Chapitre 1

(Non)-Approximability results for the multi-criteria Min and Max $TSP(1,2)$

1.1. Introduction

This chapter presents some recent results obtained by the authors ([ANGEL 04, ANGEL 05, ANGEL 05b]) about the inapproximation and the approximation properties of the multi-criteria traveling salesman problem with distances one and two. These results concerns the minimization version of the problem. We also present some new results for the maximization variant.

1.1.1. *The traveling salesman problem*

The traveling salesman problem (TSP) is one of the most studied problem in the operations research community, see for instance [JOHNSON 85]. Given a complete graph where the edges are associated with a positive distance, we search for a cycle visiting each vertex of the graph exactly once and minimizing the total distance. It is well known that the TSP problem is **NP**-hard, and that it cannot be approximated within a constant approximation ratio, unless *P=NP*. However, for the metric TSP (*i.e.* when the distances satisfy the triangle inequality), Christofides proposed an algorithm with performance ratio 3/2 [CHRISTOFIDES 76]. For more than 25 years, many researchers attempted to improve this bound but with no success. In [PAPADIMITRIOU 93], the authors studied a more restrictive version of the metric

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TSP, in which all distances are either one or two (denoted by $TSP(1,2)$), and they proposed a 7/6 approximation algorithm. Recently, Berman and Karpinski proposed a $8/7$-approximation algorithm for the $TSP(1,2)$ problem [BERMAN 06] and some inapproximability results are presented in [ENGEBRETSEN 01]. The $TSP(1,2)$ problem is a generalization of the hamiltonian cycle problem since we are asking for the tour of the graph that contains the fewest possible non-edges (edges of distance 2). In [MONNOT 02, MONNOT 03], some results for the $TSP(1,2)$ with respect to the *differential* approximation ratio are obtained. Finally, there is the maximization version of the problem, denoted by Max TSP, where the goal is to find a tour maximizing the total distance. Max TSP is also known to be **APX**-complete, even if the edge distances are either 1 or 2 (denoted by Max $TSP(1,2)$), [PAPADIMITRIOU 93] and it is approximable with expected performance ratio $25/33 - \varepsilon$ for all $\varepsilon > 0$ in the general case [HAS 00], and $7/8$ in the metric case [HAS 02].

In this chapter, we deal with a multicriteria version of the problem : the k-criteria $TSP(1,2)$. The distance between any pair of vertices is a vector of length k instead of a scalar.

1.1.2. *Multi-criteria optimization*

Multi-criteria optimization refers to problems with two or more objective functions which are in conflict. Vilfredo Pareto introduced in 1896 a concept –known today as *Pareto optimality*– that constitutes the origin of the research in this area. According to this concept, the goal in a multi-criteria optimization problem is normally not a single solution, but instead a set of *non-dominated* solutions, the so-called *Pareto curve*. From a computational point of view, the notion of Pareto curve is problematic. Two of the main reasons are :

– the size of a Pareto curve which is often exponential with respect to the size of the corresponding instance, [PAPADIMITRIOU 00, VASSILVITSKII 05] ;

– computing one Pareto optimal solution of a multi-criteria optimization problem is often an **NP**-hard problem, [EHRGOTT 00].

Approximating it with a performance guarantee, *i.e.* designing polynomial-time algorithms which return $\varepsilon-$*approximate Pareto curves*, is a motivating challenge. Inapproximability results are also needed. However, inapproximability results are not numerous in the literature of multi-criteria optimization.

In this chapter, we provide approximation and inapproximation results for several versions of the multi-criteria traveling salesman with distances one and two. In particular, we propose a way to get some negative results which works for several multi-criteria problems and we put it into practice on the multi-criteria $TSP(1,2)$. Up to our knowledge, existing multi-criteria inapproximation results were investigated only from the point of view of **NP**-hardness [DENG 02, PAPADIMITRIOU 00, VASSILVITSKII 05]. Our method is based on the following observation : in multi-criteria optimization, one tries to approximate a set of solutions (the Pareto curve) with another set of solutions (the ε-approximate Pareto curve) and the more the ε-approximate Pareto curve contains solutions, the more accurate the approximation can be. Then, the best approximation ratio that could be achieved can be related to the size of the approximate Pareto curve.

1.1.3. *Organization of the chapter*

The chapter is organized as follows : In Section 1.2, we give some definitions concerning exact and approximate Pareto curves. Section 1.3 is devoted to a method to derive some negative results (Subsection 1.3.1) for the bicriteria $TSP(1,2)$, and two polynomial time algorithms providing constant approximation of the Pareto curve. More precisely, in Subsection 1.3.2, we propose a *local search* algorithm called BLS which, with only two solutions generated in $\mathcal{O}(n^3)$, returns a $1/2$-approximate Pareto curve and in Subsection 1.3.3, we propose a *greedy* algorithm inspired of the classical *nearest neighbor* heuristic. This algorithm, called 2NN, returns two solutions generated in $\mathcal{O}(n^2)$ which constitute a $1/2$-approximation of the Pareto curve. We also give in Subsection 1.3.4, some approximation results for a restriction of the bicriteria Max $TSP(1,2)$ problem. In Section 1.4, we study the k-criteria $TSP(1,2)$ problem. More precisely, in Subsection 1.4.1, we propose several negative results for k-criteria $TSP(1,2)$. Our method is applied in the case of k-criteria $TSP(1,2)$ but it works for several other problems. In Subsection 1.4.2, we study the approximability of the k-criteria $TSP(1,2)$ by giving a generalized version of 2NN. This multi-criteria nearest neighbor heuristic, called KNN, works for any $k \geq 3$ and produces a $(k-1)/(k+1)$-approximate Pareto curve. Finally, some concluding remarks are given in Section 1.5.

1.2. Generalities

The Traveling Salesman Problem consists in finding in a weighted complete graph $G = (V, E)$ on n vertices, a Hamiltonian cycle whose total distance is minimal. For the k-criteria TSP , each edge e has a *distance* $\vec{d}(e) = (\vec{d}_1(e), \ldots, \vec{d}_k(e))$ which is a vector of length k (instead of a scalar). The *total distance* of a tour T is also a vector $\vec{D}(T)$ where $\vec{D}_j(T) = \sum_{e \in T} \vec{d}_j(e)$ and $j = 1, \ldots, k$. In fact, a tour is evaluated with k objective functions. Given this, the goal of the optimization problem could be the following : Generating a feasible solution which simultaneously minimizes (or maximizes) each coordinate. Unfortunately, such an ideal solution rarely exists since objective functions are, in general, in conflict. However a set of solutions representing all best possible trade-offs always exists : the so-called *Pareto curve*. Formally, a Pareto curve is a set of feasible solutions, each of them optimal in the sense of Pareto, which *dominates* all the others solutions outside this set. If the problem is a minimization problem, a tour T dominates another one T' iff $\vec{D}_j(T) \leq \vec{D}_j(T')$ for all

$j = 1, \ldots, k$ and, for at least one coordinate j', one has $\vec{D}_{j'}(T) < \vec{D}_{j'}(T')$. When the problem is a maximization problem, we just reverse the previous inequalities. A solution is optimal in the sense of Pareto if no solution dominates it.

Since computing Pareto curves is problematic, getting an approximation of it is more reasonable and often sufficient. For a minimization (resp. maximization) problem, an ε-approximate Pareto curve P_ε is a set of solutions such that for every solution s of the instance, there is an s' in P_ε which satisfies $\vec{D}_j(s') \leq (1+\varepsilon)\vec{D}_j(s)$ (resp. $\vec{D}_j(s') \geq (1-\varepsilon)\vec{D}_j(s)$) for all $j = 1, \ldots, k$. [1]

Interestingly, Papadimitriou and Yannakakis [PAPADIMITRIOU 00] prove that every multi-criteria problem has an ε-approximate Pareto curve that is polynomial in the size of the input, and $1/\varepsilon$, but exponential in the number k of criteria. However, computing such an approximation Pareto curve cannot always be done within polynomial-time. The design of polynomial time algorithms which generate approximate Pareto curves with performance guarantee motivated a lot of recent papers.

1.3. The bicriteria $TSP(1,2)$

We consider the bicriteria $TSP(1,2)$ with n cities. In this case, we recall that for an edge e, $\vec{d}(e) = (\vec{d}_1(e), \vec{d}_2(e)) \in \{(1,1),(1,2),(2,1),(2,2)\}$ and the objective for the minimization version, called bicriteria Min $TSP(1,2)$ (resp., for the maximization version, called bicriteria Max $TSP(1,2)$) is to find a tour T minimizing (resp., maximizing) $\vec{D}_1(T) = \sum_{e \in T} \vec{d}_1(e)$ and $\vec{D}_2(T) = \sum_{e \in T} \vec{d}_2(e)$. We mainly propose two different algorithms leading to the same approximation of the Pareto curve for the bicriteria $Min\ TSP(1,2)$: a *local search* procedure using the *2-opt neighborhood* and a *nearest neighbor* heuristic which computes a $1/2$-approximate Pareto curve. Although these two algorithms provide the same performance guarantee, it is interesting to present them since they are adaptations of well known heuristics previously used for the monocriterion TSP [JOHNSON 85, ROSENKRANTZ 77].

For the bicriteria $TSP(1,2)$ (minimization and maximization versions), it is easy to observe that the Pareto curve is composed of at most $n+1$ solutions where we recall that n is the number of vertices ; the worst case appears when the $n+1$ tours T_i with vector distance $\vec{D}(T_i) = (\vec{D}_1(T_i), \vec{D}_2(T_i)) = (n+i, 2n-i)$ for $i = 0, \cdots, n$, belongs to the instance. Moreover, notice that a 1-approximate Pareto curve can be trivially constructed, just pick any tour.

1. Here, ε is the error while $1+\varepsilon$ (resp. $1-\varepsilon$) is the approximation ratio for a minimization (resp., maximization) problem. In the chapter, we equivalently use the error and its associated approximation ratio.

We now prove that the approximability of the Pareto curve for bicriteria Min $TSP(1,2)$ and the approximability of the Pareto curve for bicriteria Max $TSP(1,2)$ are linked by an approximation preserving reduction. The idea is to modify the instance by replacing each edge (2,2) by an edge (1,1), each edge (1,1) by and edge (2,2), and each edge (1,2) by an edge (2,1) and vice et versa. It can be shown that obtaining a ε-approximate Pareto curve for the bicriteria Min $TSP(1,2)$ on this modified instance yields a $2\varepsilon/(3\varepsilon+1)$-approximate Pareto curve for the bicriteria Max $TSP(1,2)$ problem on the original instance.

