Model Checking Lower Bounds for Simple Graphs

Michael Lampis
KTH Royal Institute of Technology

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Algorithmic Meta-Theorems

Positive results
- Problem X is \textit{tractable}.

Negative results
- Problem X is \textit{hard}.
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Negative results

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- An algorithmic meta-theorem is a statement of the form: "All problems in a class C are \textbf{tractable}"
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- Meta-theorems are great! (more in a second)
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Main objective of today’s talk: barriers to meta-theorems:

“There exists a problem in class C that is \textbf{hard}”
Most famous meta-theorem: Courcelle’s theorem

All MSO-expressible properties are solvable in linear time on graphs of bounded treewidth.
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All MSO-expressible properties are solvable in linear time on graphs of bounded treewidth.

Example:

$$\exists S \forall x \forall y E(x, y) \rightarrow (x \in S \iff y \notin S)$$
Good news so far

- Most famous meta-theorem: Courcelle’s theorem
  All MSO-expressible properties are solvable in linear time on graphs of bounded treewidth.
- Can we do better?
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- Can we do better?
  
  - More graphs?
  
  - Wider classes of problems?
  
  - Faster?
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  Meta-theorems for clique-width, local treewidth, ...
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  All MSO-expressible properties are solvable in linear time on graphs of bounded treewidth.

- Can we do better?
  - More graphs? ✓
  - Wider classes of problems? ✓
  - Faster?

  This can be extended to optimization versions of MSO.
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- More graphs?
- Wider classes of problems?
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Faster than linear time?
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- Can we do better?
  - More graphs?
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  - Faster?

  Faster than linear time?

This is the main question we are concerned with today.
- Courcelle's theorem:

  There exists an algorithm which, given an MSO formula $\phi$ and a graph $G$ with treewidth $w$ decides if $G \models \phi$ in time $f(w, \phi)|G|$.
Some bad news

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  There exists an algorithm which, given an MSO formula $\phi$ and a graph $G$ with treewidth $w$ decides if $G \models \phi$ in time $f(w, \phi)|G|$.

- But the function $f$ is a tower of exponentials!
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- Unfortunately, this is not Courcelle’s fault.

  Thm: If $G \models \phi$ can be decided in $f(w, \phi)|G|^c$ for elementary $f$ then $P=NP$. [Frick & Grohe ’04]
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- In fact, Frick and Grohe’s lower bound applies to FO logic on trees!
There is still hope

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- For vertex cover, neighborhood diversity, max-leaf [L. ’10]
- For twin cover [Ganian ’11]
- For shrub-depth [Ganian et al. ’12]
- For tree-depth [Gajarský and Hliňený ’12]
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Predominant idea: Removing isomorphic parts of the graph, when we have too many
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Predominant idea: Removing isomorphic parts of the graph, when we have too many

\textbf{What’s next?}
Let’s destroy all hope!

- In this talk the pendulum swings again.

- Main goal: prove hardness results even more devastating than Frick & Grohe.

- Motivation: If we know what we can’t do, we might find things we can do.
• In this talk the pendulum swings again.

• Main goal: prove hardness results even more devastating than Frick & Grohe.

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Today: Three new hardness results.

• Threshold graphs

• Paths

• Bounded-height trees
Threshold Graphs
More background

Theorem:

- MSO$_1$ expressible properties can be decided in linear time on graphs of bounded clique-width [Courcelle, Makowsky, Rotics ’00]
Theorem:

- MSO₁ expressible properties can be decided in linear time on graphs of bounded clique-width [Courcelle, Makowsky, Rotics ’00]

- Trees have clique-width 3.
  Frick&Grohe $\rightarrow$ non-elementary dependence.

- Graphs with clique-width 1 are easy for MSO₁.
Theorem:

- MSO$_1$ expressible properties can be decided in linear time on graphs of bounded clique-width [Courcelle, Makowsky, Rotics ’00]

- Trees have clique-width 3. Frick&Grohe $\rightarrow$ non-elementary dependence.

- Graphs with clique-width 1 are easy for MSO$_1$.

What about clique-width 2?
A graph is a threshold graph if it can be constructed with the following operations:

- Add a new vertex and connect it to everything.
- Add a new vertex and connect it to nothing.
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Thm: Threshold graphs have clique-width 2.
We use the following result of Frick& Grohe:

- There is no elementary-dependence model-checking algorithm for FO logic on binary strings.

Given a string $w$ we construct a threshold graph $G$

- $w$:
- $G: uu_j$
Hardness for threshold graphs

We use the following result of Frick & Grohe:

- There is no elementary-dependence model-checking algorithm for FO logic on binary strings.

Given a string $w$ we construct a threshold graph $G$

- $w : 0$
- $G : uuj \ uu$
We use the following result of Frick & Grohe:

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- $w: \begin{array}{cc} 0 & 1 \end{array}$
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Given a string $w$ we construct a threshold graph $G$

- $w$: 0 1 1 0...

