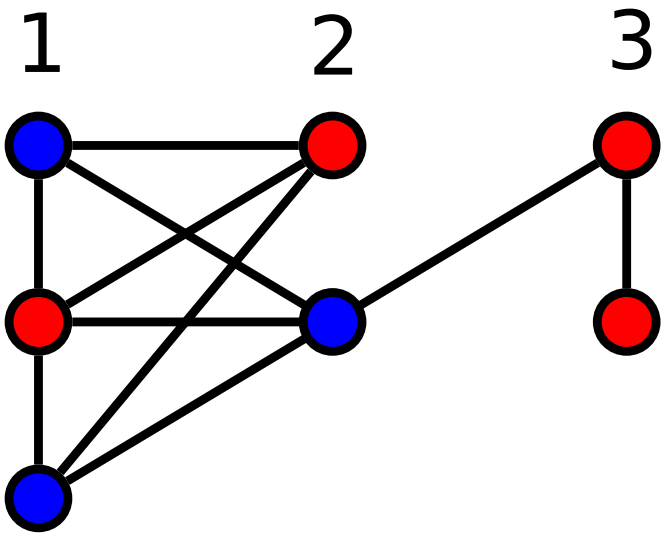
Concrete example



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$(a_1, a_2, a_3; a_C)$

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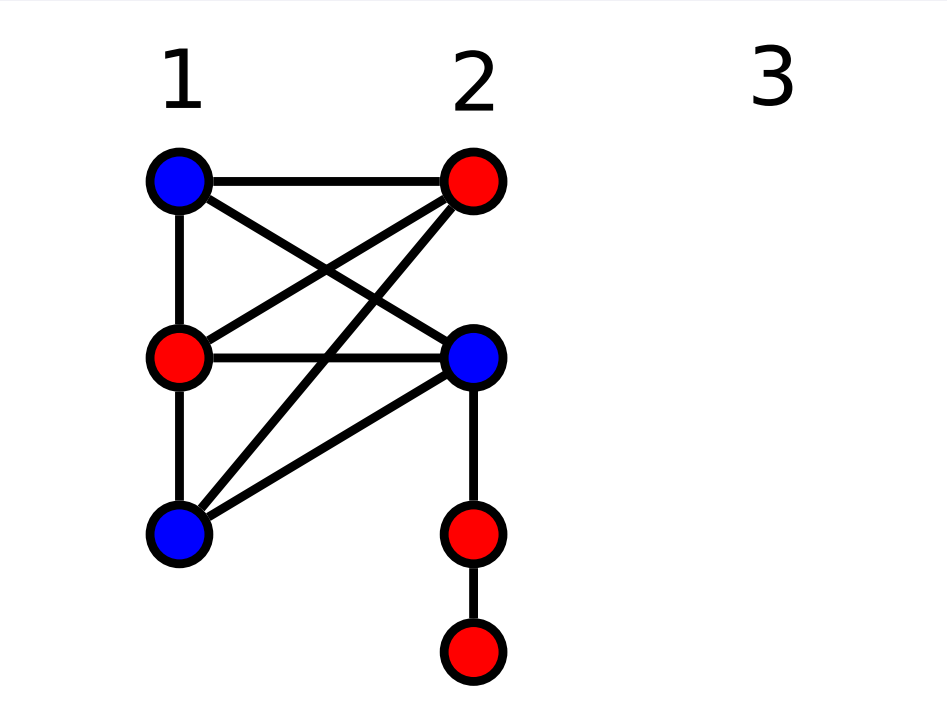
Concrete example



Example tuple: (red = L)

$(a_1, \mathbf{a_2} + \mathbf{a_3}, \mathbf{0}; a_C)$

The problem with rounding

Errors can propagate and pile up!