THEOREM 1.– *Any ε-approximate Pareto curve for the bicriteria Min $TSP(1,2)$ problem can be polynomially transformed into a $\frac{2\varepsilon}{3\varepsilon+1}$-approximate Pareto curve for the bicriteria Max $TSP(1,2)$ problem.*

PROOF.– *W*e only prove the result for the monocriterion problems, that is any ε-approximation for Min $TSP(1,2)$ can be polynomially transformed into a $\frac{2\varepsilon}{3\varepsilon+1}$-approximation for Max $TSP(1,2)$. Let $I=(G,d)$ be an instance of Max $TSP(1,2)$ where $G=(V,E)$ is a complete graph on n vertices and consider the instance $I'=(G,d')$ of Min $TSP(1,2)$ with $d'(e)=3-d(e)$ for all $e\in E$. Finally, let T^* be an optimal solution of I for Max $TSP(1,2)$ and assume that T is an ε-approximation for Min $TSP(1,2)$ on I' ; obviously, T is also a solution on I and consider the two following cases :

• if $d(T^*)\leq\frac{3\varepsilon+1}{\varepsilon+1}n$, then since $d(T)\geq n$, we have :

$$\frac{d(T)}{d(T^*)}\geq\frac{\varepsilon+1}{3\varepsilon+1}=1-\frac{2\varepsilon}{3\varepsilon+1}$$

Thus, T is a $\frac{2\varepsilon}{3\varepsilon+1}$-approximation for Max $TSP(1,2)$.

• Otherwise, $d(T^*)\geq\frac{3\varepsilon+1}{\varepsilon+1}n$. By construction, any tour T' verifies $d'(T')=3n-d(T')$. In particular, this is true for T^* and T. Hence, we deduce that T^* is also an optimal solution of I' for Min $TSP(1,2)$ and we have :

$$d'(T^*)\leq\frac{2}{\varepsilon+1}n \tag{1.1}$$

Since by hypothesis $d'(T)\leq(1+\varepsilon)d'(T^*)$, we get :

$$\frac{d(T)}{d(T^*)}=\frac{3n-d'(T)}{3n-d'(T^*)}\geq 1-\varepsilon\frac{d'(T^*)}{3n-d'(T^*)} \tag{1.2}$$

Using inequality (1.1) and since the right side of inequality (1.2) is increasing with $d'(T^*)$, we deduce $d(T)\geq(1-\frac{2\varepsilon}{3\varepsilon+1})d(T^*)$.

1.3.1. *Simple examples of the non-approximability*

Usually, non-approximability results for mono-criteria problems bring thresholds of performance guarantee under which no polynomial time algorithm is likely to exist.

Given a result of that kind for a mono-criterion problem Π, we directly get a negative result for a multi-criteria version of Π. Indeed, the multi-criteria version of Π generalizes Π. The non-approximability of the mono-criterion $TSP(1,2)$ has been studied in [ENGEBRETSEN 01, PAPADIMITRIOU 93] and the best known lower bound is $1 + 1/740 - \delta$ (for all $\delta > 0$). Consequently, for every $\delta > 0$, no polynomial time algorithm can generate a $(1/740 - \delta)$-approximate Pareto curve unless $P = NP$.

As indicated previously, in multi-criteria optimization, one tries to approximate a set of solutions (the Pareto curve) with another set of solutions (the ε-approximate Pareto curve) and the more the ε-approximate Pareto curve contains solutions, the more accurate the approximation can be. As a consequence, the best approximation ratio that could be achieved can be related to the size of the approximate Pareto curve. Formally, ε is a function of $|P_\varepsilon|$. If we consider instances for which the whole (or a large part of the) Pareto curve P is known and if we suppose that we approximate it with a set $P' \subset P$ such that $|P'| = x$ then the best approximation ratio ε such that P' is an ε-approximate Pareto curve is related to x. Indeed, there must be a solution in P' which approximates at least two (or more) solutions in P.

The question asked here is : "*What is the best approximation ratio an algorithm $\mathcal{A}$ can achieve if it outputs r solutions ?*" where r is supposed to be between 1 and $n+1$.

To get an upper bound of this approximation ratio, we consider a particular class of instances for which the distances on the edges are in $\{(1,2),(2,1)\}$. This subclass is very interesting since each tour is Pareto optimal. Actually, let G be a n nodes graph of this class ; each feasible tour has a total distance which is $(n+q, 2n-q)$ where q is the number of $(2,1)$ edges contained in the tour. Thus, no feasible tour can be dominated by another one.

THEOREM 2.– *Any ε-approximate Pareto curve P_ε for the bicriteria Min $TSP(1,2)$ problem composed of at most r tours is such that $\varepsilon \geq \frac{1}{3r-1}$.*

PROOF.– *I*n the sequel, we consider an instance of the subclass described above and assume that n is a multiple of r and all solutions T_i^* with distance $\vec{D}(T_i^*) = \big(n + \frac{i}{r}n, 2n - \frac{i}{r}n\big)$ are feasible for $i = 0, \cdots, r$. A formal proof of this claim will be given in Subsection 1.4.1 when we generalize this result to any number of criteria k ; actually, the goal of this proof is just to give an intuition of the method used. Now, assume that $\mathcal{A}$ outputs r tours T_i with $i = 1, \cdots, r$, which approximates the Pareto curve within a ratio $(1+\alpha)$ on both criteria. Since there are $r+1$ Pareto optimal tours T_i^*, at least one tour $T_i \in P_\varepsilon$ must approximate two tours T_i^* for $i = x$ and $i = y$ with $x < y$. In particular, we have $y \geq x + 1$, and thus, if $\vec{D}(T_i) = \big(n + q, 2n - q\big)$, we deduce :

$$\begin{array}{rcl} n+q & \leq & (1+\alpha)\dfrac{r+x}{r}n \\ 2n-q & \leq & (1+\alpha)\dfrac{2r-x-1}{r}n \end{array}$$

From the first inequality we derive that :

$$q \leq \frac{x}{r}n + \frac{r+x}{r}\alpha n$$

and from the second we get that :

$$q \geq \frac{x+1}{r}n - \frac{2r-x-1}{r}\alpha n$$

In fact

$$\frac{x+1}{r}n - \frac{2r-x-1}{r}\alpha n \leq q \leq \frac{x}{r}n + \frac{r+x}{r}\alpha n$$

can only be possible if $\alpha \geq \frac{1}{3r-1}$.

When the goal is the maximization of both criteria (*i.e.* the bicriteria Max $TSP(1,2)$ problem), we obtain a threshold of inapproximation of $2/(3r-2)$ using the approximation curve preserving reduction given in Theorem 1. However, applying the same proof of Theorem 2 in the context of bicriteria Max $TSP(1,2)$, we can improve this ratio to $\frac{1}{3r+1}$.

COROLLARY 1.– *Any ε-approximate Pareto curve P_ε of the bicriteria Max $TSP(1,2)$ composed of at most r tours is such that $\varepsilon \geq \frac{1}{3r+1}$.*

1.3.2. *A local search heuristic for the bicriteria $TSP(1,2)$*

In [KHANNA 98] and [MONNOT 02b], the authors have shown separately that a simple local search algorithm using the well known 2-opt neighborhood [CROES 58] returns a $3/2$-approximate tour for the monocriterion Min $TSP(1,2)$. Up to the best of our knowledge, no result was known about the ability of local search algorithms to provide solutions with performance guarantee in multi-criteria optimization. Here, we present an algorithm of this type which returns a $1/2$-approximate Pareto curve for the bicriteria $TSP(1,2)$ problem using the same neighborhood.

Given a tour T, its 2-opt neighborhood $\mathcal{N}(T)$ is the set of all Hamiltonian cycles which can be obtained by removing two non adjacent edges and inserting two new edges (see Figure 1.1).

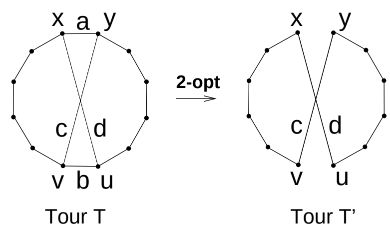

Figure 1.1. *The 2-opt move depicted here consists in replacing edges a and b by c and d. We have $T' \in \mathcal{N}(T)$*

If only one criterion is considered, a simple local search algorithm consists in starting with a feasible solution and to iteratively replace it with a neighboring solution which is better with respect to the criterion. The algorithm stops when the current solution, a *local optimum*, has no better neighbor.

If we say that a tour T is a local optimum tour with respect to the 2-opt neighborhood when no tour $T' \in \mathcal{N}(T)$ dominates T then there exist instances for which a locally optimal tour gives a poor performance guarantee for at least one criterion. Then, we introduce two symmetric preference relations defining potentially two different local optima. These preference relations, denoted by $\prec_1$ and $\prec_2$, are depicted in Figures 1.2 and 1.3. Notice that we define them considering the fact that we deal with 2-opt moves which are exchanges of couples of edges. The set of the ten possible couples of distance-vectors of the edges has been partitioned into three sets S_1, S_2 and S_3, and for any $c_1 \in S_1$, $c_2 \in S_2$, $c_3 \in S_3$, we have $c_1 \prec_1 c_2 \prec_1 c_3$ (resp. $c_1 \prec_2 c_2 \prec_2 c_3$). Intuitively, the preference relation $\prec_1$ (resp. $\prec_2$) leads to solutions which are good for the first (resp. second) criterion.

DEFINITION 1.– *We say that the tour T is a local optimum tour with respect to the 2-opt neighborhood and the preference relation $\prec_1$ (resp. $\prec_2$) if there does not exist a tour $T' \in \mathcal{N}(T)$, obtained from T by removing edges a, b and inserting edges c, d, such that $(\vec{d}(c), \vec{d}(d)) \prec_1 (\vec{d}(a), \vec{d}(b))$ (resp. $(\vec{d}(c), \vec{d}(d)) \prec_2 (\vec{d}(a), \vec{d}(b))$).*

We consider the following algorithm :

BICRITERIA LOCAL SEARCH (BLS) :

1) Let T_1 be a 2-opt local optimum tour with the preference relation $\prec_1$.

2) Let T_2 be a 2-opt local optimum tour with the preference relation $\prec_2$.

3) Return $\{T_1, T_2\}$.

