Chapitre 1

The Labeled perfect matching in bipartite graphs : complexity and (in)approximability

1.1. Introduction

This chapter presents some recent works accomplished by the author ([MONNOT 05, MONNOT 06]) about the complexity and the approximation properties of the LABELED perfect matching problems.

Let Π be a **NPO** problem accepting simple graphs $G = (V, E)$ as instances, edge-subsets $E' \subseteq E$ satisfying a given polynomial-time decidable property $Pred$ as solutions, and the solutions cardinality as objective function ; the labeled problem associated to Π, denoted by LABELED Π, seeks, given an instance $I = (G, L)$ where $G = (V, E)$ is a simple graph and L is a mapping from E to $\{c_1, \ldots, c_q\}$, in finding a subset E' satisfying $Pred$ that optimizes the size of the set $L(E') = \{L(e) : e \in E'\}$. Note that two versions of LABELED Π may be considered according to the optimization goal : LABELED Min Π that consists in minimizing $|L(E')|$ and LABELED Max Π that consists in maximizing $|L(E')|$. Roughly speaking, the mapping L corresponds to assigning a color (or a label) to each edge and the goal of LABELED Min Π (resp., Max Π) is to find an edge subset using the fewest (resp., the most) number of colors. If a given **NPO** problem Π is **NP**-hard, then the associated labeled problem LABELED Π is clearly **NP**-hard (consider a distinct color per edge). For instance, the LABELED Longest path problem or the LABELED maximum induced matching problem are both **NP**-hard. Moreover, if the decision problem derived from Π where one aims at deciding if a graph G contains an edge subset satisfying $Pred$

Chapitre rédigé par Jérôme MONNOT.

is **NP**-complete, then LABELED $Min\ \Pi$ can not be approximated within performance ratio better than $2-\varepsilon$ for all $\varepsilon > 0$ unless **P**=**NP**, even if the graph is complete. Indeed, if we color the edges from $G=(V,E)$ with a lonely color and then we complete the graph, adding a new color per edge, then it is **NP**-complete to decide between $opt(I)=1$ and $opt(I)\geq 2$, where $opt(I)$ is the value of an optimal solution. Notably, it is the case of the LABELED traveling salesman problem or the LABELED minimum partition problem into paths of length k for any $k\geq 2$.

Thus, labeled problems have been mainly studied, from a complexity and an approximability point of view, when Π is polynomial, [BROERSMA 97, BROERSMA 05, BRUGGEMANN 03, CHANG 97, KRUMKE 98, WAN 02, XIONG 05]. For example, the first labeled problem introduced in the literature is the LABELED minimum spanning tree problem, which has several applications in communication network design. This problem is **NP**-hard and many complexity and approximability results have been proposed in [BROERSMA 97, BRUGGEMANN 03, CHANG 97, KRUMKE 98, WAN 02, XIONG 05]. On the other hand, the LABELED maximum spanning tree problem has been shown polynomial in [BROERSMA 97]. Very recently, the LABELED path and the LABELED cycle problems have been studied in [BROERSMA 05, HASSIN 06] ; in particular, in [BROERSMA 05] authors prove that the LABELED minimum path problem is **NP**-hard and give some exact algorithms, whereas in [HASSIN 06] several approximation algorithms with performance guarantee are presented. Note that the **NP**-completeness also appears in [CARR 00] since the LABELED path problem is a special case of the red-blue set cover problem. Some other results can be found in [HASSIN 07] when the objective labeled function is of type bottleneck.

In this chapter, we go thoroughly into the investigation of the complexity and the approximability of labeled problems, with the analysis of the matching problem in bipartite graphs. The maximum matching problem is one of the most known combinatorial optimization problem and arises in several applications such as image analysis, artificial intelligence or scheduling. It turns out that a problem very closed to it has been studied in the literature, which is called in [ITAI 78] the restricted perfect matching problem. This latter aims at determining, given a graph $G=(V,E)$, a partition $E_1,\ldots,E_k$ of E and k positive integers $r_1,\ldots,r_k$, whether there exists a perfect matching M on G satisfying for all $j=1,\ldots,k$ the restrictions $|M\cap E_j|\leq r_j$. This problem has some relationship with the timetable problem, since a solution may be seen as a matching between classes and teachers that satisfies additional restrictions (for instance, no more that r laboratories at the same time). The restricted perfect matching problem is proved to be **NP**-complete in [ITAI 78], even if (i) $|E_j|\leq 2$, (ii) $r_j=1$, and (iii) G is a bipartite graph. On the other hand, it is shown in [YI 02] that the restricted perfect matching problem is polynomial when G is a complete bipartite graph and $k=2$; some others results of this problem can be found in [COSTA 06]. A perfect matching M only satisfying condition (ii) (that is to say $|M\cap E_i|\leq 1$) is called good in [CAMERON 97]. Thus, we deduce that the LABELED maximum perfect matching problem is **NP**-hard in bipartite graph since $opt(I) = n$ iff G contains a good matching.

In section 1.2, we analyze both the complexity and the approximability of the LABELED minimum perfect matching problem and the LABELED maximum perfect matching problem in 2-regular bipartite graphs. In particular, we deduce that these both problems are in **APX** when the graph has a maximum degree 2. Then, in section 1.3, we propose some inapproximation results when the bipartite graphs have a maximum degree at least 3 or are 3-regular. Actually, we prove first that LABELED minimum perfect matching is not in **APX** whenever the bipartite graphs have a maximum degree of 3. Hence, there is a gap of approximability between graphs of maximum degree 2 and 3. Using a weaker complexity hypothesis, we can even obtain that LABELED minimum perfect matching is not $2^{O(log^{1-\varepsilon} n)}$-approximable in bipartite graphs of maximum degree 3 on n vertices, unless **NP** $\subseteq$ **DTIME** $\left(2^{O(log^{1/\varepsilon} n)}\right)$. Dealing with the unbounded degree case, this yields to the fact that LABELED minimum perfect matching is not in **polyLog-APX**, unless **P** $=$ **NP**. Finally, section 1.4 focuses on the case of complete bipartite graphs. In particular, it is shown that a greedy algorithm picking at each iteration a monocolored matching of maximum size provides a $\frac{r+H_r}{2}$-approximation in bipartite complete graphs where r is the maximum number of times that a color appears in the graph and H_r is the r-th harmonic number.