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This allows us to interpret the string property as a graph question.
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- $w: \ 0 \ 1 \ 1 \ 0 \ldots$
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This allows us to interpret the string property as a graph question.

Thm: There is no elementary-dependence model-checking algorithm for FO logic on threshold graphs.
Recall some of the “good” graph classes we know

- Some are closed under complement (neighborhood diversity, shrub-depth)
- Some are closed under union (tree-depth)
Recall some of the “good” graph classes we know

- Some are closed under complement (neighborhood diversity, shrub-depth)
- Some are closed under union (tree-depth)
- None are closed under both operations...

Any class of graph closed under both operations must contain threshold graphs.
Paths
Why paths?

Main question:
- Is there an elementary-dependence algorithm for MSO$_1$ on paths?

Equivalent question:
- Is there an elementary-dependence algorithm for MSO$_1$ on unary strings?

Why?
- Do Frick and Grohe really need all trees?
- FO is easy on paths.
- MSO is hard on binary strings/colored paths.
- MSO for max-leaf is open!
Why would this be easy?

- MSO on paths = Regular language over unary alphabet
- FO is easy
- Reduction seems impossible...

“Normal” reduction:

- Start with $n$-variable 3-SAT
- Construct graph $G$ with $|G| = n^c$
- Construct formula $\phi$ with $|\phi| = \log^* n$
- Prove YES instance $\iff G \models \phi$

Problem: New instance would be encodable with $O(\log n)$ bits. We are making a sparse NP-hard language!
How the reduction can work

Key idea: do not use $P \neq NP$ but $EXP \neq NEXP$

- Motivation: reduction must construct exponential-size graph, so should be allowed exponential time.
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- Motivation: reduction must construct exponential-size graph, so should be allowed exponential time.

Plan:

- Start with an \( \text{NEXP} \)-complete problem and \( n \) bits of input.
- Construct a path on \( 2^{nc} \) vertices.
- Construct a formula \( \phi \) with \( |\phi| = \log^* n \).
- Prove YES instance \( \iff G \models \phi \).

Elementary parameter dependence gives \( \text{EXP} = \text{NEXP} \).
How the reduction can work

Key idea: do not use $P \neq NP$ but $EXP \neq NEXP$

- Motivation: reduction must construct exponential-size graph, so should be allowed exponential time.

Plan:

- Start with an NEXP-complete problem and $n$ bits of input.
- Construct a path on $2^{n^c}$ vertices.
- Construct a formula $\phi$ with $|\phi| = \log^* n$.
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Elementary parameter dependence gives $EXP = NEXP$.

- Formula will be somewhat larger, but still small enough.
Rough sketch

- Start with an NEXP Turing machine, $n$ bits of input. Does it accept?
- The machine runs in time $T = 2^{nc}$.
- Is there a transcript (of length $T^2$) that proves acceptance?
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![Diagram]

- \( T \)
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We have to be able to express the property “these vertices are at distance $T$”.

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• We have to do it with \( \log^* n \) quantifiers.

• This is possible by encoding counting in binary…
Consequences

Unless \( \text{EXP}=\text{NEXP} \):

- Max-leaf is hard
- Graph classes closed under edge sub-divisions are hard
- Graph classes closed under induced subgraphs with unbounded (dense)* diameter are hard
Trees of bounded height
Why trees of bounded height?

This class of graphs is important for two recent meta-theorems:

- Shrub-depth in “When trees grow low: Shrubs and fast MSO$_1$” [Ganian et al. MFCS ’12]
- Tree-depth in “Faster deciding MSO properties of trees of fixed height, and some consequences” [Gajarský and Hliňený FSTTCS ’12]

In both cases the main tool is the following:

MSO model-checking for $q$ quantifiers on trees of height $h$ colored with $t$ colors can be done in $\exp^{(h+1)}(O(q(t + q)))$ time.
Thm: $h + 1$ levels of exponentiation are exactly necessary.

Rough idea: use Frick& Grohe proof for trees, use (few colors) to cut down their height.

- Start from an $n$-variable 3-SAT instance.
- Construct a tree of height $h$. Use $t = \log^{(h)}(n)$ colors.
- Construct a formula with $q = O(h)$ quantifiers.
- Prove equivalence between instances.
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- Construct a formula with $q = O(h)$ quantifiers.
- Prove equivalence between instances.

Argument: an algorithm running in $\exp^{(h+1)}(o(t))$ would run in $2^{o(n)}$ here, disproving ETH.
Conclusions - Open problems

- Three natural barriers to future improvements.
- Paths are probably the toughest to work around.

Future work

- (Uncolored) tree-depth?
- Height of tower for paths?
Conclusions - Open problems

- Three natural barriers to future improvements.
- Paths are probably the toughest to work around.

Future work

- (Uncolored) tree-depth?
- Height of tower for paths?
- Other logics?!?
Thank you!