1.3.2.1. *Analysis of* BLS

In the following, we assume that T is any 2-opt local optimal tour with respect to the preference relation $\prec_1$. The tour O is any fixed tour. We denote by x (resp. y,z

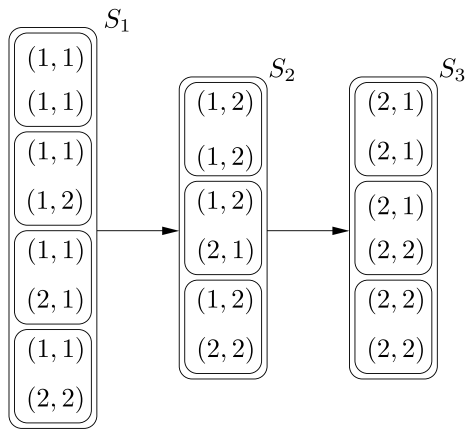

Figure 1.2. *The preference relation* $\prec_1$.

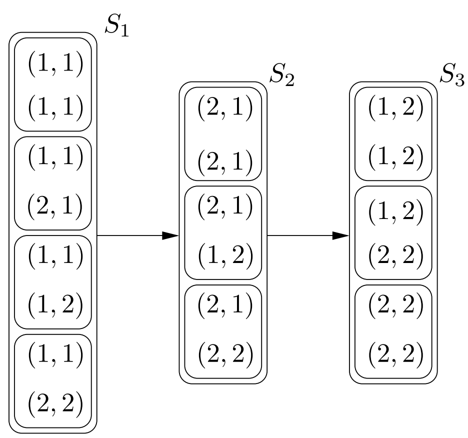

Figure 1.3. *The preference relation* $\prec_2$.

and t) the number of edges with distance vector (1,1) (resp. (1,2), (2,1) and (2,2)) in T. We denote by x' (resp. y',z' and t') the number of edges with distance vector (1,1) (resp. (1,2), (2,1) and (2,2)) in O.

LEMMA 1.– *With the preference relation $\prec_1$ one has $x \geq x'/2$.*

PROOF.– Let U_O (resp. U_T) be the set of $(1,1)$ edges in O (resp. in T). We define a function $f : U_O \rightarrow U_T$ as follows : for $e \in U_O$, $f(e) = e$ if $e \in U_T$. Otherwise, let e' and e'' the two edges adjacent to e in T as depicted in Figure 1.4 (we assume an arbitrary orientation of T and consider that the edges adjacent to e are e' and e'' but not e^4 and e^5). Let e''' be the edge forming a cycle of length 4 with e, e' and e'' (see Figure

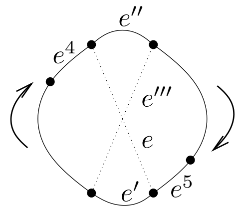

Figure 1.4. *The local optimal tour T (arbitrarily oriented).*

1.4). We claim that there is at least one edge among e' and e'' with a distance $(1,1)$ and define $f(e)$ to be one of those edges (possibly chosen arbitrarily). Otherwise, we have $\{e, e'''\} \in S_1$ and $\{e', e''\} \in S_2 \cup S_3$ (see Figures 1.2 and 1.3), contradicting the fact that T is a local optimum with respect to the preference relation $\prec_1$. Now observe that for a given edge $e'' \in U_T$, there can be at most two edges e and e' in U_O such that $f(e) = f(e') = e''$ since O is a tour. Therefore, we have $|U_T| \geq |U_O|/2$.

LEMMA 2.– *With the preference relation $\prec_1$ one has $x + y \geq (x' + y')/2$.*

PROOF.– *T*he proof is similar to the one of Lemma 1. Here, U_O (resp. U_T) is the set of $(1,1)$ and $(1,2)$ edges of O (resp. T).

LEMMA 3.– *If $\vec{D}_1(O) \leq \vec{D}_2(O)$ then $\vec{D}_1(T) \leq \frac{3}{2}\vec{D}_1(O)$ and $\vec{D}_2(T) \leq \frac{3}{2}\vec{D}_2(O)$.*

PROOF.– *W*e have $\vec{D}_1(T) = 2n - x - y$, $\vec{D}_1(O) = 2n - x' - y'$ and $\vec{D}_2(T) = 2n - x - z$, $\vec{D}_2(O) = 2n - x' - z'$. Let us consider the first coordinate. We want to show that $\frac{\vec{D}_1(T)}{\vec{D}_1(O)} = \frac{2n-x-y}{2n-x'-y'} \leq \frac{3}{2}$. Using Lemma 2 we get $\frac{2n-x-y}{2n-x'-y'} \leq \frac{2n-\frac{x'}{2}-\frac{y'}{2}}{2n-x'-y'}$. Now we have

$$
\begin{aligned}
\frac{2n - \frac{x'}{2} - \frac{y'}{2}}{2n - x' - y'} \leq \frac{3}{2} \quad &\Longleftrightarrow \quad 4n - x' - y' \leq 6n - 3x' - 3y' \\
&\Longleftrightarrow \quad x' + y' \leq n
\end{aligned}
$$

which is true since $x' + y' + z' + t' = n$ and $z', t' \geq 0$. Now, we consider the second coordinate and use the fact that $\vec{D}_1(O) \leq \vec{D}_2(O) \iff z' \leq y'$.

$$\begin{aligned}\frac{2n-x-z}{2n-x'-z'} \leq \frac{3}{2} \quad &\Longleftrightarrow \quad 4n-2x-2z \leq 6n-3x'-3z' \\ &\Longleftrightarrow \quad 3x'-2x+3z'-2z \leq 2n \\ &\Longleftrightarrow \quad 3x'-2x+3z'-2z \leq 2(x'+y'+z'+t') \\ &\Longleftrightarrow \quad x'-2x+z'-2z \leq 2y'+2t'\end{aligned}$$

which is true since $x'-2x \leq 0$ by Lemma 1, $z' \leq y'$ and $-z \leq t'$.

Now, we suppose that T is a 2-opt local optimal tour with respect to the preference relation $\prec_2$. The tour O is any fixed tour. In a way similar to the case of $\prec_1$, we can prove :

LEMMA 4.– *With the preference relation $\prec_2$ one has $x \geq \frac{x'}{2}$ and $x+z \geq (x'+z')/2$.*

LEMMA 5.– *If $\vec{D}_1(O) \geq \vec{D}_2(O)$ then $\vec{D}_1(T) \leq \frac{3}{2}\vec{D}_1(O)$ and $\vec{D}_2(T) \leq \frac{3}{2}\vec{D}_2(O)$.*

Now, we are ready to prove the main result.

THEOREM 3.– *The set of tours returned by* BLS *is a 1/2-approximate Pareto curve for the bicriteria Min $TSP(1,2)$ problem. Moreover, this bound is asymptotically sharp.*

PROOF.– *U*sing Lemma 3 and Lemma 5, we know that given a Pareto optimal tour O, if $\vec{D}_1(O) \leq \vec{D}_2(O)$ then $\vec{D}(T_1) \leq \frac{3}{2}\vec{D}(O)$, otherwise $\vec{D}(T_2) \leq \frac{3}{2}\vec{D}(O)$.

To see that this bound is asymptotically sharp, consider the instance depicted in Figure 1.5. The tour $s_1 s_2 \dots s_{2n} s_1$ is a local optimum with respect to $\prec_1$ and $\prec_2$, and it has a total distance vector $(3n, 3n)$, whereas the optimal tour

$$s_1 s_3 s_{2n} s_4 s_{2n-1} \dots s_{n-1} s_{n+4} s_n s_{n+3} s_{n+1} s_{n+2} s_2 s_1$$

has a total distance vector $(2n+1, 2n+1)$.

Concerning the time complexity, we can show that BLS runs in time $\mathcal{O}(n^3)$ since searching the 2-opt neighborhood of a tour is done in $\mathcal{O}(n^2)$ and at most $\mathcal{O}(n)$ 2-opt moves are done to reach a local optimum.

The result given in Theorem 3 can also be applied to the bicriteria version of the Max $TSP(1,2)$ problem. We recall that in this problem, the objective is the maximization of the length of the tour. For the monocriterion case, the best approximation algorithm known has a performance ratio of 7/8 [MONNOT 02, MONNOT 03]. Using

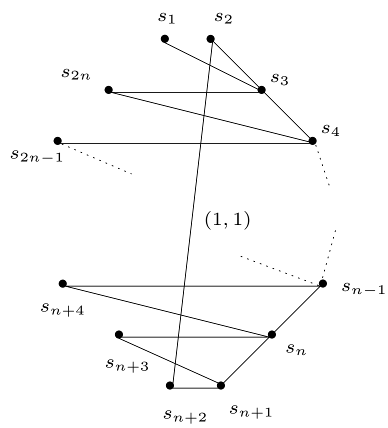

Figure 1.5. *The edges represented have a distance vector* $(1,1)$*, whereas non represented edges have a distance vector* $(2,2)$*.*

Theorems 1 and 3, we obtain a 2/5-approximate Pareto curve for the bicriteria Max $TSP(1,2)$ problem. However, if we call BMAXLS the algorithm which consists in using modified preference relations $\prec'_1$ and $\prec'_2$ obtained from $\prec_1$ and $\prec_2$ by replacing each edge (a,b) by an edge $(3-a,3-b)$, one can show that the inequalities obtained in Theorem 3 allow us to obtain a 1/3-approximate Pareto curve.

COROLLARY 2.– *The set of solutions returned by* BMAXLS *is a 1/3-approximate Pareto curve for the bicriteria Max* $TSP(1,2)$ *problem. Moreover, this bound is asymptotically sharp.*

1.3.3. *A nearest neighbor heuristic for the bicriteria* $TSP(1,2)$

We now propose a *nearest neighbor* heuristic which computes in $\mathcal{O}(n^2)$ a $1/2$-approximate Pareto curve for the bicriteria $Min\ TSP(1,2)$. The idea of this classical heuristic, applied to a monocriterion TSP instance, consists in starting from a randomly chosen node and greedily insert non-visited vertices, chosen as the closest ones from the last inserted vertex [ROSENKRANTZ 77]. We adapt this algorithm to the bicriteria Min $TSP(1,2)$. As done before with BLS, we build two solutions using two symmetric preference relations denoted by $\prec_1$ and $\prec_2$.