- The new value we would like to store $(a_2 + a_3)$ is not necessarily "round" (integer power of $(1 + \delta)$).
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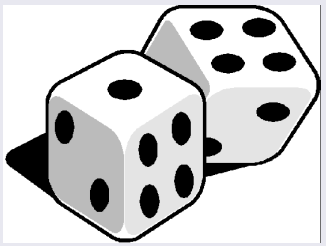

How to measure errors

- Plan so far:
  - Start with exact DP. Run it with approximate values.
  - TBD: how to re-round non-round intermediate values.
- There is a value x calculated by the exact DP
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- Define

$$Error(x, y) := \log_{(1+\delta)}(\max\{\frac{x}{y}, \frac{y}{x}\})$$

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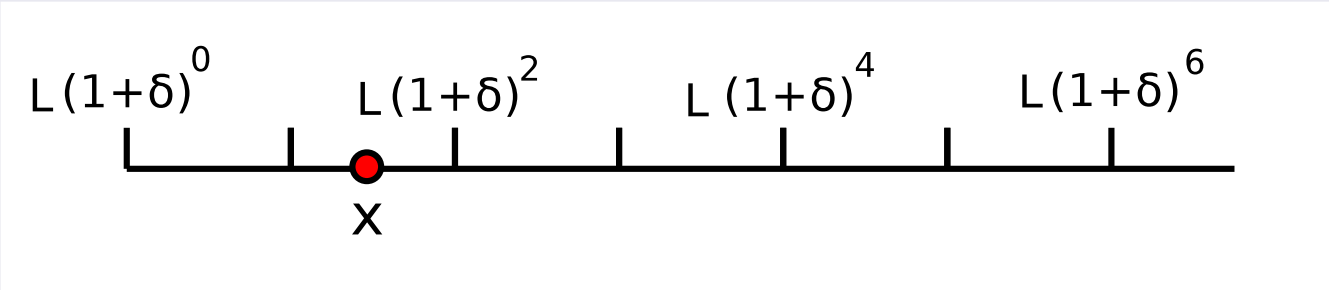

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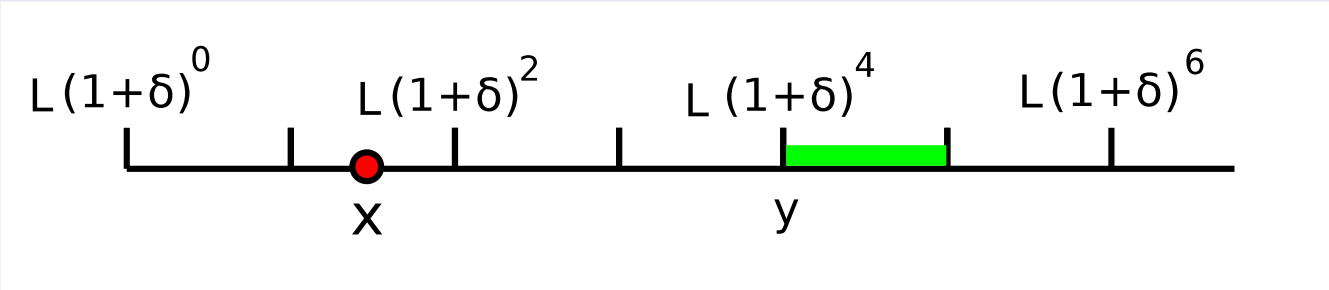

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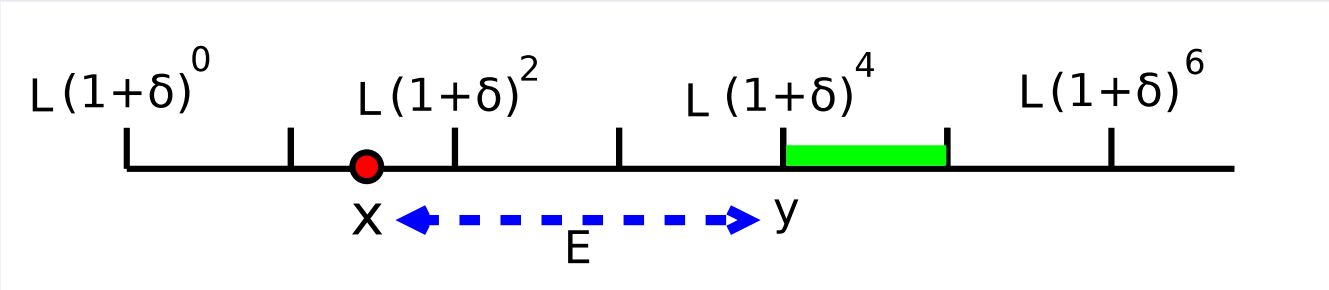