Now, we introduce some terminology and notations that will be used in the chapter. A *matching* M on a graph $G = (V, E)$ is a subset of edges that are pairwise non adjacent ; M is said a *perfect matching* if it covers the vertex set of G. A graph $G = (V, E)$ is *bipartite* if the vertex set can be partitioned into two sets, the left set L and the right set R such that every edge of G has an endpoint in L and the other in R. In the labeled perfect matching problem (LABELED PM in short), we are given a simple graph $G = (V, E)$ on $|V| = 2n$ vertices which contains a perfect matching together with a color (or label) function $L : E \rightarrow \{c_1, \dots, c_q\}$ on the edge set of G. For $i = 1, \dots, q$, we denote by $L^{-1}(\{c_i\}) \subseteq E$ the set of edges of color c_i. The goal of LABELED $Min\ PM$ (resp., $Max\ PM$) is to find a perfect matching on G using a minimum (resp., a maximum) number of colors. An equivalent formulation of LABELED $Min\ PM$ could be the following : if $G[\mathcal{C}]$ denotes the subgraph induced by the edges of colors $\mathcal{C} \subseteq \{c_1, \dots, c_q\}$, then LABELED $Min\ PM$ aims at finding a subset $\mathcal{C}$ of minimum size such that $G[\mathcal{C}]$ contains a perfect matching. The restriction of LABELED PM to the case where each color occurs at most r times in $I = (G, L)$ (i.e., $|L^{-1}(\{c_i\})| \leq r$ for $i = 1, \dots, q$) will be denoted by LABELED PM_r. The LABELED $Min\ PM$ problem has some relationship with the timetable problem, since a solution may be seen as a matching between classes and teachers that satisfies additional restrictions (for instance, a color corresponds to a school where we assume that a professor may teach in several schools). An inspector would like to assess all teachers during one lecture of each one of them and it would be desirable that (s)he visits not twice the same class. Hence the lectures to be attended would form a maximum matching. For convenience the inspector would like these lectures to take place in the smallest possible number of schools. Then clearly the inspector has to construct a maximum matching meeting a minimum number of colors in the graph associated with the lectures. In [RICHEY 92] a generalization, called perfect matching under categorization, has been studied. In this framework, each edge e has also a non-negative weight $w(e)$, and the colors are called categories (thus, q indicates the number of categories). The goal is to find a perfect matching M of E minimizing $\sum_{i=1}^{q} \max_{e\in\mathcal{L}_i\cap M} w(e)$. In [RICHEY 92], it is shown that, on the one hand, the problem is polynomial when the number of categories (i.e., colors) is fixed, and on the other hand, the problem is **NP**-hard when the weights take values 0 or 1 and the graph is a collection of disjoint 4-cycles. Note that the case $w(e) = 1, \forall e \in E$ corresponds to LABELED $Min\ PM$.

We denote by $opt(I)$ and $apx(I)$ the value of an optimal and an approximate solution, respectively. We say that an algorithm $\mathcal{A}$ is an ε-approximation of LABELED $Min\ PM$ with $\varepsilon \geq 1$ (resp., $Max\ PM$ with $\varepsilon \leq 1$) if $apx(I) \leq \varepsilon \times opt(I)$ (resp., $apx(I) \geq \varepsilon \times opt(I)$) for any instance $I = (G, L)$.

1.2. The 2-regular bipartite case

In this section, we deal with a particular class of graphs that consist in a collection of pairwise disjoints cycles of even length ; note that such graphs are 2-regular bipartite graphs.

THEOREM 1.– LABELED $Min\ PM_r$ *is **APX**-complete in 2-regular bipartite graphs for any* $r \geq 2$.

PROOF.– *O*bserve that any solution of LABELED $Min\ PM_r$ is an r-approximation. The rest of the proof will be done via an approximation preserving reduction from the minimum balanced satisfiability problem with clauses of size at most r, MIN BALANCED r-SAT for short. An instance $I = (\mathcal{C}, X)$ of MIN BALANCED r-SAT consists in a collection $\mathcal{C} = (C_1, \ldots, C_m)$ of clauses over the set $X = \{x_1, \ldots, x_n\}$ of boolean variables, such that each clause C_j has at most r literals and each variable appears positively as many time as negatively ; let B_i denotes this number for any $i = 1, \ldots, n$. The goal is to find a truth assignment f satisfying a minimum number of clauses. MIN BALANCED 2-SAT where $2 \leq B_i \leq 3$ has been shown **APX**-complete by the way of an L-reduction from MAX BALANCED 2-SAT where $B_i = 3$, [BERMAN 98, KARPINSKI 05].

We only prove the case $r = 2$. Let $I = (\mathcal{C}, X)$ be an instance of MIN BALANCED 2-SAT on m clauses $\mathcal{C} = \{C_1, \ldots, C_m\}$ and n variables $X = \{x_1, \ldots, x_n\}$ such that each variable x_i has either 2 occurrences positive and 2 occurrences negative,

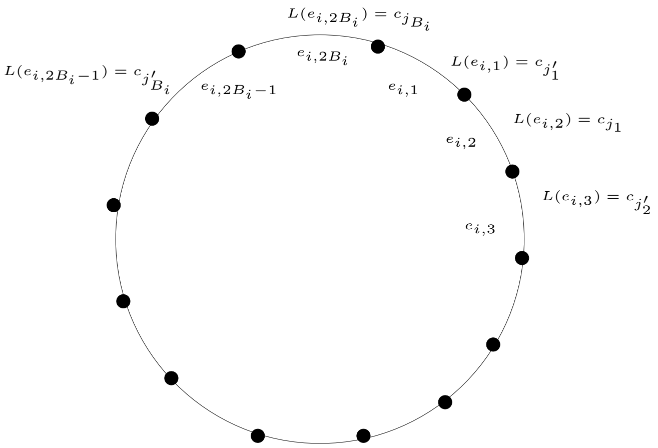

Figure 1.1. *The gadget $H(x_i)$ and the color of its edges.*

or 3 occurrences positive and 3 occurrences negative. We build the instance $I' = (H, L)$ of LABELED $Min\ PM_2$ where H is a collection of pairwise disjoints cycles $\{H(x_1), \ldots, H(x_n)\}$ and L colors edges of H with colors $c_1, \ldots, c_j, \ldots, c_m$, by applying the following process :

• For each variable x_i, create the $2B_i$-long cycle $H(x_i) = \{e_{i,1}, \ldots, e_{i,k}, \ldots, e_{i,2B_i}\}$.

• Color the edges of $H(x_i)$ as follows : if x_i appears positively in clauses $C_{j_1}, \ldots, C_{j_{B_i}}$ and negatively in clauses $C_{j'_1}, \ldots, C_{j'_{B_i}}$, then set $L(e_{i,2k}) = c_{j_k}$ and $L(e_{i,2k-1}) = c_{j'_k}$ for $k = 1, \ldots, B_i$.

Figure 1.1 provides an illustration of the gadget $H(x_i)$. Clearly, H is made of n disjoint cycles and is painted with m colors. Moreover, each color appears at most twice.

Let f^* be an optimal truth assignment on I satisfying m^* clauses and consider the perfect matching $M = \cup_{i=1}^{n} M_i$ where $M_i = \{e_{i,2k} | k = 1, \ldots, B_i\}$ if $f(x_i) = true$, $M_i = \{e_{i,2k-1} | k = 1, \ldots, B_i\}$ otherwise ; M uses exactly m^* colors and thus :

$$opt(I) \leq m^* \qquad [1.1]$$

Conversely, let M' be a perfect matching on H using $apx(I) = m'$ colors ; if one sets $f'(x_i) = true$ if $e_{i,2} \in M'$, $f'(x_i) = false$ otherwise, we can easily observe that the truth assignment f' satisfies m' clauses.

$$apx(I) = val(f') \qquad [1.2]$$

Hence, using inequalities [1.1] and [1.2] the result follows.

Trivially, the problem becomes obvious when each color is used exactly once. We now show that we have a 2-approximation in 2-regular bipartite graphs, showing that the restriction of LABELED $Min\ PM$ to 2-regular bipartite graphs is as easy as approximate as MINSAT.

THEOREM 2.– *There exists an approximation preserving reduction from* LABELED $Min\ PM$ *in 2-regular bipartite graphs to* MINSAT *of expansion* $c(\varepsilon) = \varepsilon$.

PROOF.– The result comes from the reciprocal of the previous transformation. Let $I = (G, L)$ be an instance of LABELED $Min\ PM$ where $G = (V, E)$ is a collection $\{H_1, \ldots, H_n\}$ of disjoint cycles of even length and $L(E) = \{c_1, \ldots, c_m\}$ defines the label set, we describe every cycle H_i as the union of two matchings M_i and $\overline{M_i}$. We construct an instance $I' = (\mathcal{C}, X)$ of the satisfiability problem, MINSAT where $\mathcal{C} = \{C_1, \ldots, C_m\}$ is a set of m clauses and $X = \{x_1, \ldots, x_n\}$ is a set of n variables, as follows. The clause set $\mathcal{C}$ is in one to one correspondence with the color set $L(E)$ and the variable set X is in one to one correspondence with the connected components of G ; a literal x_i (resp., $\overline{x_i}$) appears in C_j iff $c_j \in L(M_i)$ (resp., $c_j \in L(\overline{M_i})$). We easily deduce that any truth assignment f on I' that satisfies k clauses can be converted into a perfect matching M_f on I that uses k colors.