DEFINITION 2.– *For two edges* e *and* e'*,* $\vec{d}(e) \prec_1 \vec{d}(e')$ *if* $\vec{d_1}(e) < \vec{d_1}(e')$ *or,* $\vec{d_1}(e) = \vec{d_1}(e')$ *and* $\vec{d_2}(e) < \vec{d_2}(e')$*. Symmetrically,* $\vec{d}(e) \prec_2 \vec{d}(e')$ *if* $\vec{d_2}(e) < \vec{d_2}(e')$ *or,* $\vec{d_2}(e) = \vec{d_2}(e')$ *and* $\vec{d_1}(e) < \vec{d_1}(e')$*. When* $\vec{d}(e) \not\prec_1 \vec{d}(e')$ *and* $\vec{d}(e) \not\prec_2 \vec{d}(e')$*, we say that* $\vec{d}(e)$ *are* $\vec{d}(e')$ *incomparable.*

2NN : bicriteria Nearest Neighbor
Input : $I = (G, \vec{d})$ instance of the bicriteria $TSP(1,2)$;
Output : Two tours p_1 and p_2 of I ;

Take arbitrarily $v^\star \in V$;
Set $v_s = v^\star$
Set $S = \{v_s\}$ and $u = v_s$;
Until $S \neq V$ Do
  Take $r \in V - S$ such that $\nexists t \in V - S$ s.t. $\vec{d}([u,t]) \prec_1 \vec{d}([u,r])$;
  Set $p_1(u) = r$ and $u = r$;
End Until ;
Set $p_1(r) = v_s$;

Take arbitrarily $v^{\star\star} \in V$;
Set $v_s = v^{\star\star}$
Set $S = \{v_s\}$ and $u = v_s$;
Until $S \neq V$ Do
  Take $r \in V - S$ such that $\nexists t \in V - S$ s.t. $\vec{d}([u,t]) \prec_2 \vec{d}([u,r])$;
  Set $p_2(u) = r$ and $u = r$;
End Until ;
Set $p_2(r) = v_s$;

Return $\{p_1, p_2\}$;

Tableau 1.1. *Algorithm* 2NN.

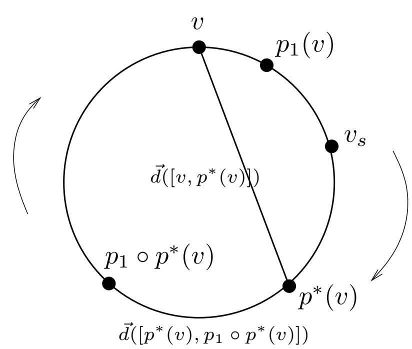

Figure 1.6. *Case 1.*

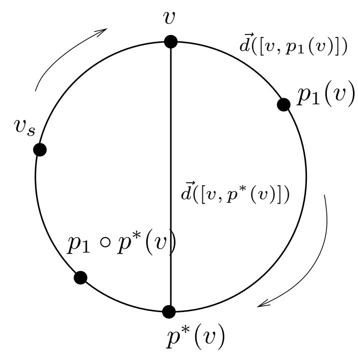

Figure 1.7. *Case 2.*

The algorithm proposed is called 2NN (*Bicriteria Nearest Neighbor*) and is given in Table 1.1. This algorithm returns two tours p_1 and p_2. We assume that for each node $v \in V$, $p_1(v)$ (resp. $p_2(v)$) represents the node which immediately follows v in p_1 (resp. p_2). Here, p^* denotes a Pareto optimal tour. Before proving that the two tours returned by 2NN constitute a $1/2$-approximate Pareto curve for the bicriteria $TSP(1,2)$, we introduce some notations. Let x_1 (resp. y_1, z_1, t_1) be the number of edges with distance $(1,1)$ (resp. $(1,2)$, $(2,1)$, $(2,2)$) of p_1. Let x_2 (resp. y_2, z_2, t_2) be the number of edges with distance $(1,1)$ (resp. $(1,2)$, $(2,1)$, $(2,2)$) of p_2. Let x^* (resp. y^*, z^*, t^*) be the number of edges with distance $(1,1)$ (resp. $(1,2)$, $(2,1)$, $(2,2)$) of p^*.

Since a tour has exactly n edges, we know that :

$$n = x_1 + y_1 + z_1 + t_1 = x_2 + y_2 + z_2 + t_2 = x^* + y^* + z^* + t^*.$$

LEMMA 6.– *For p_1 and p^*, one has $x_1 \geq x^*/2$.*

PROOF.– Let U_{p_1} (resp. U_{p^*}) be the set of $(1,1)$ edges of p_1 (resp. p^*). We define a function $f : U_{p^*} \rightarrow U_{p_1}$ in the following way. Let $[v, v']$ be an edge in U_{p^*}. If $[v, v'] \in U_{p_1}$ then $f([v, v']) = [v, v']$. Otherwise, we claim that among $[v, p_1(v)]$ and $[v', p_1(v')]$, there is at least one edge with distance $(1,1)$. Thus, $f([v, v']) = [v, p_1(v)]$ if the distance vector of $[v, p_1(v)]$ is $(1,1)$, otherwise $f([v, v']) = [v', p_1(v')]$. To see that, we consider two possibilities :

– $p_1(v) = p^*(v)$: If $\vec{d}([v, p^*(v)]) = (1,1)$ then $\vec{d}([v, p_1(v)]) = (1,1)$ and $f([v, p^*(v)]) = [v, p_1(v)]$.

– $p_1(v) \neq p^*(v)$: In case 1 (see Figure 1.6), if $\vec{d}([v, p^*(v)]) = (1,1)$ then $\vec{d}([p^*(v), p_1(p^*(v))]) = (1,1)$ and $f([v, p^*(v)]) = [p^*(v), p_1(p^*(v))]$. In case 2 (see

Figure 1.7), if $\vec{d}([v,p^*(v)]) = (1,1)$ then $\vec{d}([v,p_1(v)]) = (1,1)$ and $f([v,p^*(v)]) = [v,p_1(v)]$.

Thus, each $(1,1)$ edge of p_1 has at most two antecedents of distance $(1,1)$.

LEMMA 7.– *For p_2 and p^*, one has $x_2 \geq x^*/2$.*

PROOF.– See the proof of Lemma 6 and replace p_1 by p_2.

LEMMA 8.– *For p_1 and p^*, one has $x_1 + y_1 \geq (x^* + y^*)/2$.*

PROOF.– Let U_{p_1} (resp. U_{p^*}) be the set of $(1,1)$ and $(1,2)$ edges of p_1 (resp. p^*). We define a function $f : U_{p^*} \rightarrow U_{p_1}$ in the following way. Let $[v,v']$ be an edge in U_{p^*}. If $[v,v'] \in U_{p_1}$ then $f([v,v']) = [v,v']$. Otherwise, we claim that among $[v,p_1(v)]$ and $[v',p_1(v')]$, there is at least one edge with distance $(1,1)$ or $(1,2)$. Thus, $f([v,v']) = [v,p_1(v)]$ if the distance of $[v,p_1(v)]$ is $(1,1)$, otherwise $f([v,v']) = [v',p_1(v')]$. To see that, we consider two possibilities :

– $p_1(v) = p^*(v)$: If $\vec{d}([v,p^*(v)]) = (1,1)$ then $\vec{d}([v,p_1(v)]) = (1,1)$ or $\vec{d}([v,p_1(v)]) = (1,2)$ and $f([v,p^*(v)]) = [v,p_1(v)]$.

– $p_1(v) \neq p^*(v)$: In case 1 (see Figure 1.6), if $\vec{d}([v,p^*(v)]) = (1,1)$ or $\vec{d}([v,p^*(v)]) = (1,2)$ then $\vec{d}([p^*(v),p_1(p^*(v))]) = (1,1)$ or $\vec{d}([p^*(v),p_1(p^*(v))]) = (1,2)$ and $f([v,p^*(v)]) = [p^*(v),p_1(p^*(v))]$. In case 2 (see Figure 1.7), if $\vec{d}([v,p^*(v)]) = (1,1)$ or $\vec{d}([v,p^*(v)]) = (1,2)$ then $\vec{d}([v,p_1(v)]) = (1,1)$ or $\vec{d}([v,p_1(v)]) = (1,2)$ and $f([v,p^*(v)]) = [v,p_1(v)]$.

Thus, each edge of distance in $\{(1,1),(1,2)\}$ of p_1 has at most two antecedents of distance $(1,1)$ or $(1,2)$.

LEMMA 9.– *For p_2 and p^*, one has $x_2 + z_2 \geq (x^* + z^*)/2$.*

PROOF.– See the proof of Lemma 8.

THEOREM 4.– *The two tours returned by* 2NN *constitute a $1/2$-approximate Pareto curve for the bicriteria $TSP(1,2)$. Moreover, this bound is asymptotically sharp.*

PROOF.– The proof is completely similar to the one given in Theorem 3. Moreover, this bound is asymptotically attained for the particular instance given in Figure 1.5. Indeed, 2NN can produce two identical tours $s_1\, s_2\, \ldots s_{2n-1}\, s_{2n}$ of total distance $(3n,3n)$ while tour $_1\, s_3\, s_{2n}\, s_4 \ldots s_{n+3}\, s_{n+1}\, s_{n+2}\, s_2$ has a total distance of $(2n+1, 2n+1)$.

In the next subsection, we deal with a subcase of the bicriteria Max $TSP(1,2)$ problem.

AL1 :
Input : $I = (G, d)$ instance of the bicriteria Max $TSP(1, 2)$ where $d(e) \in \{(1, 2), (2, 1)\}$;
Output : Three tours T_1, T_2 and T_3 of I ;

1. Produce a tour T_1 using the nearest neighbor algorithm where edges $(1, 2)$ are prefered to edges $(2, 1)$;
2. Produce a tour T_2 using the nearest neighbor algorithm where edges $(2, 1)$ are prefered to edges $(1, 2)$;
3. Find a third tour T_3 by the following way
   3.1 Find a maximum matching M_1 in the subgraph induced by the $(1, 2)$ edges ;
   3.2 Find a maximum matching M_2 in the subgraph induced by the $(2, 1)$ edges ;
   3.3 $C_1, \ldots, C_q$ and $P_1, \ldots, P_r$ respectively denote the cycles and the paths in the graph induced by $M_1 \cup M_2$;
   3.4 For each C_{2i} with $1 \leq 2i \leq q$, delete one edge with distance vector $(1, 2)$ and let P'_{2i} denotes this path ;
   3.5 For each C_{2i+1} with $1 \leq 2i + 1 \leq q$, delete one edge with distance vector $(2, 1)$ and let P'_{2i+1} denotes this path ;
   3.6 Build T_3 by adding arbitrarily chosen edges to $\left(\bigcup_{i=1}^{r} P_i\right) \cup \left(\bigcup_{i=1}^{q} P'_i\right)$;
4. Return T_1, T_2 and T_3 ;

Tableau 1.2. *Algorithm* AL1.