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End goal:

- Would like $Error(x, y) \leq \epsilon/\delta$ for all x, y.
  - Approximation ratio $= (1+\delta)^{Error} \leq (1+\delta)^{\epsilon/\delta} \approx 1 + \epsilon$

What we know about errors

- Consider values x_1, x_2 and their approximations y_1, y_2 with Errors E_1, E_2.

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- DPs relying on additions are the “Easiest Target”.

From now on only Additive DPs considered.

- Fortunately, there are plenty...
- E.g. Max Cut, Capacitated Dominating Set

Two roads to success

Obliviously round in some way. Hope for the best!

Probabilistically round. Prove that good things happen whp.

The lucky man's solution

Consider a DP that only uses additions.

- Trivial observation: each level of the given clique-width expression/tree decomposition increases maximum Error by at most 1.
  - Error can only be introduced in re-rounding.
- What if the given decomposition is **balanced**? Then it has logarithmic height!
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Thm [Bodlaender and Hagerup SICOMP '98]: Every graph with treewidth w has a balanced tree decomposition with width $3w$.

Using our gift

1. Set $\delta = \epsilon / \log n$.
2. Balance decomposition.
3. Run approximate DP, rounding arbitrarily.

This works! (As long as we only do additions/comparisons)

- Approximation ratio $\leq (1+\delta)^{\log n} \approx (1+\epsilon)$.
- Running time $(\log n/\epsilon)^{O(k)}$.

Application approximation schemes:

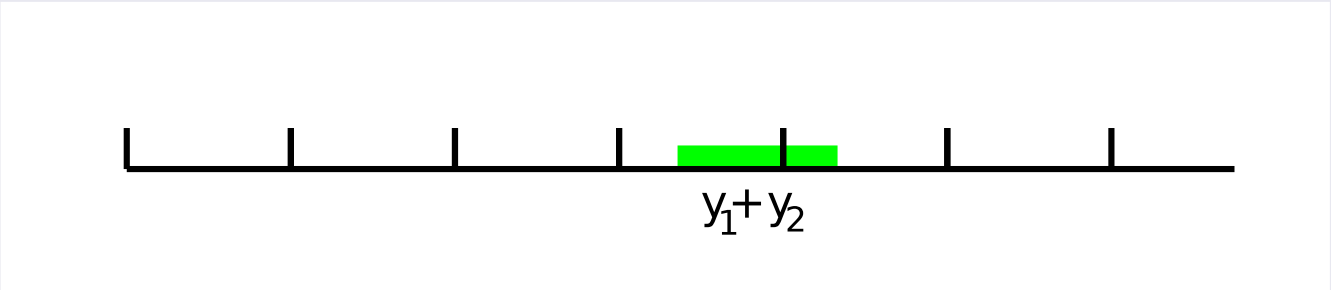
- Capacitated Dom. Set (bi-criteria)
- Capacitated Vertex Cover (bi-criteria)
- Bounded Degree Deletion (bi-criteria)
- Equitable Coloring (bi-criteria)
- Graph Balancing

Back to the Interesting Part

We have to round

- What about Max Cut on clique-width?
  - Best known balancing theorem blows up number of labels to 2^k
- Must round in a way that works for n steps.
- Intuition: randomization “evens out” the errors.

