Super-polynomial time approximability of inapproximable problems

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STACS, Feb 18, 2016
Consider Time-Approximation Trade-offs for Clique.
Clique is $\tilde{\Theta}(n)$-approximable in P and optimally solvable in $\lambda^n$. 

Optimal under ETH?
Clique is $r$-approximable in time $2^{n/r}$.
Is this the correct algorithm? For every $r$?
Minimization subset problems

$I, n$
Minimization subset problems

\[ I, n \leq n/r \]
Minimization subset problems

\[
I, n \leq \frac{n}{r}
\]
Minimization subset problems

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Minimization subset problems

If a solution is found, it is an optimal solution.
Minimization subset problems

If a solution is found, it is an optimal solution.
If not, any feasible solution is an \( r \)-approximation.
Weakly monotone maximization subset problems

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Weakly monotone maximization subset problems

$I, n \leq \frac{n}{r}$
Weakly monotone maximization subset problems

$I, n \leq n/r$

If a solution is found, it is an $r$-approximation.
Weakly monotone maximization subset problems

If a solution is found, it is an $r$-approximation.
If not, there is no feasible solution.
The $r$-approximation takes time
\[ O^*(\binom{n}{n/r}) = O^*(\left(\frac{en}{n/r}\right)^{n/r}) = O^*((er)^{n/r}) = O^*(2^{n\log(er)/r}). \]
The $r$-approximation takes time
\[ O^*\left(\left(\frac{n}{r}\right)^n\right) = O^*\left(\left(\frac{en}{n/r}\right)^{n/r}\right) = O^*\left((er)^{n/r}\right) = O^*\left(2^{n\log(er)/r}\right). \]

Can we improve this time to $O^*(2^{n/r})$?
The $r$-approximation takes time
$$O^*\left(\left(\frac{n}{r}\right)^n\right) = O^*\left(\left(\frac{en}{r}\right)^{n/r}\right) = O^*\left((er)^{n/r}\right) = O^*\left(2^{n\log(er)/r}\right).$$

Can we improve this time to $O^*(2^{n/r})$?

- In this talk we don’t care! (?? sort of)
- Bottom line: $r^{n/r}$ is a Base-line Trade-off.
- When can we do better?
- When is it optimal?
Min Asymmetric Traveling Salesman Problem
Min ATSP in polytime

- $O(\log n)$-approximation [FGM ’82].
- $O\left(\frac{\log n}{\log \log n}\right)$-approximation [AGMOS ’10].

Our goal:

**Theorem**

$\forall r \leq n$, *Min ATSP* is $\log r$-approximable in time $O^*(2^{n/r})$. 
A circuit cover of minimum length can be found in polytime.
Pick any vertex in each cycle and recurse.
This can only decrease the total length (triangle inequality).
ratio = recursion depth: $\log n$ for polytime; $\log r$ for time $2^{n/r}$. 
Is this optimal? NO!
Is this close to optimal? No idea!
Inapproximability in super-polynomial time

(Randomized) Exponential Time Hypothesis:
There is no (randomized) $2^{o(n)}$-time algorithm solving 3-SAT.

**Theorem (CLN ’13)**

*Under randomized ETH, $\forall \varepsilon > 0$, for all sufficiently big $r < n^{1/2-\varepsilon}$, Max Independent Set is not $r$-approximable in time $2^{n^{1-\varepsilon}/r^{1+\varepsilon}}$.***
Inapproximability in super-polynomial time

(Randomized) Exponential Time Hypothesis:
There is no (randomized) $2^{o(n)}$-time algorithm solving 3-SAT.

**Theorem (CLN ’13)**

Under randomized ETH, $\forall \varepsilon > 0$, for all sufficiently big $r < n^{1/2-\varepsilon}$,

Max Independent Set is not $r$-approximable in time $2^{n^{1-\varepsilon}/r^{1+\varepsilon}}$.