1.3.4. *On the bicriteria Max* $TSP(1, 2)$

In Corollary 2, we saw that BMAXLS gives a 1/3-approximate Pareto curve for the bicriteria Max $TSP(1, 2)$ problem. In this subsection we improve this performance ratio, but only when we restrict the distances vector to edges with distance vectors $(1, 2)$ and $(2, 1)$. Actually, using Corollary 1, we observe that this restriction is already hard to approximate with $\varepsilon < \frac{1}{3r+1}$ if we use at most r tours. In particular, a single tour never provides a $(1/4 + \varepsilon)$-approximation of this restriction, for any $\varepsilon > 0$. We now prove how we can reach this performance ratio using only 3 tours. The algorithm presented below, starts with the two solutions produced by a maximization version of 2NN together with a third solution based on matchings.

THEOREM 5.– *Algorithm* AL1 *(see Table 1.2) returns a* $\frac{1}{4}$*-approximate Pareto curve for the bicriteria Max* $TSP(1, 2)$ *when we restrict the distance vectors to* $(1, 2)$ *and* $(2, 1)$.

PROOF.– *W*e start with the notations introduced in Subsection 1.3.3. The following equalities can be easily proved :

$$y_1 + z_1 = y_2 + z_2 = y^* + z^* = n \tag{1.3}$$

$$\begin{aligned}
\vec{D}(T_1) &= (n + z_1, n + y_1),\\
\vec{D}(T_2) &= (n + z_2, n + y_2),\\
\vec{D}(T^*) &= (n + z^*, n + y^*).
\end{aligned}$$

By Lemma 8, we have the following property for T_1 :

$$2y_1 \geq y^*. \tag{1.4}$$

Thus, using (1.3) and (1.4), we get :

$$\begin{aligned}
\vec{D}_2(T_1) = n + y_1 &\geq n + y^*/2\\
&= 3n/4 + 3y^*/4 + n/4 - y^*/4\\
&= \frac{3}{4}(n + y^*) + z^*/4\\
&\geq \frac{3}{4}\vec{D}_2(T^*)
\end{aligned}$$

By Lemma 9, we have the following property for T_2 :

$$2z_2 \geq z^*. \tag{1.5}$$

Thus, using (1.3) and (1.5), we get :

$$\begin{aligned}\vec{D}_1(T_2) = n + z_2 &\geq n + z^*/2 \\ &= 3n/4 + 3z^*/4 + n/4 - z^*/4 \\ &= \frac{3}{4}(n + z^*) + y^*/4 \\ &\geq \frac{3}{4}\vec{D}_1(T^*)\end{aligned}$$

(i) Assume $z_1 \geq n/4$. If $2n \geq 3z^*$ then $\vec{D}_1(T_1) \geq \frac{3}{4}\vec{D}_1(T^*)$ and T_1 approximates the whole Pareto curve. Indeed, we have :

$$\begin{aligned}z_1 &\geq n/4 \\ n + z_1 &\geq 3n/4 + n/2 \\ n + z_1 &\geq 3n/4 + 3z^*/4 \\ \vec{D}_1(T_1) &\geq \frac{3}{4}\vec{D}_1(T^*).\end{aligned}$$

In case $2n \leq 3z^*$, we have $2y^* \leq z^*$ and $n \geq 3y^*$ since $n = y^* + z^*$. Moreover, $3n + 4y_2 \geq 3n$ is always true since $y_2 \geq 0$. If we add $n \geq 3y^*$ to $3n + 4y_2 \geq 3n$, we get :

$$\begin{aligned}4n + 4y_2 &\geq 3n + 3y^* \\ \vec{D}_2(T_2) &\geq \frac{3}{4}\vec{D}_2(T^*).\end{aligned}$$

As a consequence, T_2 approximates the whole Pareto curve.

(ii) Assume $y_2 \geq n/4$. If $2n \geq 3y^*$ then $\vec{D}_2(T_2) \geq \frac{3}{4}\vec{D}_2(T^*)$ and T_2 approximates the whole Pareto curve. Indeed, we have :

$$\begin{aligned}y_2 &\geq n/4 \\ n + y_2 &\geq 3n/4 + n/2 \\ n + y_2 &\geq 3n/4 + 3y^*/4 \\ \vec{D}_2(T_2) &\geq \frac{3}{4}\vec{D}_2(T^*).\end{aligned}$$

In case $2n \leq 3y^*$, we have $2z^* \leq y^*$ and $n \geq 3z^*$ since $n = y^* + z^*$. Moreover, $3n + 4z_1 \geq 3n$ is always true since $z_1 \geq 0$. If we add $n \geq 3z^*$ to $3n + 4z_1 \geq 3n$, we get :

$$\begin{aligned} 4n + 4z_1 &\geq 3n + 3z^* \\ \vec{D}_1(T_1) &\geq \frac{3}{4}\vec{D}_1(T^*). \end{aligned}$$

As a consequence, T_1 approximates the whole Pareto curve.

(iii) Now assume that we simultaneously have $z_1 \leq n/4$ and $y_2 \leq n/4$. In this case, T_1 has at least $3n/4$ $(1,2)$ edges and T_2 has at least $3n/4$ $(2,1)$ edges. We deduce that M_1 (resp. M_2) has at least $3n/8$ edges of distance $(1,2)$ (resp. $(2,1)$). On the other hand, when we add M_1 to M_2, there is at most $n/4$ cycles and then, we delete at most $n/8$ edges of distance vector $(1,2)$ and $n/8$ edges of distance vector $(2,1)$. In conclusion, if z_3 (resp., y_3) denotes the set of $(2,1)$-edges (resp., $(1,2)$-edges) of T_3, we deduce $z_3 \geq 3n/8 - n/8 = n/4$ and $y_3 \geq 3n/8 - n/8 = n/4$. Hence, we get :

$$\vec{D}(T_3) \geq (n + z_3, n + y_3) \geq (5n/4, 5n/4) \tag{1.6}$$

We also assume that $z^* \leq 2y^*$ and $y^* \leq 2z^*$. Indeed, when $z^* \geq 2y^*$, we deduce that $y^* \leq n/3$ since $z^* + y^* = n$. Thus $\vec{D}_2(T^*) \leq 4n/3$. On the other hand, trivially $\vec{D}_2(T_2) \geq n$ and then $\vec{D}_2(T_2) \geq \frac{3}{4}\vec{D}_2(T^*)$. In conclusion, since previously we proved that $\vec{D}_1(T_2) \geq \frac{3}{4}\vec{D}_1(T^*)$ always holds, we get $\vec{D}(T_2) \geq \frac{3}{4}\vec{D}(T^*)$. When $y^* \geq 2z^*$, the same inequality holds for T_1. Hence, in conclusion one can assume that $z^* \leq 2y^*$ and $y^* \leq 2z^*$. Then, $2n \geq 3z^*$ and $2n \geq 3y^*$. We derive from these two inequalities that :

$$\begin{aligned} n/3 \leq \quad & y^* \quad \leq 2n/3 \\ n/3 \leq \quad & z^* \quad \leq 2n/3 \end{aligned}$$

So, we obtain :

$$\vec{D}(T^*) \leq (5n/3, 5n/3). \tag{1.7}$$

Thus, in the worst case, T_3 approximates the whole Pareto curve within a ratio $3/4$.

1.4. k-criteria $TSP(1,2)$

In this section, we present a generalization of the previous results when k, the number of criteria, is larger than 2. We first give some non-approximability results related to the number of generated solutions for the k-criteria $TSP(1,2)$. Afterwards, a generalization of 2NN, which has a better complexity than BLS, is proposed. This generalization, called KNN, computes in $\mathcal{O}(n^2 k!)$ time a $\frac{k-1}{k+1}$-approximate Pareto curve for the k-criteria $TSP(1,2)$ when $k \geq 3$.

Let us observe here that the dependence of the time complexity on $k!$ is not surprising since the size of the approximate ε-Pareto curve is not necessarily polynomial on the number of the optimization criteria [PAPADIMITRIOU 00].

1.4.1. *Non-approximability related to the number of generated solutions*

We give in this section a non trivial generalization of the results given in Section 1.3.1. Thus, we propose a way to get some negative results which works for several multi-criteria problems and we put it into practice on the k-criteria $TSP(1,2)$.

In the following, we explicitly give a family of instances (denoted by $I_{n,r}$) of the k-criteria $TSP(1,2)$ for which we know a lot of different Pareto optimal tours covering a large spectrum of the possible values.

We first consider an instance I_n with $n \geq 2k+1$ vertices where distances belong to $\{(1,2,\ldots,2),\ (2,1,2,\ldots,2),\ldots,(2,\ldots,2,1)\}$. We suppose that for any $i = 1,\ldots,k$, the subgraph of I_n induced by the edges whose distance is 1 only on coordinate i is Hamiltonian (T_i denotes this tour). Using an old result [LUCAS 92], we know that K_n is Hamiltonian cycles decomposable into k disjoint tours if $n \geq 2k+1$ and then, I_n exists.

We duplicate the instance I_n r times to get $I_{n,r}$. We denote by v_a^c the vertex v_a of the c-th copy of I_n. Between two copies with $1 \leq c_1 < c_2 \leq r$, we set $\vec{d}([v_a^{c_1}, v_b^{c_2}]) = \vec{d}([v_a, v_b])$ if $a \neq b$ and $\vec{d}([v_a^{c_1}, v_a^{c_2}]) = (1,2,\ldots,2)$.

LEMMA 10.– *There are $\binom{r+k-1}{r}$ Pareto optimal tours in $I_{n,r}$ (denoted by $T_{c_1,\ldots,c_{k-1}}$ where c_i for $1 \leq i \leq k-1$ are $k-1$ indexes in $\{0,\ldots,r\}$) satisfying :*

(i) $\forall i = 1,\ldots,k-1$, $c_i \in \{0,\ldots,r\}$ *and* $\sum_{i=1}^{k-1} c_i \leq r$.

(ii) $\forall i = 1,\ldots,k-1$, $\vec{D}_i(T_{c_1,\ldots,c_{k-1}}) = 2rn - c_i n$ *and* $\vec{D}_k(T_{c_1,\ldots,c_{k-1}}) = rn + n(\sum_{i=1}^{k-1} c_i)$.