Process:



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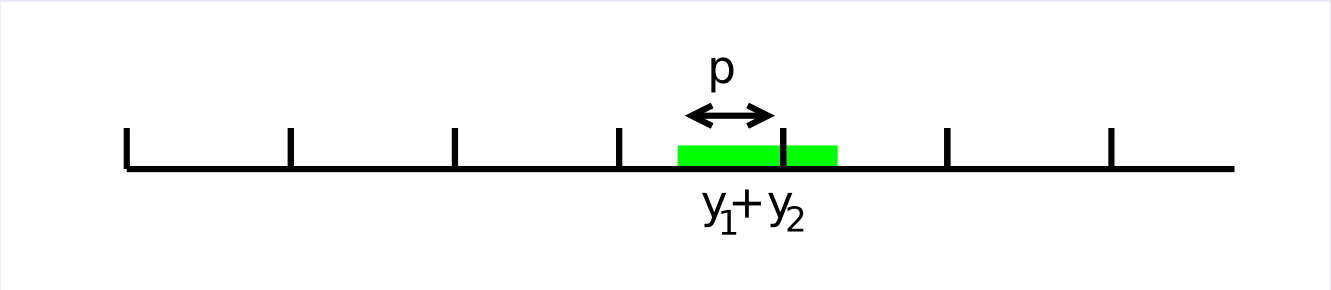

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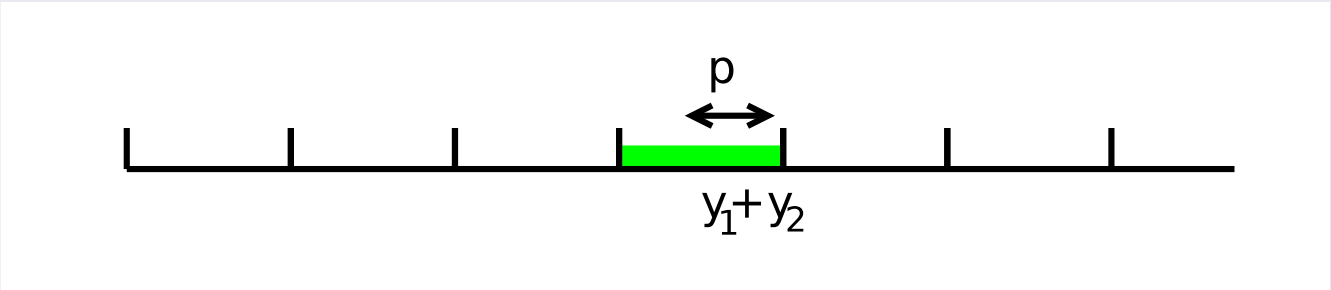

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Addition Trees

- We want this process to work whp for $\delta = \Omega(1/poly(\log n))$.
- This is complicated. So we abstract it out.

Definition: An Addition Tree (AT) is a binary tree with positive integers on the leaves. The value of each node is the sum of its children.

Definition: An Approximate Addition Tree (AAT) is an Addition Tree where additions are replaced by the $\oplus$ operation.

- Motivation: If AATs are good whp, we can use this as a black box for any DP that only does additions.

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Theorem: For any n-vertex AAT T and any $\epsilon > 0$, there exists $\delta = \Omega(\epsilon^2/\log^6 n)$ such that:

$$Pr\left[\exists v \in T : Error(v) > 1 + \epsilon\right] \leq n^{-\log n}$$

Black Box Applications

Application approximation schemes for clique-width:

- Max Cut
- Edge Dominating Set
  - Is DP additive?
- Capacitated Dom. Set (bi-criteria)
- Bounded Degree Deletion (bi-criteria)
- Equitable Coloring (bi-criteria)

- Running times $(\log n/\epsilon)^{O(k)}$
- Recall: last three are W-hard even for treewidth

AAT theorem proof sketch

Intuition for main Approximate Addition Tree theorem.
Two main cases:

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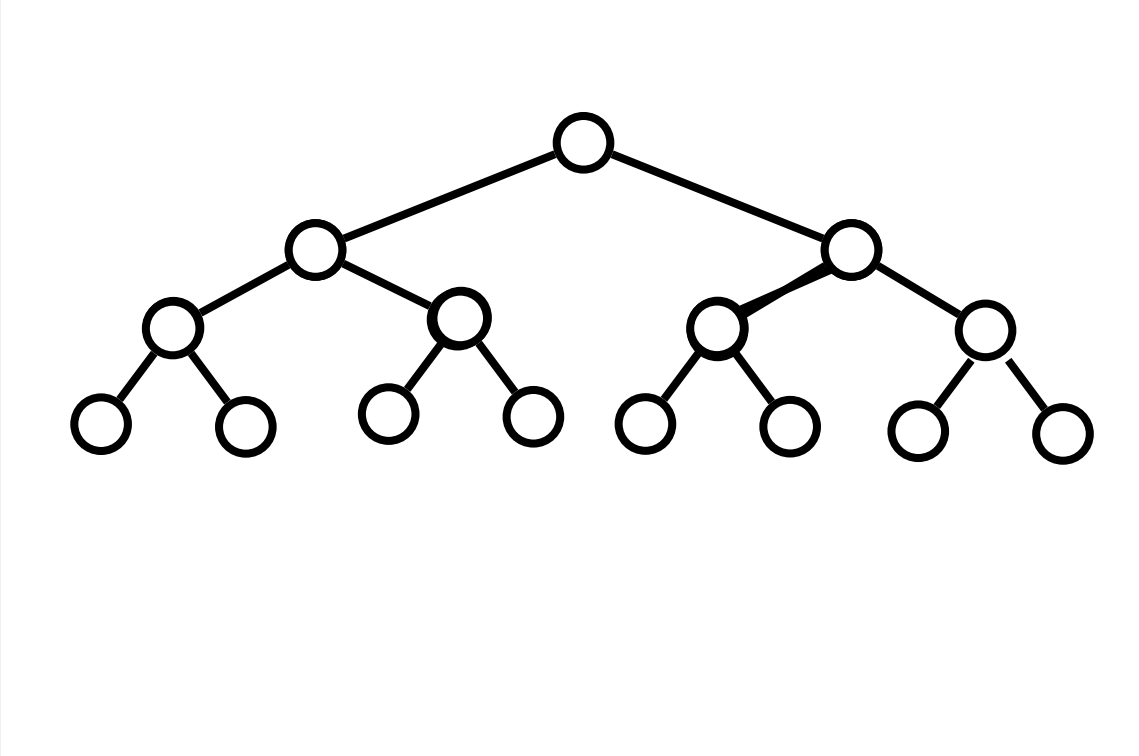

Balanced Tree: easy

AAT theorem proof sketch

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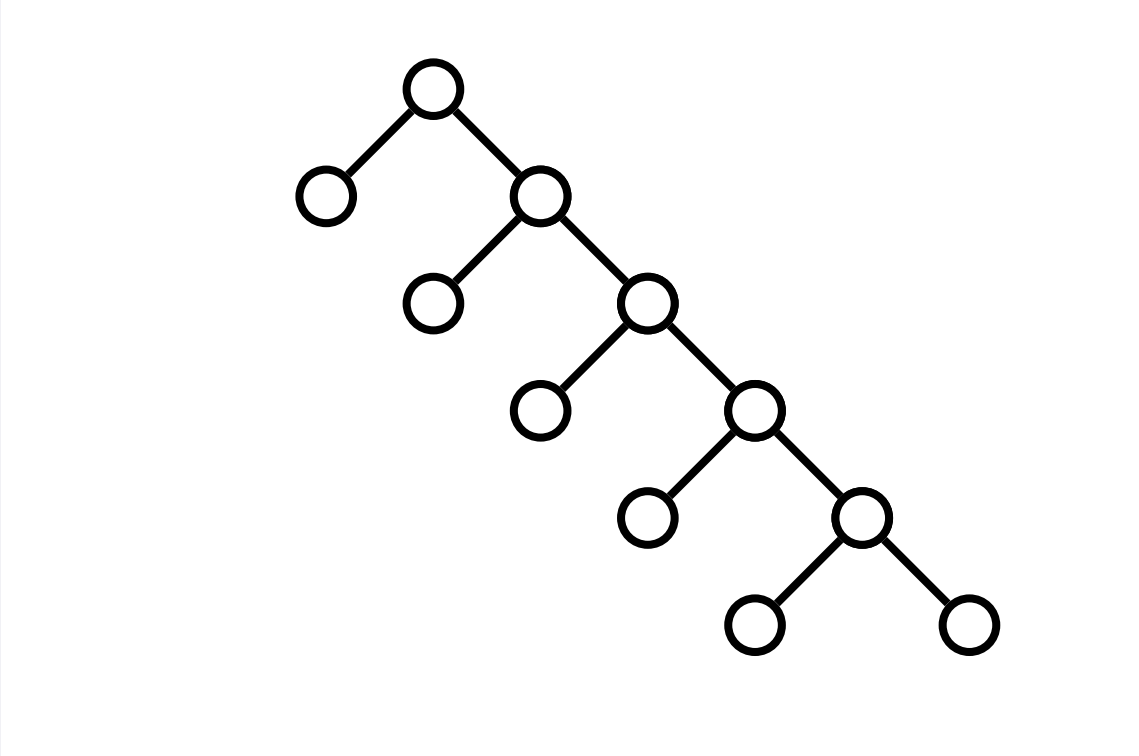