SAT formula $\phi$ with $N$ variables $\mapsto$ graph $G$ with $r^{1+\varepsilon}N^{1+\varepsilon}$ vertices

- $\phi$ satisfiable $\Rightarrow \alpha(G) \approx rN^{1+\varepsilon}$.
- $\phi$ unsatisfiable $\Rightarrow \alpha(G) \approx r^\varepsilon N^{1+\varepsilon}$. 
Inapproximability in super-polynomial time

Goal: Assuming ETH, Π is not $r$-approximable in time $2^{o\left(n/f(r)\right)}$
Inapproximability in super-polynomial time

Goal: Assuming ETH, $\Pi$ is not $r$-approximable in time $2^{o(n/f(r))}$

SAT formula $\phi$ with $N$ variables $\leadsto I$ instance of $\Pi$ s.t.

- $|I| \approx f(r)N$
- $\phi$ satisfiable $\Rightarrow \text{val}(\Pi) \approx a$
- $\phi$ unsatisfiable $\Rightarrow \text{val}(\Pi) \approx ra$
Min Independent Dominating Set
Inapproximability in polytime \([I \ '91, H \ '93]\)

Satifiable CNF formula with \(N\) variables and \(CN\) clauses
Inapproximability in polytime [I ’91, H ’93]
Inapproximability in polytime [I ’91, H ’93]

\begin{align*}
\text{Unsatifiable CNF formula with } N \text{ variables and } CN \text{ clauses}
\end{align*}
Inapproximability in polytime [I ’91, H ’93]

MIDS of size greater than \( rN \)
Inapproximability in polytime \([I \ '91, H \ '93]\)

\[
\begin{align*}
C_1 & \quad & C_2 & \quad & C_3 & \quad & C_4 & \quad & C_5 \\
\end{align*}
\]

\[
\begin{align*}
\neg x_1 & \quad & \neg x_2 & \quad & \neg x_3 & \quad & \neg x_4
\end{align*}
\]

Set \( r = N^{9998} \approx n^{\frac{9998}{10000}} \geq n^{0.999} \)

As \( n = 2N + CrN^2 \approx N^{1000} \)
(In)approximability in subexponential time

Our goal:

**Theorem**  
*Under ETH, $\forall \varepsilon > 0, \forall r \leq n,$*  

\[
\text{MIDS is not } r\text{-approximable in time } O^*(2^{n^{1-\varepsilon} / r^{1+\varepsilon}}).
\]

almost matching the $r$-approximation in time $O^*(2^{n\log(er)/r})$. 
(In)approximability in subexponential time

Our goal:

**Theorem**

*Under ETH, ∀ε > 0, ∀r ≤ n,*

MIDS is not r-approximable in time $O^*(2^{n^{1-\varepsilon}/r^{1+\varepsilon}})$.

- In the previous reduction, $\frac{n^{1-\varepsilon}}{r^{1+\varepsilon}} \approx N^{2-\varepsilon'}$. We need to build a graph with $n \approx rN$ vertices.
(In)approximability in subexponential time

Our goal:

**Theorem**

*Under ETH, $\forall \varepsilon > 0$, $\forall r \leq n$,*

$$MIDS \text{ is not } r\text{-approximable in time } O^*(2^{n^{1-\varepsilon}/r^{1+\varepsilon}}).$$

- In the previous reduction, $\frac{n^{1-\varepsilon}}{r^{1+\varepsilon}} \approx N^{2-\varepsilon'}$.
  We need to build a graph with $n \approx rN$ vertices.

- Can we use only $r$ vertices per independent set $C_i$ and use the inapproximability of a CSP to boost the gap?
Almost linear PCP with perfect completeness?