PROOF.– Let $c_1,\ldots,c_{k-1}$ be integers satisfying (i). We build the tour $T_{c_1,\ldots,c_{k-1}}$ by applying the following process : On the c_1 first copies, we take the tour T_1, on the c_2

second copies, we take the tour T_2 and so on. Finally, for the $r - \sum_{i=1}^{k-1} c_i$ last copies, we take T_k. For any $1 \leq l_1 < l_2 \leq r$, and any tours T and T', we patch T on copy l_1 with T' on copy l_2 by replacing the edges $[v_i^{l_1}, v_j^{l_1}] \in T$ and $[v_j^{l_2}, v_m^{l_2}] \in T'$ by the edges $[v_i^{l_1}, v_j^{l_2}]$ and $[v_m^{l_2}, v_j^{l_1}]$. Observe that the resulting tour has a total distance $\vec{D}(T') + \vec{D}(T)$. So, by applying r times this process, we can obtain a tour $T_{c_1,\ldots,c_{k-1}}$ satisfying (ii). Moreover, the number of tours is equal to the number of choices of $k-1$ elements among $r + (k-1)$.

THEOREM 6.– *For any $k \geq 2$, any ε-approximate Pareto curve for the k-criteria $TSP(1,2)$ containing at most x solutions satisfies :*

$$\begin{aligned} \varepsilon &\geq max_{i=2,\ldots,k}\{\frac{1}{(2i-1)r(i,x)-1}\} \\ \textit{where } r(i,x) &= min\{r|\ x \leq \binom{r+i-1}{r} - 1\}. \end{aligned}$$

PROOF.– Let $r(k,x) = r$ be the smallest integer such that $x \leq \binom{r+k-1}{r} - 1$ and consider the instance $I_{n,r}$. Since $x \leq \binom{r+k-1}{r} - 1$, there exists two distinct tours $T_{c_1,\ldots,c_{k-1}}$ and $T_{c'_1,\ldots,c'_{k-1}}$ and a tour T in the approximate Pareto curve such that :

$$\vec{D}(T) \leq (1+\varepsilon)\vec{D}(T_{c_1,\ldots,c_{k-1}}) \text{ and } \vec{D}(T) \leq (1+\varepsilon)\vec{D}(T_{c'_1,\ldots,c'_{k-1}}) \tag{1.8}$$

Let $l_i = \max\{c_i, c'_i\}$ for $i = 1,\ldots,k-1$ and $l_k = \min\{\sum_{i=1}^{k-1} c_i, \sum_{i=1}^{k-1} c'_i\}$. By construction, we have $l_k \leq \sum_{i=1}^{k-1} l_i - 1$. Moreover, the total distance of T can be written $\vec{D}_i(T) = 2rn - q_i$ for $i = 1,\ldots,k-1$ and $\vec{D}_k(T) = rn + \sum_{i=1}^{k-1} q_i$ for some value of q_i (q_i is the number of edges of T where the distance has a 1 on coordinate i and 2 on the others). Thus, using inequalities (1.8), we deduce that for $i = 1,\ldots,k-1$, we have $2nr - q_i \leq (1+\varepsilon)(2rn - l_i n)$ which is equivalent to

$$q_i \geq l_i n(1+\varepsilon) - 2rn\varepsilon. \tag{1.9}$$

Using inequalities (1.8), we also have $rn + \sum_{i=1}^{k-1} q_i \leq (1+\varepsilon)(rn + l_k n)$ which is equivalent to

$$\sum_{i=1}^{k-1} q_i \leq \varepsilon rn + l_k n(1+\varepsilon). \tag{1.10}$$

Adding inequalities (1.9) for $i = 1,\ldots,k-1$ and by using inequality (1.10) and $l_k \leq \sum_{i=1}^{k-1} l_i - 1$, we deduce :

$$\varepsilon \geq \frac{1}{(2k-1)r(k,x)-1}. \tag{1.11}$$

k \ x	1	2	3	4	5	6	7	8	9
2	0.500	0.200	0.125	0.090	0.071	0.058	0.050	0.043	0.038
3	0.500	0.250	0.125	0.111	0.111	0.071	0.071	0.071	0.071
4	0.500	0.250	0.166	0.111	0.111	0.076	0.076	0.076	0.076

Tableau 1.3. *Numerical values of ε according to Theorem 6.*

Finally, since an ε-approximation for the k-criteria $TSP(1,2)$ is also an ε-approximation for the i-criteria $TSP(1,2)$ with $i = 2, \dots, k-1$, we can apply $k-1$ times the inequality (1.11) and the result follows.

The Table 1.3 illustrates the Theorem 6 for some values of k and x. From Theorem 6, we are able to give a more explicit but less powerful result.

COROLLARY 3.– *For any $k \geq 2$, any ε-approximate Pareto curve for the k-criteria $TSP(1,2)$ containing at most x solutions satisfies :*

$$\varepsilon \geq \frac{1}{(2k-1)\Big(x(k-1)!\Big)^{1/(k-1)} - 1}.$$

PROOF.– By construction of $r(k,x) = r$, we have $x \geq \binom{r(k,x)-1+k-1}{k-1}$. Since

$$\binom{r-1+k-1}{k-1} \geq \frac{r^{k-1}}{(k-1)!},$$

we deduce

$$r \leq \Big(x(k-1)!\Big)^{1/(k-1)}.$$

Thus, using the inequality (1.11), we obtain the expected result.

More generally, if we write $R_k(x) = \Big((2k-1)(x(k-1)!)^{1/(k-1)} - 1\Big)^{-1}$, then observe that the following property holds : $\forall k \geq 2, \exists x_0, \forall x \geq x_0$ we have $R_{k+1}(x) \geq R_k(x)$. In other words, between two different versions of the k-criteria $TSP(1,2)$, the negative bound increases with k. So, these bounds are interesting when k is a fixed constant and x is an arbitrarily large integer (indeed, when $k = o(x)$).

On the other hand, we can also obtain other bounds when x is fixed and k grows to infinity ($x = o(k)$). In particular, when the ε-approximate Pareto curve contains just x solutions, we obtain $\varepsilon \geq 1/(2x-1) - \delta$ for the k-criteria $TSP(1,2)$ with k arbitrarily large.

THEOREM 7.– *For any $k \geq 2$, any ε-approximate Pareto curve for the k-criteria $TSP(1,2)$ containing at most $x \leq k$ solutions satisfies :*

$$\varepsilon \geq 1 - \frac{1}{\lceil k/x \rceil}.$$

PROOF.– Let x be an integer smaller than or equal to k and consider the instance $I_{n,r}$ used in Lemma 10 with $n \geq 2k+1$ and $r = 1$. The instance $I_{n,r}$ admits (at least) k Pareto optimal tours denoted by T_j (the tour T_j only uses edges with a 1 on coordinate j and a 2 on the others). By the construction, we know that $\vec{D}_i(T_j) = 2n$ when $i \neq j$ and $\vec{D}_i(T_j) = n$ otherwise.

Now, consider an ε-approximate Pareto curve that contains at most x solutions. One of these x solutions, denoted by T', approximates at least $p = \lceil \frac{k}{x} \rceil$ Pareto optimal tours. W.l.o.g., we suppose that T' ε-approximates the tours $T_1, T_2, \ldots T_p$:

$$1 + \varepsilon \geq \max_{i=1,\ldots,k \text{ and } j=1,\ldots p} \frac{\vec{D}_i(T')}{\vec{D}_i(T_j)}.$$

Since $\vec{D}_i(T_j) = 2n$ when $j \in \{1, \ldots, p\}$ and $p < i \leq k$, we get :

$$\max_{i=1,\ldots,k \text{ and } j=1,\ldots p} \frac{\vec{D}_i(T')}{\vec{D}_i(T_j)} = \max_{i=1,\ldots,p \text{ and } j=1,\ldots p} \frac{\vec{D}_i(T')}{\vec{D}_i(T_j)}.$$

Since $\vec{D}_j(T_j) = n$ when $j \in \{1, \ldots, p\}$, we get :

$$\max_{i=1,\ldots,p \text{ and } j=1,\ldots p} \frac{\vec{D}_i(T')}{\vec{D}_i(T_j)} = \frac{1}{n}\Big(\max_{i=1,\ldots,p} \vec{D}_i(T')\Big).$$

Then, we have :

$$1 + \varepsilon \geq \frac{1}{n}\Big(\max_{i=1,\ldots,p} \vec{D}_i(T')\Big) \tag{1.12}$$

We know that any feasible tour T of $I_{n,r}$ satisfies :

$$\sum_{i=1}^{k} \vec{D}_i(T) = n(2k-1).$$

Indeed, the distance of each edge of the instance has exactly one coordinate equal to 1. Thus, any tour T satisfies :

$$\sum_{i=1}^{p} \vec{D}_i(T) \geq \sum_{i=1}^{k} \vec{D}_i(T) - 2n(k-p) = n(2p-1).$$

In particular, one can observe that :

$$\max_{i=1,\dots,p} \vec{D}_i(T') \geq \frac{\sum_{i=1}^{p} \vec{D}_i(T')}{p} \geq n(2 - \frac{1}{p}) \tag{1.13}$$

Thus, inequalities (1.12) and (1.13) give $1 + \varepsilon \geq 2 - 1/p$ which means that $\varepsilon \geq 1 - (\lceil k/x \rceil)^{-1}$.

The method presented in this section can be applied to several other multi-criteria problems. For instance, it works with problems where all feasible solutions have the same size ($|V|$ for a Hamiltonian cycle, $|V| - 1$ for a spanning tree, etc).