Using the 2-approximation of MINSAT [MARATHE 96] and the Theorem 2, we deduce :

COROLLARY 1.– LABELED $Min\ PM$ *in 2-regular bipartite graphs is 2-approximable.*

Dealing with LABELED $Max\ PM_r$, the result of [ITAI 78] shows that computing a *good matching* is **NP**-hard even if the graph is bipartite and each color appears at most twice ; a good matching M is a perfect matching using $|M|$ colors. Thus, we deduce from this result that LABELED $Max\ PM_r$ is **NP**-hard for any $r \geq 2$. We strengthen this result using a reduction from MAX BALANCED 2-SAT.

THEOREM 3.– $Max\ PM_r$ *is* ***APX****-complete in 2-regular bipartite graphs for any* $r \geq 2$.

In the same way, using the approximate result for MAXSAT [ASANO 02], we obtain

COROLLARY 2.– LABELED $Max\ PM$ *in 2-regular bipartite graphs is 0.7846-approximable.*

1.3. Some inapproximation results

In order to simplify the proofs exposed in the rest of the section, the results concern a variation of LABELED $Min\ PM$, where the value of each perfect matching M is given by $val_1(M) = val(M) - 1$. This problem is denoted LABELED $Min\ PPM$ and we have for any instance I, $apx_1(I) = apx(I) - 1$ and $opt_1(I) = opt(I) - 1$. It is important to note that a $\rho(n)$-approximation of LABELED $Min\ PM$ becomes a $2\rho(n)$-approximation of LABELED $Min\ PPM$, and conversely a $\rho(n)$-approximation of LABELED $Min\ PPM$ remains a $\rho(n)$-approximation of LABELED $Min\ PM$. Actually, since LABELED $Min\ PM$ is simple, [PAZ 81] (i.e., the restriction to $opt(I) \leq k$ is polynomial), we can see that LABELED $Min\ PM$ and LABELED $Min\ PPM$ are asymptotically equivalent to approximate. Hence, the proposed results for LABELED $Min\ PPM$ also hold LABELED $Min\ PM$.

We now propose a self improving operation for some classes of instances $\mathcal{P}_k$ described as follows. $I = (H, \mathcal{L}) \in \mathcal{P}_k$ where $H = (V, E)$ if and only if the following properties are satisfied :

(i) H is planar of maximum degree k and connected.

(ii) $\exists u, v \in V$ such that $[u, u_1]$ and $[v, v_1]$ for some $u_1, v_1 \in V$ are the only edges incident to u and v. Moreover, these two edges have color c_0, i.e., $\mathcal{L}([u, u_1]) = \mathcal{L}([v, v_1]) = c_0$.

(iii) H is bipartite and admits a perfect matching.

(iv) $H[\{c_0\}]$, the subgraph induced by edges of color c_0 does not have any perfect matching and the subgraph $H[\mathcal{L}(E) \setminus \{c_0\}]$ induced by edges of colors different from c_0 is acyclic.

(v) if $H' = H \setminus \{u, v\}$ denotes the subgraph induced by $V \setminus \{u, v\}$, then $H'[\{c_0\}]$ has a perfect matching denoted by M_{c_0}.

We have $\mathcal{P}_1 = \emptyset$ and $\mathcal{P}_2$ is the set of odd paths from u to v alternating matchings M and M_{c_0} where M_{c_0} is only colored by color c_0. Finally, we define the class $\mathcal{P}$ by $\mathcal{P} = \cup_k \mathcal{P}_k$.

Restricted label squaring operation. Given an instance $I = (H, \mathcal{L}) \in \mathcal{P}_k$ of LABELED $Min\ PM$, its *label squaring instance* is $I^2 = (H^2, \mathcal{L}^2)$ with $H^2 = (V^2, E^2)$, where

1) The graph H^2 is created by removing each edge $e = [x, y]$ of H with color different from c_0 and placing instead of it a copy $H(e)$ of H, such that x and y are now identified with u and v of H, respectively.

2) For each copy $H(e)$ of H and for an edge e' in $H(e)$ with color different from c_0, the new color of e' is $\mathcal{L}^2(e') = (\mathcal{L}(e), \mathcal{L}(e'))$. The remaining edges of copy $H(e)$ keep their color c_0, that is if $\mathcal{L}(e') = c_0$, then $\mathcal{L}^2(e') = c_0$.

Let us prove that classes $\mathcal{P}_k$ are closed under the restricted label squaring operation.

LEMMA 1.– *If $I \in \mathcal{P}_k$, then $I^2 \in \mathcal{P}_k$.*

PROOF.– Let $I \in \mathcal{P}_k$. The proofs of (i) and (ii) are obvious.

For (iii), since H and $H \setminus \{u, v\}$ admit a perfect matching, we deduce that $u \in L$ and $v \in R$ where (L, R) is the bipartition of H. Thus, we can extend the bipartition to H^2 by taking for each $H(e)$ a copy of the bipartition. Finally, it is easy to verify that H^2 admits a perfect matching if H does.

For (iv) assume the reverse, that is $H^2[\{c_0\}]$ admits a perfect matching M and $H[\{c_0\}]$ not. By hypothesis, in each copy $H([x, y])$, the vertices x and y are not saturated by M and then the edges of M which do not traverse copies $H(e)$ form a perfect matching of $H[\{c_0\}]$, contradiction. Moreover, using property (ii), it is easy to verify that the subgraph $H^2[\mathcal{L}^2(E^2) \setminus \{c_0\}]$ is acyclic whenever $H[\mathcal{L}(E) \setminus \{c_0\}]$ is acyclic.

For (v) let M_{c_0} be a perfect matching of $H' = H \setminus \{u, v\}$ only using color c_0. We complete M_{c_0} by taking for each copy $H(e)$ a copy of M_{c_0}. In this way, we obtain a perfect matching of $H^2 \setminus \{u, v\}$ that uses only color c_0.

We now propose an approximation preserving reduction using the label squaring operation on $\mathcal{P}_k$.

THEOREM 4.– *Let $I = (H, \mathcal{L}) \in \mathcal{P}_k$. If there exists a (polynomial) ρ-approximation of I^2 for* LABELED $Min\ PPM$*, then there exists a $\sqrt{\rho}$-approximation of I for* LABELED $Min\ PPM$.

PROOF.– Let M^* be an optimal perfect matching of $I \in \mathcal{P}_k$ using $opt(I)$ colors and let $e_1, \cdots, e_p$ be the edges of H using colors different from c_0. For each copy $H(e_i)$ we take a copy of M^* using colors $(\mathcal{L}(e_i), \mathcal{L}(e_j))$ for $j = 1, \cdots, p$ and color c_0. For the remaining copies, we take a copy of M_{c_0} (a perfect matching on $H \setminus \{u, v\}[\{c_0\}]$) and we complete this matching into a perfect matching of H^2 using the remaining edges of M^*. This matching uses $(opt(I) - 1)^2 + 1$ colors and thus

$$opt_1(I^2) \leq opt_1^2(I) \tag{1.3}$$

Now, consider an approximate perfect matching M^2 of H^2 with value $apx(I^2)$ and let $H(e_1), \cdots, H(e_p)$ be the copies of H such that the restriction of M^2 to $H(e_i)$ is a perfect matching. Hence, we may always assume that $M^2 \setminus (\cup_{i=1}^{p} H(e_i))$ only uses color c_0. Therefore, if we denote $\mathcal{L}' = \{\mathcal{L}(e_i) : i = 1, \cdots, p\}$, then for any

$c_j \in \mathcal{L}'$ there exists a perfect matching $M_{c_j,k} \subseteq M^2$ in copy $H(e_k)$ such that edge e_k has color c_j. Let M_{c_j} be a matching of H minimizing $|\mathcal{L}(M_{c_j,k})|$ for any $c_j \in \mathcal{L}'$ and let M_0 be a perfect matching of H containing edges $\{e_1, \dots, e_p\}$ and some other edges of color c_0.