UnBalanced Tree: not so easy

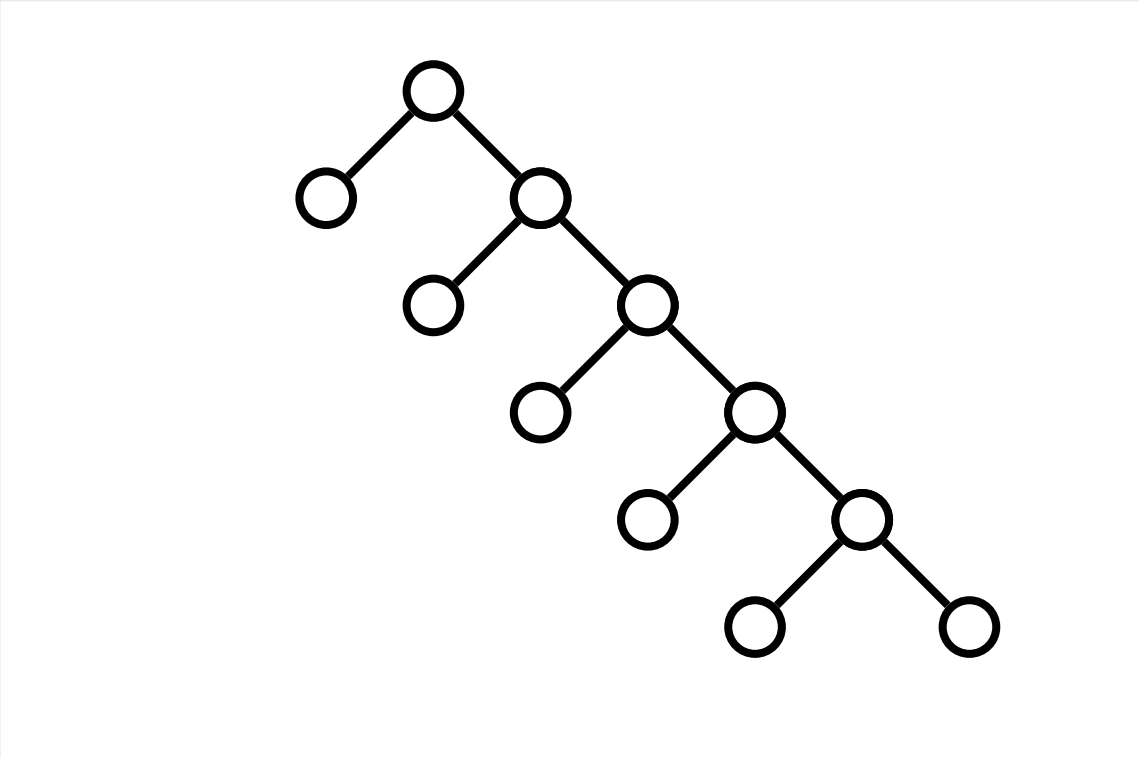
AAT theorem proof sketch

Intuition for main Approximate Addition Tree theorem.
Proof Strategy:

- Prove the theorem for UnBalanced Trees
  - Main part
- Define notion of balanced height
- Use induction
  - Base case: UnBalanced trees
  - Inductive step similar to UnBalanced case

Unbalanced case

Intuition: self-correcting random walk

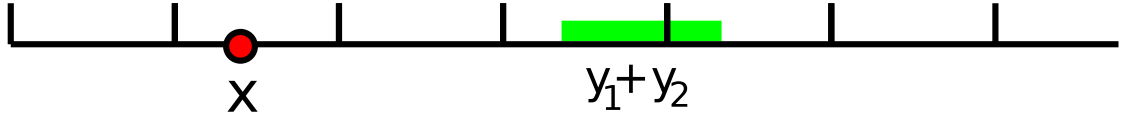


- n addition + rounding, each can increase Error by 1.
- In the end we should have error at most $\log^c n$

Unbalanced case

Intuition: self-correcting random walk

Observation 1: Each rounding step has in expectation no effect.



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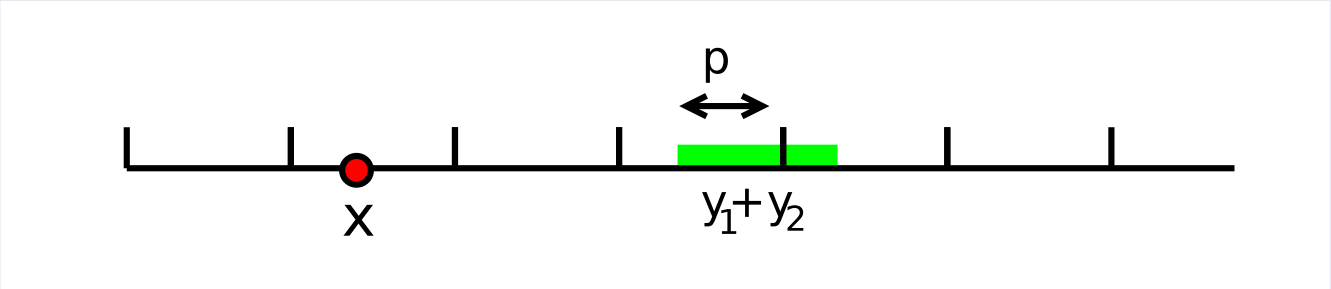

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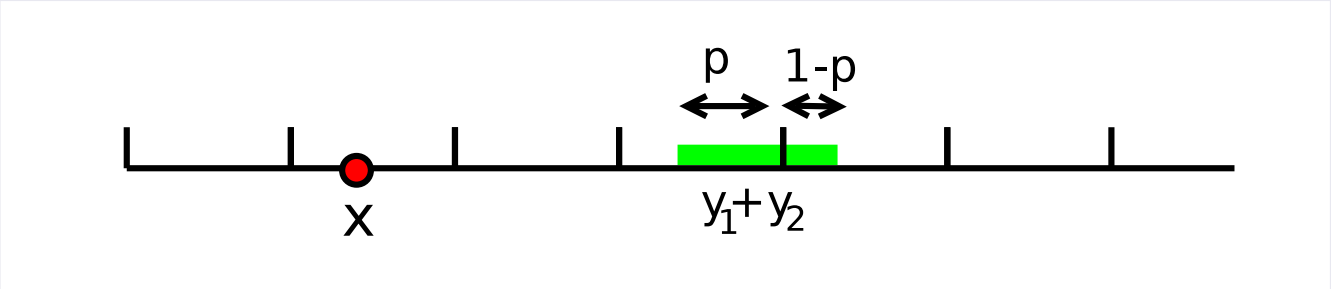