Lemma (D ’05, BS ’04)

$\exists c_1, c_2 > 0$, we can transform $\phi$ a SAT instance of size $N$ into a constraint graph $G = \langle (V, E), \Sigma, E \rightarrow 2^\Sigma^2 \rangle$ such that:

- $|V| + |E| \leq N(\log N)^{c_1}$ and $|\Sigma| = O(1)$.
- $\phi$ satisfiable $\Rightarrow$ UNSAT$(G) = 0$.
- $\phi$ unsatisfiable $\Rightarrow$ UNSAT$(G) \geq 1/(\log N)^{c_2}$.
Constraint graph
Constraint graph
Introduction Min Independent Dominating Set Max Induced Path/Forest/Tree Max Minimal Vertex Cover
$I_{vw}$
vertices

$I_{vx}$
vertices

$I_{wx}$
vertices

$I_{wy}$
vertices

$I_{xy}$
vertices

$l_{st,ab} \leftrightarrow t[b']$ if $ab'$ satisfies $st$
Min Independent Dominating Set

Max Induced Path/Forest/Tree

Max Minimal Vertex Cover

\[ l_{vw}, l_{vx}, l_{wx}, l_{wy}, l_{xy} \]

\[ r \text{ vertices} \]

\[ l_{st} \leftrightarrow s[a] \text{ if } \exists b, ab \text{ satisfies } st \]
\( st \) is satisfied by the coloration iff \( l_{st} \) and \( \bigcup_{a,b} l_{st,ab} \) are dominated.
Take for instance $vw$ satisfied by $\bullet\bullet$. 
\(v[\bullet] \) dominates \(I_{vw} \) (\(\exists \bullet, \bullet \) satisfies \(vw\)).
\( v[\bullet] \) dominates \( I_{vw,\bullet} \) (and potentially all the \( I_{vw,ab} \) with \( a \neq \bullet \)).
$w[\bullet]$ dominates $l_{vw,\bullet}$ (and potentially all the $l_{vw,ab}$ with $a = \bullet$).
Reciprocally, \( I_{st} \) needs \( s[a] \) with \( ab \) satisfying \( st \) for some \( b \).
Then, \( I_{st,ab} \) can only be dominated by \( t[b'] \) (if \( ab' \) satisfies \( st \)).
SAT ($\phi$) $\leadsto$ CG ($V, E$) $\leadsto$ MIDS ($V', E'$)

Recall $|V| + |E| \leq N(\log N)^{c_1}$ and $\Sigma = O(1)$.

- $\phi$ satisfiable $\Rightarrow$ MIDS of size $|V| \approx N$.
- $\phi$ unsatisfiable $\Rightarrow$ MIDS of size $|V| + r \frac{|E|}{(\log N)^{c_2}} \approx rN$
- $n := |V'| \leq (|\Sigma| + 1)|V| + (1 + |\Sigma|^2)r|E| \approx rN$
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Max Induced Path/Forest/Tree
Theorem

Under ETH, $\forall \varepsilon > 0, \forall r \leq n^{1/2-\varepsilon}$,

$Max \ Induced \ Forest \ has \ no \ r \text{-approximation \ in \ time} \ 2^{n^{1-\varepsilon}/(2r)^{1+\varepsilon}}$.

A max induced forest has size in $[\alpha(G), 2\alpha(G)]$. 
**Theorem**

*Under ETH, $\forall \varepsilon > 0$, $\forall r \leq n^{1/2-\varepsilon}$, Max Induced Forest has no $r$-approximation in time $2^{n^{1-\varepsilon} / (2r)^{1+\varepsilon}}$.*

A max induced forest has size in $[\alpha(G), 2\alpha(G)]$.

- An independent set is a special forest.
- A forest has an independent set of size at least the half.
Theorem

Under ETH, \( \forall \varepsilon > 0, \forall r \leq n^{1/2-\varepsilon} \),

\[
\text{Max Induced Tree has no } r\text{-approximation in time } 2^{n^{1-\varepsilon}/(2r)^{1+\varepsilon}}.
\]

Add a universal vertex \( v \) to the gap instances of MIS: \( G \leadsto G' \).
**Theorem**

*Under ETH, \( \forall \varepsilon > 0, \forall r \leq n^{1/2-\varepsilon} \),*

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\]

Add a universal vertex \( v \) to the gap instances of MIS: \( G \leadsto G' \).