1.4.2. *A nearest neighbor heuristic for the k-criteria $TSP(1,2)$*

Adapting the nearest neighbor heuristic to the k-criteria $TSP(1,2)$ gives rise to two questions : How can we translate the notion of closeness when multiple objectives are considered ? How many solutions must be generated to get an approximation of the Pareto curve ? In the following, we propose a way which simultaneously brings an answer to both questions. Given the problem, the total distance of a Pareto optimal tour T^* is enclosed in a k-dimensional distance space. The way to generate a tour T which approximates T^*, and also the notion of closeness, depend on where $\vec{D}(T^*)$ is located in the distance space. The idea is to partition the distance space into a fixed number of parts. Then, with each part we associate an appropriate notion of closeness. Given a part and its proper notion of closeness, we can generate with the nearest neighbor rule a tour which approximates any Pareto optimal solution whose total distance is in this part. For any instance of the k-criteria $TSP(1,2)$, we propose to divide the distance space into $k!$ parts as follows : Each part is identified by a permutation of $\{1, \dots, k\}$. Given a permutation L of $\{1, \dots, k\}$, a tour T is in the part identified by L if $\vec{D}_{L(1)}(T) \leq \dots \leq \vec{D}_{L(k)}(T)$. For the notion of closeness, we introduce a preference relation over all possible distance vectors which looks like a lexicographic order. This preference relation which depends on L (denoted by $\prec_L$) is defined by using $k+1$ sets $S_1, \dots, S_{k+1}$:

$$\begin{aligned} S_q &= \{\vec{a} \in \{1,2\}^k \mid \forall j \leq k+1-q \quad \vec{a}_{L(j)} = 1\}, \ \text{ for } 1 \leq q \leq k \\ S_{k+1} &= \{1,2\}^k. \end{aligned}$$

COROLLARY 3.– *For any edge e, we say that e is S_q-*preferred *(for $\prec_L$) if $\vec{d}(e) \in S_q \backslash S_{q-1}$ (where $S_0 = \emptyset$). For two edges e and e' such that e is S_q-preferred and e' is*

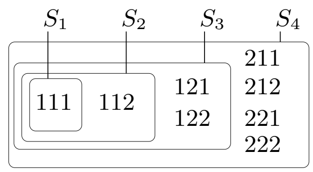

Figure 1.8. *One has*
$111 \prec_L 112 \prec_L 121 \preccurlyeq_L 122 \prec_L 211 \preccurlyeq_L 212 \preccurlyeq_L 221 \preccurlyeq_L 222$.

KNN : k-criteria Nearest Neighbor
$P := \emptyset$;
For each permutation L of $\{1, 2, \dots, k\}$ do
  Take arbitrarily $v \in V$;
  $W := \{v\}$; $u := v$;
  While $W \neq V$ do
    Take $r \in V \backslash W$ s.t. r is the closest vertex to u by $\preccurlyeq_L$;
    $W := W \cup \{r\}$;
    $p(u) := r$; $u := r$;
  End While ;
  $p(r) := v$;
  $P := P \cup \{p\}$;
End do ;
Return P ;

Tableau 1.4. *For $v \in V$ and p a tour, $p(v)$ denotes the node which immediately follows v in p.*

$S_{q'}$-preferred, we say that $\vec{d}(e)$ is preferred (resp., weakly preferred) to $\vec{d}(e')$ and we note $\vec{d}(e) \prec_L \vec{d}(e')$ (resp., $\vec{d}(e) \preccurlyeq_L \vec{d}(e')$) iff $q < q'$ (resp., $q \leq q'$).

An example where $k = 3$ and L is the identity permutation is given in Figure 1.8.

The algorithm that we propose for the k-criteria $TSP(1,2)$ is given in Table 1.4. Called KNN for k-criteria Nearest Neighbor, it is composed of $k!$ steps. A permutation L of $\{1, 2, \dots, k\}$ is determined at each step. With a permutation L, we build a preference relation $\prec_L$ and finally, a solution is greedily generated with the nearest neighbor rule.

1.4.2.1. *Analysis of* KNN

We prove that KNN returns a $(k-1)/(k+1)$-approximate Pareto curve for the k-criteria $TSP(1,2)$ when $k \geq 3$. The proof of this result requires some notations and intermediate lemmata.

In the following, we consider two particular tours p and p^*. We assume that p is the tour generated by KNN with the preference relation $\prec_L$ and that p^* is a Pareto optimal tour satisfying

$$\vec{D}_{L(1)}(p^*) \leq \vec{D}_{L(2)}(p^*) \leq \ldots \leq \vec{D}_{L(k)}(p^*). \tag{1.14}$$

The set of all possible distance vectors $\{1,2\}^k$ is denoted by Ω. For every $j \leq k$, we introduce $U_j = \{\vec{a} \in \Omega \mid \vec{a}_j = 1\}$ and $\overline{U}_j = \{\vec{a} \in \Omega \mid \vec{a}_j = 2\}$. For $\vec{a} \in \Omega$, we note $X_{\vec{a}} = \{v \in V \mid \vec{d}([v, p(v)]) = \vec{a}\}$ and $X^*_{\vec{a}} = \{v \in V \mid \vec{d}([v, p^*(v)]) = \vec{a}\}$. Finally, $x_{\vec{a}}$ (resp. $x^*_{\vec{a}}$) denotes the cardinality of $X_{\vec{a}}$ (resp. $X^*_{\vec{a}}$).

If n is the number of vertices then by construction we have $\sum_{\vec{a}\in\Omega} x_{\vec{a}} = \sum_{\vec{a}\in\Omega} x^*_{\vec{a}} = n$, $\vec{D}_j(p) = 2n - \sum_{\vec{a}\in U_j} x_{\vec{a}}$ and $\vec{D}_j(p^*) = 2n - \sum_{\vec{a}\in U_j} x^*_{\vec{a}}$.

LEMMA 11.– *The following holds for any $q \leq k$:*

$$2 \sum_{\vec{a}\in\bigcap_{j=1}^{k+1-q} U_{L(j)}} x_{\vec{a}} \geq \sum_{\vec{a}\in\bigcap_{j=1}^{k+1-q} U_{L(j)}} x^*_{\vec{a}}.$$

PROOF.– We define $F_q = \{v \in V \mid \vec{d}([v, p(v)]) \in S_q\}$ and $F^*_q = \{v \in V \mid \vec{d}([v, p^*(v)]) \in S_q\}$. Then, we have to prove that $2|F_q| \geq |F^*_q|$. The key result is to see that $p^*[F^*_q \backslash F_q] \subseteq F_q$ where $p^*[W] = \bigcup_{v\in W}\{p^*(v)\}$. Take a vertex v in $F^*_q \backslash F_q$ (see Figure 1.9). Then, $\vec{d}([v, p^*(v)]) \in S_q$, $\vec{d}([v, p(v)]) \in S_{q'}$ and $q' > q$. During the computation of p, suppose that v is the current node and that $p^*(v)$ is not already visited. We get a contradiction (the nearest neighbor rule is violated) since $p(v)$ immediately follows v in p and $\vec{d}([v, p^*(v)]) \prec_L \vec{d}([v, p(v)])$. Now, suppose that $p^*(v)$ was already visited. It directly precedes $p(p^*(v))$ in p and then $\vec{d}([p*(v), p(p^*(v))]) \preccurlyeq_L \vec{d}([v, p^*(v)])$. As a consequence, $\vec{d}([p^*(v), p(p^*(v))]) \in S_{q''}$ such that $q'' \leq q$ and $p^*(v) \in F_q$ since $S_{q''} \subseteq S_q$.

Since $|p^*[F^*_q \backslash F_q]| = |F^*_q \backslash F_q|$, $|F^*_q| = |F^*_q \backslash F_q| + |F^*_q \cap F_q|$ and $|F_q| \geq |F^*_q \cap F_q|$, we deduce $|F^*_q| = |p^*[F^*_q \backslash F_q]| + |F^*_q \cap F_q| \leq 2|F_q|$. Finally, since $\bigcap_{j=1}^{k+1-q} U_{L(j)} = S_q$, $|F_q| = \sum_{\vec{a}\in S_q} x_{\vec{a}}$ and $|F^*_q| = \sum_{\vec{a}\in S_q} x^*_{\vec{a}}$, the result follows.

The following inequality is equivalent to (1.14) :

$$\sum_{\vec{a}\in U_{L(1)}} x^*_{\vec{a}} \geq \sum_{\vec{a}\in U_{L(2)}} x^*_{\vec{a}} \geq \ldots \geq \sum_{\vec{a}\in U_{L(k)}} x^*_{\vec{a}}.$$

We easily deduce that for any couple j_1, j_2 such that $j_1 < j_2$ we have :

$$\sum_{\vec{a}\in U_{L(j_2)}\backslash U_{L(j_1)}} x^*_{\vec{a}} \leq \sum_{\vec{a}\in U_{L(j_1)}\backslash U_{L(j_2)}} x^*_{\vec{a}}. \tag{1.15}$$

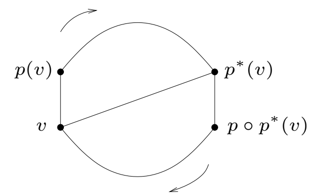

Figure 1.9. *The tour p generated by* KNN. *The edge $[v, p^*(v)]$ belongs to p^*.*

Let b_1, b_2, j and m be such that $b_1 \in \{1,2\}$, $b_2 \in \{1,2\}$, $1 \leq j \leq k$ and $1 \leq m < j$. Let $R(b_1, j, m, b_2)$ be the set of all $\vec{a} \in \Omega$ such that $\vec{a}_{L(j)} = b_1$ and there exists exactly m distinct coordinates of $\vec{a}$ among $\{\vec{a}_{L(1)}, \vec{a}_{L(2)}, \dots, \vec{a}_{L(j-1)}\}$ which are equal to b_2. Remark that $R(b_1, j, m, b_2) = R(b_1, j, j-1-m, \overline{b_2})$ where $\overline{b_2} = 3 - b_2$.

LEMMA 12.– *For any $j \leq k$, one has :*

$$\sum_{q=1}^{j-1} \Big(q \sum_{\vec{a} \in R(1,j,q,2) \cup R(2,j,q,2)} x^*_{\vec{a}} \Big) \leq (j-1) \sum_{q=0}^{j-1} \Big(\sum_{\vec{a} \in R(2,j,q,1)} x^*_{\vec{a}} \Big).$$

PROOF.– We sum up inequality (1.15) with $j_1 \in \{1, \dots, j-1\}$ and $j_2 = j$. We get the following inequality :

$$\sum_{q=1}^{j-1} \Big(\sum_{\vec{a} \in U_{L(j)} \setminus U_{L(q)}} x^*_{\vec{a}} \Big) \leq \sum_{q=1}^{j-1} \Big(\sum_{\vec{a} \in U_{L(q)} \setminus U_{L(j)}} x^*_{\vec{a}} \Big). \quad (1.16)$$

We also have the following equality :

$$\forall j \leq k, \ \sum_{q=1}^{j-1} \Big(\sum_{\vec{a} \in U_{L(j)} \setminus U_{L(q)}} x^*_{\vec{a}} \Big) = \sum_{q=1}^{j-1} \Big(q \sum_{\vec{a} \in R(1,j,q,2)} x^*_{\vec{a}} \Big). \quad (1.17)$$

Let $\vec{a}$ be a distance vector in $R(1, j, q, 2)$. By definition, $\vec{a}_{L(j)} = 1$ and there exists a set $\{i_1, \dots, i_q\}$ with $1 \leq i_1 < i_2 < \dots < i_q < j$ such that $\vec{a}_{L(i_1)} = \vec{a}_{L(i_2)} = \dots = \vec{a}_{L(i_q)} = 2$. Moreover, for all $j' \leq j-1$ such that $j' \notin \{i_1, \dots, i_q\}$, we have $\vec{a}_{L(j')} = 1$. Thus, $\vec{a} \in U_{L(j)} \setminus U_{L(g)}$ iff $g \in \{i_1, i_2, \dots, i_q\}$.