The approximate perfect matching M of I will be given by one of the matchings M_{c_j} or M_0 with value $apx(I) = \min\{|\mathcal{L}(M_0)|, |\mathcal{L}(M_{c_j})| : c_j \in \mathcal{L}'\}$. Thus, we deduce that $apx_1(I) = apx(I) - 1 = \min\{|\mathcal{L}(M_0)| - 1, |\mathcal{L}(M_{c_j})| - 1 : c_j \in \mathcal{L}'\}$ and hence :

$$\begin{aligned} apx_1^2(I) &\leq (|\mathcal{L}(M_0)| - 1) \min\{|\mathcal{L}(M_{c_j})| - 1 : c_j \in \mathcal{L}'\}; \\ &\leq \sum_{c_j \in \mathcal{L}'} \big(|\mathcal{L}(M_{c_j})| - 1\big) \leq apx_1(I^2) \end{aligned} \qquad [1.4]$$

Applying inequality [1.4] with an optimal perfect matching M^2 of H^2, we obtain $opt_1^2(I) \leq opt_1(I^2)$. Using inequality [1.3], we deduce $opt_1^2(I) = opt_1(I^2)$ and the expected result follows.

THEOREM 5.– LABELED $Min\ PPM$ *is not* $c \log n$ *approximable for some constant* $c > 0$ *for instances in* $\mathcal{P}$ *having* $2n$ *vertices, unless* ***P*=*NP***.

PROOF.– Given a family $\mathcal{S} = \{S_1, \dots, S_{n_0}\}$ of subsets of a ground set $X = \{x_1, \dots, x_{m_0}\}$ (we assume that $\cup_{i=1}^{n_0} S_i = X$), a set cover of X is a sub-family $\mathcal{S}' = \{S_{f(1)}, \dots, S_{f(p)}\} \subseteq \mathcal{S}$ such that $\cup_{i=1}^{p} S_{f(i)} = X$; the *set cover problem* denoted by MINSC is the problem of determining a minimum-size set cover $\mathcal{S}^* = \{S_{f^*(1)}, \dots, S_{f^*(q)}\}$ of X. Given an instance $I_0 = (\mathcal{S}, X)$ of MINSC, its *characteristic graph* $G_{I_0} = (L_0, R_0; E_{I_0})$ is a bipartite graph with a left set $L_0 = \{l_1, \dots, l_{n_0}\}$ that represents the members of the family $\mathcal{S}$ and a right set $R_0 = \{r_1, \dots, r_{m_0}\}$ that represents the elements of the ground set X ; the edge-set E_{I_0} of the characteristic graph is defined by $E_{I_0} = \{[l_i, r_j] : x_j \in S_i\}$.

From I_0, we construct the instance $I = (H, \mathcal{L})$ of LABELED $Min\ PPM$ containing $(n_0 + 1)$ colors $\{c_0, c_1, \cdots, c_{n_0}\}$, described as follows :

- For each element $x_j \in X_0$, we build a gadget $H(x_j)$ that consists of a bipartite graph of $2(d_{G_{I_0}}(r_j) + 3)$ vertices and $3d_{G_{I_0}}(r_j) + 4$ edges, where $d_{G_{I_0}}(r_j)$ denotes the degree of vertex $r_j \in R$ in G_{I_0}. The graph $H(x_j)$ is illustrated in Figure 1.2.
- Assume that vertices $\{l_{f(1)}, \dots, l_{f(p)}\}$ are the neighbors of r_j in G_{I_0}, then color $H(x_j)$ as follows : for any $k = 1, \dots, p$, $L(v_{3,j}, l_{j,f(k)}) = c_{f(k)}$ and the other edges receive color c_0.

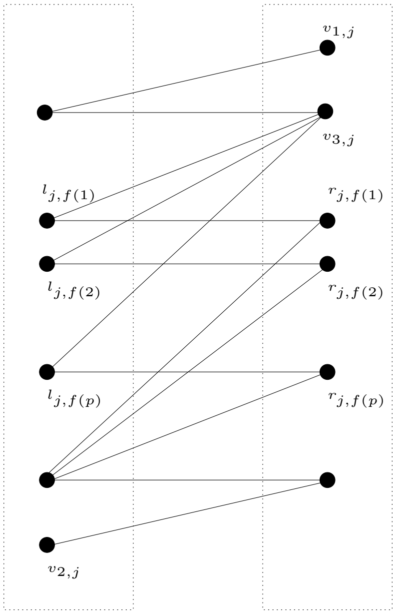

Figure 1.2. *The gadget* $H(x_j)$.

– We complete $H = \cup_{x_j \in X} H(x_j)$ by adding edges $[v_{2,j}, v_{1,j+1}]$ with color c_0 for $j = 1, \cdots, m_0 - 1$.

– Finally, we set $u = v_{1,1}$ and $v = v_{2,m_0}$.

Clearly, $I \in \mathcal{P}$ and has $2n = 2\sum_{r_j \in R}(d_{G_{I_0}}(r_j) + 3) = 2|E_{I_0}| + 6m_0$ vertices.

Let $\mathcal{S}^*$ be an optimal set cover on I_0. From $\mathcal{S}^*$, we can easily construct a perfect matching M^* of $I = (H, \mathcal{L})$ that uses exactly $(|\mathcal{S}^*| + 1)$ colors. Conversely, let M be a perfect matching on I ; by construction, the subset $\mathcal{S}' = \{S_k : c_k \in \mathcal{L}(M)\}$ of $\mathcal{S}$ is a set cover of X using $(|\mathcal{L}(M)| - 1)$ sets.

Now, it is well known that the set cover problem is **NP**-hard to approximate within factor $c \log n_0$ for some constant $c > 0$. This result also applies to instances $(X, \mathcal{S})$ when $|X|$ and $|\mathcal{S}|$ are polynomially related (i.e., $|X|^q \leq |\mathcal{S}| \leq |X|^p$ for some constants p, q).

Hence, given such an instance $I_0 = (X, \mathcal{S})$, from any algorithm A solving LABELED $Min\ PPM$ within a performance ratio $\rho_{\mathrm{A}}(I) \leq \frac{c}{q+1} \times \log(n)$ for a bipartite graph on $2n$ vertices, we can deduce an algorithm for MINSC that guarantees the performance ratio $c\frac{1}{q+1}\log(n) \leq c\frac{1}{q+1}\log(n_0^{q+1}) = c\log(n_0)$, contradiction.

Starting from the **APX**-completeness result for the vertex cover problem in cubic graphs, [ALIMONTI 00], we are able to obtain the following result.

COROLLARY 3.– LABELED $Min\ PPM$ *for instances in* $\mathcal{P}_3$ *is not in* ***PTAS****.*

PROOF.– *S*tarting from the restriction of set cover where each element x_i is covered by exactly two sets (this case is usually called the *vertex cover problem* and denoted by MINVC), we apply the same proof as in Theorem 5. The instance I becomes an element of $\mathcal{P}_3$, and using for instance the hardness result of [ALIMONTI 00], the expected result follows.