- \( G' \) has an induced tree of size \( \alpha(G) + 1 \).
- If \( T \) is an induced tree of \( G' \), \( \alpha(G) \geq |T|/2 \).
PCP-free inapproximability

Our goal:

**Theorem**

*Under ETH, \( \forall \varepsilon > 0 \) and \( \forall r \leq n^{1-\varepsilon} \),

Max Induced Path has no \( r \)-approximation in time \( 2^{o(n/r)} \).*
Walking through partial satisfying assignments

Contradicting edges are not represented
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Max Minimal Vertex Cover
Approximability in polytime [BDP '13]

- MMVC admits a $n^{1/2}$-approximation,
- but no $n^{1/2-\varepsilon}$-approximation for any $\varepsilon > 0$, unless P=NP.
Approximability in polytime [BDP ’13]

- MMVC admits a $n^{1/2}$-approximation,
- but no $n^{1/2-\varepsilon}$-approximation for any $\varepsilon > 0$, unless $P=NP$.

Our goal:

**Theorem**

For any $r \leq n$, \textit{MMVC is $r$-approximable in time $O^*(3^{n/r^2})$}.

**Theorem**

Under ETH, $\forall \varepsilon > 0$, $\forall r \leq n^{1/2-\varepsilon}$,

\textit{MMVC is not $r$-approximable in time $O^*(2^{n^{1-\varepsilon}/r^{2+\varepsilon}})$}.
Compute any maximal matching $M$. 
If $|M| \geq n/r$, then any (minimal) vertex cover contains $\geq n/r$. 
Otherwise split $M$ into $r$ parts $(A_1, A_2, \ldots, A_r)$ of size $\leq n/r^2$. 
For each of the $\leq 3^{n/r^2}$ independent sets of each $G[A_i]$,
add all the non dominated vertices of $I$, 
and compute a minimal vertex cover from the complement.
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and compute a minimal vertex cover from the complement.
An optimal solution \( R = N(\overline{R}) = N(\overline{R} \cap I) \cup \bigcup_i N(\overline{R} \cap A_i) \).
\[ \exists i, \quad |N(\overline{R} \cap I) \cup N(\overline{R} \cap A_i)| \geq \frac{|N(\overline{R})|}{r}. \]
$\overline{R} \cap A_i$ will be tried, and completed with a superset of $\overline{R} \cap I$. 
MIS (≈ $rN$ vertices) $\leadsto$ MMVC (≈ $r^2N$ vertices)
\[
\text{MIS (} \approx rN \text{ vertices)} \Rightarrow \text{MMVC (} \approx r^2N \text{ vertices)}
\]

\[
\phi \text{ satisfiable } \Rightarrow |\text{IS}| \approx rN; \quad \phi \text{ unsatisfiable } \Rightarrow |\text{IS}| \approx N.
\]
MIS (≈ $rN$ vertices) $\sim$ MMVC (≈ $r^2N$ vertices)

$\phi$ satisfiable $\Rightarrow |\text{MVC}| \approx r^2N$; $\phi$ unsatisfiable $\Rightarrow |\text{MVC}| \approx rN$. 
Open questions

▷ Is there an $r$-approximation in $O^*(2^{n/r})$ for MIDS? for Max Induced Matching?
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- Is there an $r$-approximation in $O^*(2^{n/r})$ for MIDS? for Max Induced Matching?

- Set Cover is log $r$-approximable in time $O^*(2^{n/r})$ [CKW '09] but not in time $O^*(2^{(n/r)^\alpha})$ for some $\alpha$ [M' 11]. Can we tighten this lower bound?
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- For Set Cover, we know a polytime $\sqrt{m}$-approximation [N ’07] but only an $r$-approximation in time $O^*(2^{m/r})$ [CKW ’09]. Can we match the upper and lower bounds?
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Thank you for your attention!