$1 - p$ is the probability of rounding up

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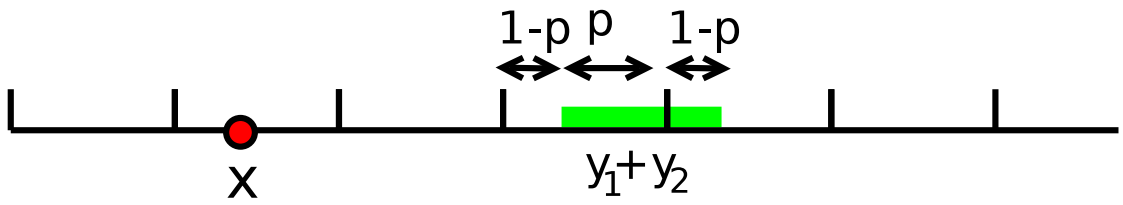

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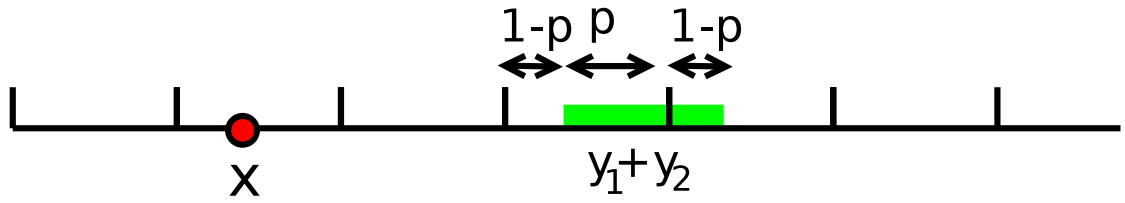

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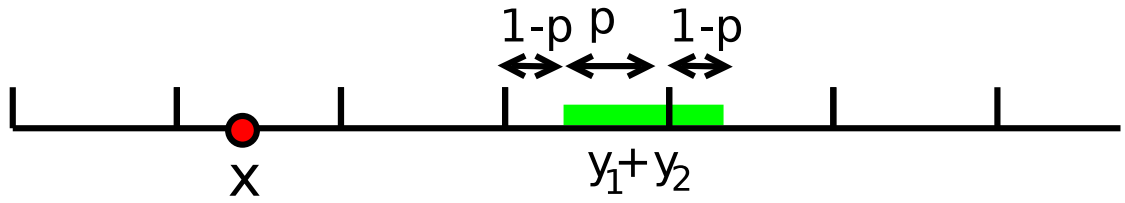

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Expected change: $-p(1-p) + (1-p)p = 0$

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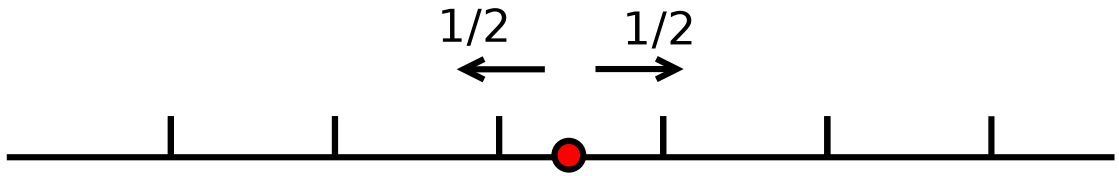

Token will end up at distance $\sqrt{n}$ whp.

We need distance $\leq \epsilon/\delta \leq \log^c n$

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Observation 2: In UnBalanced tree, initial approximate value $y_1 + y_2$ always has improved error.

- Informally: one value is known without error
- $y_1 = (1+\delta)^{E_1} x_1$
- $y_2 = (1+\delta)^{0} x_2$
- $\Rightarrow y_1 + y_2 = (1+\delta)^{E_1} x_1 + x_2 < (1+\delta)^{E_1}(x_1 + x_2)$

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- Step 1: Obtain initial approximation $\Rightarrow$ improves Error
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UnBalanced Trees are OK

Summary – Further Work

Recap:

- (Randomized) Parameterized Approximation Algorithms for several problems.
- General Approximation Result for AATs.

Further questions:

- Concrete: Hamiltonicity on clique-width
- General: Deal with other operations (subtraction?)
- Soft: Other applications of AATs?
  - Problems W-hard on trees? (e.g. parameterized by degree)

Thank you!

Questions?