By applying the well known method of self improving, we obtain the two following results :

THEOREM 6.– LABELED $Min\ PPM$ *for instances in* $\mathcal{P}_3$ *is not in* ***APX****, unless* ***P*** $=$ ***NP****.*

PROOF.– *A*ssume the reverse and let A be a polynomial algorithm solving LABELED $Min\ PPM$ within a constant performance ratio ρ. Let $\varepsilon > 0$ (with $\varepsilon < \rho - 1$) and choose the smallest integer q such that :

$$q \geq \log\log\rho - \log\log(1+\varepsilon) \qquad [1.5]$$

Consider now an instance $I = (H, \mathcal{L}) \in \mathcal{P}_3$ and use the restricted label squaring operation on I. We produce the instance $I^2 = (H^2, \mathcal{L}^2)$ and by repeating q times this operation on I^2, we obtain thanks to Lemma 1 the instance $I^{2^q} = (H^{2^q}, \mathcal{L}^{2^q}) \in \mathcal{P}_3$, in time $P(|I|)$ for some polynomial P since on the one hand, I^2 is obtained from I in time $O(|I|^2)$ (we have $|V(H^2)| = O(|V(H)|^2)$ and $|\mathcal{L}^2(E(H^2))| = O(|\mathcal{L}(E(H))|^2)$) and on the other hand, we repeat this operation a constant number of times. Using Theorem 4, from the ρ-approximation on I^{2^q} given by A , we obtain a $\rho^{2^{-q}}$-approximation on I. Thanks to inequality [1.5], we deduce $\rho^{2^{-q}} \leq 1+\varepsilon$. Hence, we obtain a polynomial time approximation scheme for instances in $\mathcal{P}_3$, contradiction with Corollary 3.

THEOREM 7.– *For any* $\varepsilon > 0$ LABELED $Min\ PPM$ *is not* $2^{O(log^{1-\varepsilon} n)}$*-approximable for instances in* $\mathcal{P}_3$ *on* n *vertices, unless* ***NP*** $\subseteq$ ***DTIME*** $\left(2^{O(log^{1/\varepsilon} n)}\right)$.

PROOF.– *L*et $\varepsilon > 0$ and $I = (H, \mathcal{L}) \in \mathcal{P}_3$ where H has n vertices. Choose the smallest integer p such that $n^{2^p} \geq 2^{\log^{1/\varepsilon} n}$. Thus, $2^{2^p \times \log n} \geq 2^{\log^{1/\varepsilon} n}$ and then,

$$2^{p\times\varepsilon} \geq \log^{1-\varepsilon} n \qquad [1.6]$$

Using the restricted label squaring operation on I, we produce the instance $I^2 = (H^2, \mathcal{L}^2)$. By repeating p times this operation on I^2, we obtain the instance $I^{2^p} = (H^{2^p}, \mathcal{L}^{2^p}) \in \mathcal{P}_3$. Since H has n vertices, we derive from property (iv) of Lemma 1 that the number n' of vertices of H^{2^p} and the number $|\mathcal{L}^{2^p}(E(H^{2^p}))|$ of colors of H^{2^p} satisfy :

$$n' \leq n^{2^p} \text{ and } |\mathcal{L}^{2^p}(E(H^{2^p}))| \leq |\mathcal{L}(E(H))|^{2^p} \qquad [1.7]$$

Now, assume that we have a $f(n')$-approximation on I^{2^p} where $f(n') \leq 2^{c \times \log^{1-\varepsilon} n'}$ for some $c > 0$. Using Theorem 4, we obtain a $f(n')^{2^{-p}}$-approximation on I. Using inequalities [1.6] and [1.7], we deduce :

$$\begin{array}{rcl} apx_1(I) & \leq & f(n')^{2^{-p}} opt_1(I) \\ & \leq & 2^{c \times \frac{\log^{1-\varepsilon} n'}{2^p}} opt_1(I) \\ & \leq & 2^{c \times \frac{\log^{1-\varepsilon} n}{2^{\varepsilon \times p}}} opt_1(I) \\ & \leq & 2^c opt_1(I) \ , \end{array}$$

Thus, using inequality [1.7], we obtain a constant approximation in time $poly(n') = 2^{O(\log^{1/\varepsilon} n)}$, and thus, a contradiction with Theorem 6.

It is natural to wonder whether the problem is easier in 3-regular bipartite graphs (also called cubic bipartite graphs) or not. Here, we prove that the answer is negative.

THEOREM 8.– LABELED *$Min\ PPM$ is not in **APX** in connected planar cubic bipartite graphs, unless **P** $=$ **NP**.*

PROOF.– *T*he proof consists of two steps. First, using a quite similar reduction to the one of Corollary 3, we prove that Theorem 7 also holds for the sub-family $\mathcal{P}'_3$ of $\mathcal{P}_3$ where each vertex has a degree 3, except u and v. Then, we transform any instance of $\mathcal{P}'_3$ into a connected planar cubic bipartite graph.

Let $G = (V, E)$ with $V = \{v_1, \cdots, v_n\}$ and $E = \{e_1, \cdots, e_n\}$ be an instance of vertex cover. We transform any edge $e_j = [x, y]$ into gadget $H(e_j)$ described in Figure 1.3. All edges of $H(e_j)$, except $[v_{3,j}, l_{j,x}]$ and $[v_{3,j}, l_{j,y}]$, have color c_0. We

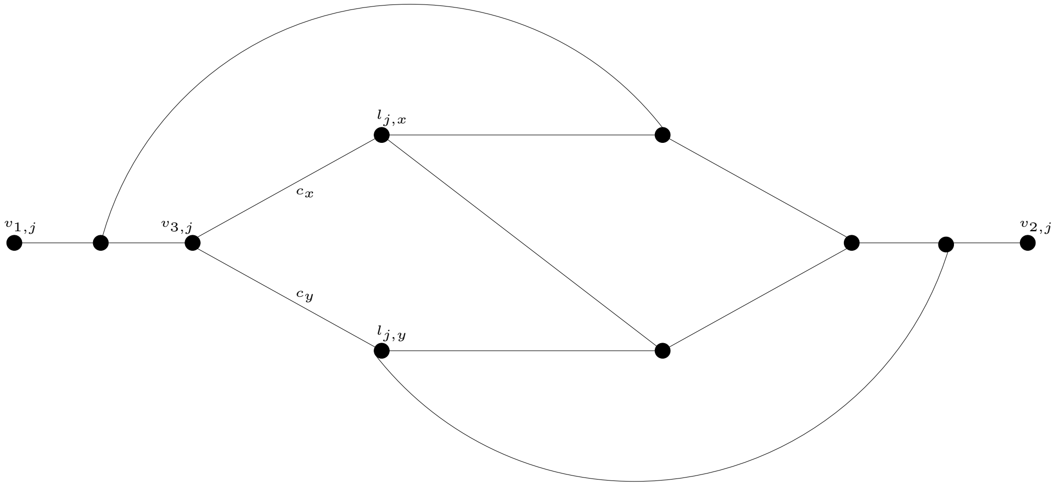

Figure 1.3. *The gadget* $H(e_j)$ *for* $e_j = [x, y]$.

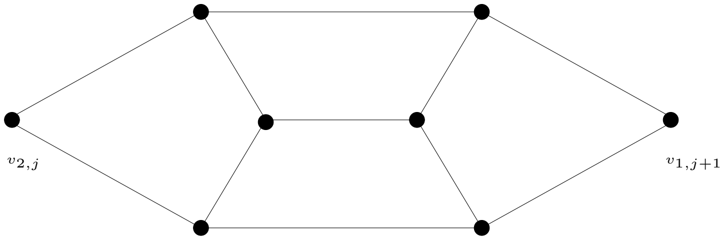

Figure 1.4. *The gadget linking* $H(e_j)$ *to* $H(e_{j+1})$.

have $\mathcal{L}([v_{3,j}, l_{j,x}]) = c_x$ and $\mathcal{L}([v_{3,j}, l_{j,y}]) = c_y$. Finally, $H(e_j)$ is linked to $H(e_{j+1})$ using the graph depicted in Figure 1.4 where each edge is colored with c_0.

Clearly, LABELED $Min\ PPM$ is **APX**-hard in class $\mathcal{P}'_3$. Since the restricted label squaring operation also preserves the membership in $\mathcal{P}'_3$, we deduce that LABELED $Min\ PPM$ is not in **APX** when the instances are restricted to $\mathcal{P}'_3$. Finally, given $I \in \mathcal{P}'_3$ with $I = (G, \mathcal{L})$, we consider the instance I' where G is duplicated 3 times into G_1, G_2, G_3. If u_i, v_i denote the extreme vertices of G_i, we shrink vertices u_1, u_2, u_3 into u and v_1, v_2, v_3 into v. Clearly, this new graph G' is connected, bipartite, planar and cubic. Finally, since we can restrict ourselves to perfect matchings M' of G' that use only color c_0 for exactly two copies of G, the result follows.

Dealing with the unbounded degree case (that is instances of $\mathcal{P}$), we can deduce the following stronger result :

THEOREM 9.– LABELED $Min\ PPM$ *for instances in* $\mathcal{P}$ *is not in* ***polyLog-APX****, unless* ***P*** $=$ ***NP****.*

PROOF.– *A*ssume the reverse, that is LABELED $Min\ PPM$ is $f(n)$-approximable with $f(n) \leq c \log^k n$ for some constants $c > 0$ and $k \geq 1$. Let $I = (H, \mathcal{L}) \in \mathcal{P}$ where H has $2n$ vertices. Let $p = \lceil logk \rceil + 1$. Using as previously 2^p times the restricted label squaring operation on I, we produce in polynomial-time the instance $I^{2^p} = (H^{2^p}, \mathcal{L}^{2^p}) \in \mathcal{P}$. The same arguments as in Theorem 7 allow us to obtain a contradiction with Theorem 5.

1.4. The complete bipartite case

When considering complete bipartite graphs, we obtain several results :

THEOREM 10.– LABELED $Min\ PM_r$ *is* ***APX****-complete in complete bipartite graphs* $K_{n,n}$ *for any* $r \geq 6$*.*

PROOF.– *W*e give an approximation preserving L-reduction (cf. Papadimitriou & Yannakakis [PAPADIMITRIOU 91]) from a restriction of the set cover problem, called MINSC$_3$. In this restriction each set is of size at most 3 and each element x_j appears in at most 3 and at least 2 different sets. MINSC$_3$ has been proved **APX**-complete in [PAPADIMITRIOU 91].

Let G_{I_0} be the characteristic graph corresponding to an instance $I_0 = (\mathcal{S}, X)$ of MINSC$_3$ (see the proof of Theorem 5 for a formal definition of MINSC and G_{I_0}). Note that G_{I_0} has a maximum degree 3. From I_0, we construct the instance $I = (K_{n,n}, L)$ of LABELED $Min\ PM_6$ using a slight modification of the construction given in Theorem 5). First, we start from a bipartite graph having m_0 connected components $H(x_j)$ and $n_0 + m_0$ colors $\{c_1, \ldots, c_{n_0+m_0}\}$, described as follows :

- For each element $x_j \in X$, we build a gadget $H(x_j)$ that consists in a bipartite graph of $2(d_{G_{I_0}}(r_j) + 1)$ vertices and $3d_{G_{I_0}}(r_j)$ edges, where $d_{G_{I_0}}(r_j)$ denotes the degree of vertex $r_j \in R$ in G_{I_0}. The graph $H(x_j)$ is illustrated in Figure 1.5.
- Assume that vertices $\{l_{f(1)}, \ldots, l_{f(p)}\}$ are the neighbors of r_j in G_{I_0}, then color $H(x_j)$ as follows : for any $k = 1, \ldots, p$, $L(v_{1,j}, s_{1,j,f(k)}) = L(v_{2,j}, s_{2,j,f(k)}) = c_{f(k)}$ and $L(s_{1,j,f(k)}, s_{2,j,f(k)}) = c_{n_0+j}$.

– We complete $H = \cup_{x_j \in X} H(x_j)$ into $K_{n,n}$, by adding a new color per edge.

Clearly, $K_{n,n}$ is complete bipartite and has $2n = 2\sum_{r_j \in R}(d_{G_{I_0}}(r_j) + 1) = 2|E_{I_0}| + 2m_0$ vertices. Moreover, each color is used at most 6 times.

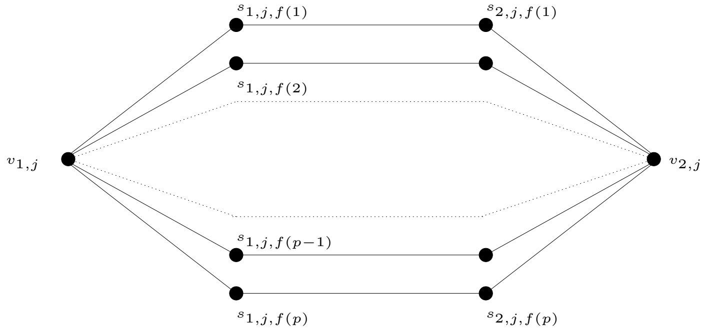

Figure 1.5. *The gadget* $H(x_j)$.

Let $\mathcal{S}^*$ be an optimal set cover on I_0. From $\mathcal{S}^*$, we can easily construct a perfect matching M^* on I using exactly $|\mathcal{S}^*|+m_0$ colors (since we assume that each element appears in at least 2 sets) and thus :

$$opt_{\text{LABELED } Min\ PM_6}(I) \leq opt_{\text{MINSC}_3}(I_0) + m_0 \qquad [1.8]$$

Conversely, we show that any perfect matching M may be transformed into a perfect matching $M"$ using the edges of H and satisfying : $|L(M")| \leq |L(M)|$. Let M be a perfect matching on I and consider M_1 the subset of edges from M that link two different gadgets $H(x_j)$; we denote by G the multi-graph of vertex set $\cup_j V_{H(x_j)}$ (each vertex v_j of G corresponds to the gadget $H(x_j)$) and of edge set M_1. So, G is obtained from $K_{n,n}$ where the vertices in each gadget $H(x_j)$ is shrunk into a vertex of G. Remark that each connected component of G is Eulerian. Each cycle C on G may be completed into a $2|C|$-long cycle C' on K_{2n} in such a way that the two endpoints of each edge from $C' \setminus C$ do belong to the same gadget $H(x_j)$. Here, for purely formal reasons of the proof, we assume that each gadget $H(x_j)$ is a complete graph by adding a new color per missing edge. Thus, there are edges linking two any vertices $s_{1,j,f(k)}$ and $s_{1,j,f(k')}$ (or $s_{2,j,f(k)}$ and $s_{2,j,f(k')}$). If one swaps the edges from each cycle C by the edges from $C' \setminus C$, we obtain a new perfect matching M' where every edge of which has its two endpoints in a same gadget $H(x_j)$ and that satisfies $|L(M')| = |L(M)|$. For the moment, note that as indicated previously, the perfect matching M' is not necessarily a matching of $K_{n,n}$ since some edges linking 2 vertices of the same part of gadget $H(x_j)$ may exist. Now consider for any j the set M'_j of edges from $M' \cap H(x_j)$, we set $M"_j = \{[v_{1,j}, s_{1,j,f(k)}], [v_{2,j}, s_{2,j,f(k)}]\} \cup \{[s_{1,j,f(i)}, s_{2,j,f(i)}] | i = 1, \ldots, p\}$ for some k such that $[v_{1,j}, s_{1,j,f(k)}] \in M'_j$ or $[v_{2,j}, s_{2,j,f(k)}] \in M'_j$ (if such a k does not exist, set $k = 1$). In any case, $M" = (M' \setminus M'_j) \cup M"_j$ is a perfect matching of $K_{n,n}$ that uses no more colors than M' does. Applying this procedure for any $j = 1, \dots, m_0$, we obtain the expected matching M" with value $apx(I)$. From such a matching, we may obtain a set cover $\mathcal{S}" = \{S_k | c_k \in L(M")\}$ on I_0 satisfying :

$$|\mathcal{S}"| = apx(I) - m_0 \qquad [1.9]$$

Using [1.8] and [1.9], we deduce $opt_{\text{LABELED } Min\ PM_6}(I) = opt_{\text{MINSC}_3}(I_0) + m_0$ and $|\mathcal{S}"| - opt_{\text{MINSC}_3}(I_0) \leq |L(M))| - opt_{\text{LABELED } Min\ PM_6}(I)$. Finally, since $opt_{\text{MINSC}_3}(I_0) \geq \frac{m_0}{3}$ the result follows.

Applying the same kind of proof to the vertex cover problem in cubic graphs [ALIMONTI 00], we obtain that LABELED $Min\ PM_r$ in $K_{n,n}$ is **APX**-complete for any $r \geq 3$. In order to establish this fact and starting from a cubic graph $G = (V, E)$, we associate to each edge $e = [x, y] \in E$ a 4-long cycle $\{a_{1,e}, a_{2,e}, a_{3,e}, a_{4,e}\}$ together with a coloration L given by : $L(a_{1,e}) = c_x$, $L(a_{2,e}) = c_y$ and $L(a_{3,e}) = L(a_{4,e}) = c_e$. We complete this graph into a complete bipartite graph, adding a new color per edge. Each color c_x ($\forall x \in V$) appears 3 times, c_e ($\forall e \in E$) twice and any other color, once. Hence, the application of the proof that was made in Theorem 10 leads to the announced result. Unfortunately, we can not apply the proof of Theorem 2 since in this latter, on the one hand, we have some cycles of size 6 and, on the other hand, a color may occur in different gadgets. One open question concerns the complexity of LABELED $Min\ PM_2$ in bipartite complete graphs. Moreover, from Theorem 10, we can also obtain a stronger inapproximability result concerning the general problem LABELED $Min\ PM$: one can not compute in polynomial-time an approximate solution of LABELED $Min\ PM$ that uses less than $(1/2 - \varepsilon)\ln(opt_{\text{LABELED } Max\ PM}(I))$ colors in complete bipartite graphs where $opt_{\text{LABELED } Max\ PM}(I)$ is the value of an optimal solution of LABELED $Max\ PM$, i.e., the maximum number of colors used by a perfect matching.

COROLLARY 4.– *For any* $\varepsilon > 0$, LABELED $Min\ PM$ *is not* $(\frac{1}{2} - \varepsilon) \times \ln(n)$ *approximable in complete bipartite graphs* $K_{n,n}$, *unless* ***NP***$\subset$***DTIME***$(n^{loglogn})$.

PROOF.– First, we apply the construction made in Theorem 10, except that $I_0 = (\mathcal{S}, X)$ is an instance of MINSC such that the number of elements m_0 is strictly larger than the number of sets n_0. From I_0, we construct n_0 instances $I'_1, \dots, I'_{n_0}$ of LABELED $Min\ PM$ where $I'_i = (H, L_i)$. The colors $L_i(E)$ are the same as $L(E)$, except that we replace colors $c_{n_0+1}, \dots, c_{n_0+m_0}$ by c_i. Finally, as previously, we complete each instance I'_i into a complete bipartite graph $K_{n,n}$ by adding a new color by edge.

Let $\mathcal{S}^*$ be an optimal set cover on I_0 and assume that $S_i \in \mathcal{S}^*$, we consider the instance I_i of LABELED $Min\ PM$. From $\mathcal{S}^*$, we can easily construct a perfect matching M_i^* of I_i that uses exactly $|\mathcal{S}^*|$ colors. Conversely, let M_i be a perfect matching on I_i ; by construction, the subset $\mathcal{S}' = \{S_k : c_k \in L(M_i)\}$ of $\mathcal{S}$ is a set cover of X using

$|L(M_i)|$ sets. Finally, let A be an approximate algorithm for LABELED $Min\ PM$, we compute n_0 perfect matchings M_i, applying A on instances I_i. Thus, if we pick the matching that uses the minimum number of colors, then we can polynomially construct a set cover on I_0 of cardinality this number of colors.

Since $n_0 \leq m_0 - 1$, the size n of a perfect matching of $K_{n,n}$ satisfies : $n = |E_{I_0}| + m_0 \leq n_0 \times m_0 + m_0 \leq m_0(m_0 - 1) + m_0 = m_0^2$. Hence, from any algorithm A solving LABELED $Min\ PM$ within a performance ratio $\rho_{\text{A}}(I) \leq \frac{1}{2} \times \ln(n)$, we can deduce an algorithm for MINSC that guarantees the performance ratio $\frac{1}{2}\ln(n) \leq \frac{1}{2}\ln(m_0^2) = \ln(m_0)$. Since the negative result of [FEIGE 98] holds when $n_0 \leq m_0 - 1$, i.e., MINSC is not $(1-\varepsilon) \times \ln(m_0)$ approximable for any $\varepsilon > 0$, unless **NP**$\subset$**DTIME**$(n^{loglogn})$, we obtain a contradiction.

On the other hand, dealing with LABELED $Max\ PM_r$ in $K_{n,n}$, the result of [CAMERON 97] shows that the case $r = 2$ is polynomial, whereas it becomes **NP**-hard when $r = \Omega(n^2)$. Indeed, it is proved in [CAMERON 97] that, on the one hand, we can compute a good matching in $K_{n,n}$ within polynomial-time when each color appears at most twice and, on the other hand, there always exists a good matching in such a graph if $n \geq 3$. An interesting question is to determine the complexity and the approximability of LABELED $Max\ PM_r$ when r is a constant greater than 2.

1.4.1. *Approximation algorithm for* LABELED $Min\ PM_r$

Let us consider the greedy algorithm for LABELED $Min\ PM_r$ in complete bipartite graphs that iteratively picks the color that induces the maximum-size matching in the current graph and delete the corresponding vertices. Formally, if $L(G')$ denotes the colors that are still available in the graph G' at a given iteration and if $G'[c]$ (resp., $G'[V']$) denotes the subgraph of G' that is induced by the edges of color c (resp., by the vertices V'), then the greedy algorithm consists in the following process :

```
Greedy
1 Set $\mathcal{C}' = \emptyset$, $V' = V$ and $G' = G$ ;
2 While $V' \neq \emptyset$ do
  2.1 For any $c \in L(G')$, compute a maximum matching $M_c$ in $G'[c]$ ;
  2.2 Select a color $c^*$ maximizing $|M_c|$ ;
  2.3 $\mathcal{C}' \leftarrow \mathcal{C}' \cup \{c^*\}$, $V' \leftarrow V' \setminus V(M_{c^*})$ and $G' = G[V']$ ;
3 output $\mathcal{C}'$ ;
```

THEOREM 11.– *`Greedy` is an $\frac{H_r+r}{2}$-approximation of* LABELED $Min\ PM_r$ *in complete bipartite graphs where H_r is the r-th harmonic number $H_r = \sum_{i=1}^{r} \frac{1}{i}$, and this ratio is tight.*

PROOF.– Let $I = (G, L)$ be an instance of LABELED $Min\ PM_r$. We denote by $\mathcal{C}'_i$ for $i = 1, \ldots, r$ the set of colors of the approximate solution which appears exactly i times in $\mathcal{C}'$ and by p_i its cardinality (thus, $\forall c \in \mathcal{C}'_i$ we have $|M_c| = i$ in $G'[c]$) ; finally, let M_i denote the matching with colors $\mathcal{C}'_i$. If $apx(I) = |\mathcal{C}'|$, then we have :

$$apx(I) = \sum_{i=1}^{r} p_i \qquad [1.10]$$

Let $\mathcal{C}^*$ be an optimal solution corresponding to the perfect matching M^* of size $opt(I) = |\mathcal{C}^*|$; we denote by E_i the set of edges of M^* that belong to $G[\cup_{k=1}^{i} V(M_k)]$, the subgraph induced by $\cup_{k=1}^{i} V(M_k)$ and we set $q_i = |E_i \setminus E_{i-1}|$ (where we assume that $E_0 = \emptyset$). For any $i = 1, \ldots, r-1$, we get :

$$opt(I) \geq \frac{1}{i} \sum_{k=1}^{i} q_k \qquad [1.11]$$

Indeed, $\sum_{k=1}^{i} q_k = |E_i|$ and by construction, each color appears at most i times in $G[\cup_{k=1}^{i} V(M_k)]$.

We also have the following inequality for any $i = 1, \ldots, r-1$:

$$opt(I) \geq \frac{1}{r} \left(2 \sum_{k=1}^{i} k \times p_k - \sum_{k=1}^{i} q_k \right) \qquad [1.12]$$

Since M^* is a perfect matching, the quantity $2 \sum_{k=1}^{i} k \times p_k - \sum_{k=1}^{i} q_k$ counts the edges of M^* of which at least one endpoint belongs to $G[\cup_{k=1}^{i} V(M_k)]$. Because each color appears on at most r edges, the result follows.

Finally, since $\sum_{k=1}^{r} k \times p_k$ is the size of a perfect matching of G, the following inequality holds :

$$opt(I) \geq \frac{1}{r} \sum_{k=1}^{r} k \times p_k \qquad [1.13]$$

Using equality [1.10] and adding inequalities [1.11] with coefficient $\alpha_i = \frac{1}{2(i+1)}$ for $i = 1, \ldots, r-1$, inequalities [1.12] with coefficient $\beta_i = \frac{r}{2i(i+1)}$ for $i = 1, \ldots, r-1$ and inequality [1.13], we obtain :

$$apx(I) \leq \left(\frac{H_r + r}{2}\right) opt(I) \qquad [1.14]$$

Indeed, $\sum_{i=1}^{r-1} \alpha_i = \frac{1}{2}H_r - \frac{1}{2}$ and $\sum_{i=1}^{r-1} \beta_i = \frac{r}{2} - \frac{1}{2}$. Thus, $\sum_{i=1}^{r-1} (\alpha_i + \beta_i) + 1 = \frac{H_r+r}{2}$.

The quantity p_j appears in inequality [1.13] and inequality [1.12] for $i = j, \ldots, r-1$. Its total contribution is : $\frac{1}{r} j \times p_j + \frac{2}{r}\left(\sum_{i=j}^{r-1} \beta_i\right) j \times p_j = p_j$. The quantity q_j appears in inequality [1.11] for $i = j, \ldots, r-1$ and inequality [1.12] for $i = j, \ldots, r-1$. We have : $\left(\sum_{i=j}^{r-1} \frac{\alpha_i}{i}\right) - \frac{1}{r}\left(\sum_{i=j}^{r-1} \beta_i\right) q_j = 0$. Thus, using equality [1.10], the inequality [1.14] holds.

In order to show the tightness of this bound, consider the instance $I = (K_{n,n}, L)$ where the left set A and the right set B of vertices of the complete bipartite graph are given by $A = \{a_{i,j} : i = 1, \ldots, r,\ j = 1, \ldots, n_i\}$ and $B = \{b_{i,j} : i = 1, \ldots, r,\ j = 1, \ldots, n_i\}$, with $n_1 = (r+1)!$ and $n_i = r!$ for $i = 2, \ldots, r$. Moreover, the edge coloration satisfies :

- For any $i = 1, \ldots, r$ and for any $j = 1, \ldots, n_i$, $L(a_{i,j}, b_{i,j}) = c_{i,\lceil \frac{j}{i} \rceil}$.
- For any $i = 2, \ldots, r$ and for any $j = 1, \ldots, r!$, $L(a_{i,j}, b_{1,i-1+(r-1)(j-1)}) = c^*_{1,j}$ and $L(b_{i,j}, a_{1,i-1+(r-1)(j-1)}) = c^*_{2,j}$.
- For any $j = 1, \ldots, r!$, $L(b_{1,j+(r-1)\times r!}, a_{1,(r+1)!-j+1}) = c^*_{1,j}$ and $L(a_{1,j+(r-1)\times r!}, b_{1,(r+1)!-j+1}) = c^*_{2,j}$.
- We associate a new color to each missing edge.

I is clearly an instance of LABELED $Min\ PM_r$. The set of colors $\mathcal{C}' = \{c_{i,\lceil \frac{j}{i} \rceil} : i = 1, \ldots, r,\ j = 1, \ldots, n_i\}$ is the approximate solution outputted by `Greedy` and it uses $apx(I) = (H_r + r) \times r!$ colors, whereas $\mathcal{C}^* = \{c^*_{i,j} : i = 1, 2,\ j = 1, \ldots, r!\}$ is the set of colors that are used by an optimal solution ; this latter satisfies $opt(I) = 2 \times r!$. The Figure 1.6 describes the instance I for $r = 2$.

1.5. Bibliographie

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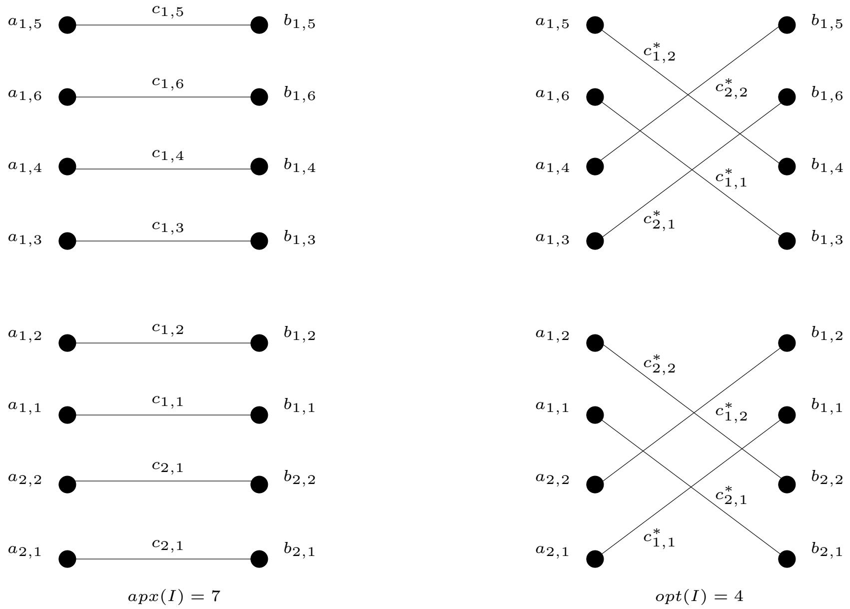

Figure 1.6. *The instance I when $r = 2$.*

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Chapitre 2

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