Using a similar argument, we obtain :

$$\forall j \leq k, \ \sum_{q=1}^{j-1} \Big(\sum_{\vec{a} \in U_{L(q)} \setminus U_{L(j)}} x^*_{\vec{a}} \Big) = \sum_{q=1}^{j-1} \Big(q \sum_{\vec{a} \in R(2,j,q,1)} x^*_{\vec{a}} \Big). \quad (1.18)$$

Then, using (1.16), (1.17) and (1.18) we get :

$$\sum_{q=1}^{j-1}\Big(q\sum_{\vec{a}\in R(1,j,q,2)} x^*_{\vec{a}}\Big) \leq \sum_{q=1}^{j-1}\Big(q\sum_{\vec{a}\in R(2,j,q,1)} x^*_{\vec{a}}\Big). \quad (1.19)$$

Since $R(2,j,q,2) = R(2,j,j-1-q,1)$, the following equality holds :

$$\sum_{q=1}^{j-1}\Big(q\sum_{\vec{a}\in R(2,j,q,1)} x^*_{\vec{a}}\Big) = (j-1)\sum_{q=0}^{j-1}\Big(\sum_{\vec{a}\in R(2,j,q,1)} x^*_{\vec{a}}\Big) - \sum_{q=1}^{j-1}\Big(q\sum_{\vec{a}\in R(2,j,q,2)} x^*_{\vec{a}}\Big). \quad (1.20)$$

So, Lemma 12 follows from (1.19) and (1.20).

THEOREM 8.– kNN *returns a* $(k-1)/(k+1)$*-approximate Pareto curve for the* k*-criteria* $TSP(1,2)$ *when* $k \geq 3$*.*

PROOF.– In the following, we consider that L is any permutation of $\{1,\ldots,k\}$, p^* is a Pareto optimal tour satisfying (1.14) and p is built with the nearest neighbor rule and the preference relation $\prec_L$. Then, we have to show that if $j \geq 3$ then $\vec{D}_{L(j)}(p) \leq (1+\frac{j-1}{j+1})\vec{D}_{L(j)}(p^*)$.

The previous inequality holds if we have the following inequality :

$$-(j+1)\sum_{\vec{a}\in U_{L(j)}} x_{\vec{a}} \leq 2(j-1)\sum_{\vec{a}\in \overline{U}_{L(j)}} x^*_{\vec{a}} - 2\sum_{\vec{a}\in U_{L(j)}} x^*_{\vec{a}}. \quad (1.21)$$

$$\begin{aligned}
\vec{D}_{L(j)}(p) \leq \frac{2j}{j+1}\vec{D}_{L(j)}(p^*) \quad &\Leftrightarrow \quad (j+1)\Big(2n - \sum_{\vec{a}\in U_{L(j)}} x_{\vec{a}}\Big) \leq 2j\Big(2n - \sum_{\vec{a}\in U_{L(j)}} x^*_{\vec{a}}\Big)\\
&\Leftrightarrow \quad -(j+1)\sum_{\vec{a}\in U_{L(j)}} x_{\vec{a}} \leq 2(j-1)n - 2j\sum_{\vec{a}\in U_{L(j)}} x^*_{\vec{a}}\\
&\Leftrightarrow \quad -(j+1)\sum_{\vec{a}\in U_{L(j)}} x_{\vec{a}} \leq 2(j-1)\sum_{\vec{a}\in \overline{U}_{L(j)}} x^*_{\vec{a}} - 2\sum_{\vec{a}\in U_{L(j)}} x^*_{\vec{a}},
\end{aligned}$$

using $n = \sum_{\vec{a}\in U_{L(j)}} x^*_{\vec{a}} + \sum_{\vec{a}\in \overline{U}_{L(j)}} x^*_{\vec{a}}$.

Let us denote by $\mathcal{A}$ and $\mathcal{B}$ the following quantities :

$$\begin{aligned}\sum_{\vec{a}\in U_{L(j)}} x_{\vec{a}} &= \sum_{\vec{a}\in U_{L(j)}\setminus(\bigcap_{m\leq j-1} U_{L(m)})} x_{\vec{a}} + \sum_{\vec{a}\in \bigcap_{m\leq j} U_{L(m)}} x_{\vec{a}} = \mathcal{A}\\ \sum_{\vec{a}\in U_{L(j)}} x^*_{\vec{a}} &= \sum_{\vec{a}\in U_{L(j)}\setminus(\bigcap_{m\leq j-1} U_{L(m)})} x^*_{\vec{a}} + \sum_{\vec{a}\in \bigcap_{m\leq j} U_{L(m)}} x^*_{\vec{a}} = \mathcal{B}.\end{aligned}$$

Then, inequality (1.21) becomes :

$$-(j+1)\mathcal{A} \leq 2(j-1)\sum_{\vec{a}\in \overline{U}_{L(j)}} x^*_{\vec{a}} - 2\mathcal{B}. \tag{1.22}$$

To prove (1.22), we propose the following decomposition :

$$\mathcal{C} = 2(j-1)\sum_{\vec{a}\in \overline{U}_{L(j)}} x^*_{\vec{a}} - 2\sum_{\vec{a}\in U_{L(j)}\setminus \bigcap_{m\leq j-1} U_{L(m)}} x^*_{\vec{a}} - 4\sum_{\vec{a}\in \bigcap_{m\leq j} U_{L(m)}} x_{\vec{a}} \tag{1.23}$$

$$-(j+1)\mathcal{A} \leq \mathcal{C} \tag{1.24}$$

$$\mathcal{C} \leq 2(j-1)\sum_{\vec{a}\in \overline{U}_{L(j)}} x^*_{\vec{a}} - 2\mathcal{B} \tag{1.25}$$

Thus, (1.24) becomes :

$$\begin{aligned}-(j+1)\sum_{\vec{a}\in U_{L(j)}\setminus \bigcap_{m\leq j-1} U_{L(m)}} x_{\vec{a}} - (j-3)\sum_{\vec{a}\in \bigcap_{m\leq j} U_{L(m)}} x_{\vec{a}} \leq\\ \leq 2(j-1)\sum_{\vec{a}\in \overline{U}_{L(j)}} x^*_{\vec{a}} - 2\sum_{\vec{a}\in U_{L(j)}\setminus \bigcap_{m\leq j-1} U_{L(m)}} x^*_{\vec{a}}\end{aligned}$$

Since the left part of this inequality is negative, we want to prove that the right part is positive :

$$0 \leq 2(j-1)\sum_{\vec{a}\in \overline{U}_{L(j)}} x^*_{\vec{a}} - 2\sum_{\vec{a}\in U_{L(j)}\setminus \bigcap_{m\leq j-1} U_{L(m)}} x^*_{\vec{a}} \tag{1.26}$$

$$\sum_{\vec{a}\in U_{L(j)}\setminus \bigcap_{m\leq j-1} U_{L(m)}} x^*_{\vec{a}} \leq (j-1)\sum_{\vec{a}\in \overline{U}_{L(j)}} x^*_{\vec{a}} \tag{1.27}$$

We also have :

$$\sum_{\vec{a}\in U_{L(j)}\setminus\bigcap_{m\le j-1}U_{L(m)}} x^*_{\vec{a}} = \sum_{q=1}^{j-1}\Big(\sum_{\vec{a}\in R(1,j,q,2)} x^*_{\vec{a}}\Big) \text{ and}$$

$$(j-1)\sum_{\vec{a}\in\overline{U}_{L(j)}} x^*_{\vec{a}} = (j-1)\sum_{q=0}^{j-1}\Big(\sum_{\vec{a}\in R(2,j,q,1)} x^*_{\vec{a}}\Big).$$

The first equality follows from $U_{L(j)}\setminus\bigcap_{m\le j-1}U_{L(m)} = \bigcup_{q=1}^{j-1}R(1,j,q,2)$ since $\vec{a}\in U_{L(j)}\setminus\bigcap_{m\le j-1}U_{L(m)}$ iff $\vec{a}_{L(j)}=1$ and there exists exactly q indexes $\{i_1,\dots i_q\}$ such that $1\le q\le j-1$ and $\vec{a}_{L(i_1)}=\vec{a}_{L(i_2)}=\dots=\vec{a}_{L(i_q)}=2$, which is equivalent to $\vec{a}\in R(1,j,q,2)$. The second equality follows from $\overline{U}_{L(j)}=\bigcup_{q=0}^{j-1}R(2,j,q,1)$ because $\vec{a}\in\overline{U}_{L(j)}$ means $\vec{a}_{L(j)}=2$.

As a consequence, (1.27) becomes :

$$\sum_{q=1}^{j-1}\Big(\sum_{\vec{a}\in R(1,j,q,2)} x^*_{\vec{a}}\Big) \le (j-1)\sum_{q=0}^{j-1}\Big(\sum_{\vec{a}\in R(2,j,q,1)} x^*_{\vec{a}}\Big).$$

With Lemma 12, we have :

$$\sum_{q=1}^{j-1}\Big(q\sum_{\vec{a}\in R(1,j,q,2)\cup R(2,j,q,2)} x^*_{\vec{a}}\Big) \le (j-1)\sum_{q=0}^{j-1}\Big(\sum_{\vec{a}\in R(2,j,q,1)} x^*_{\vec{a}}\Big)$$

and (1.27) follows from

$$\sum_{q=1}^{j-1}\Big(q\sum_{\vec{a}\in R(1,j,q,2)\cup R(2,j,q,2)} x^*_{\vec{a}}\Big) \ge \sum_{q=1}^{j-1}\Big(\sum_{\vec{a}\in R(1,j,q,2)} x^*_{\vec{a}}\Big).$$

By Lemma 11 with $q=k+1-j$ we have :

$$2\sum_{\vec{a}\in\bigcap_{m\le j}U_{L(m)}} x_{\vec{a}} \ge \sum_{\vec{a}\in\bigcap_{m\le j}U_{L(m)}} x^*_{\vec{a}}$$

which is exactly (1.25).

1.5. Concluding remarks

Negative results for multi-criteria optimization problems were not extensively investigated though their approximability motivated a lot of articles. By connecting the size of the approximate Pareto curve and the best approximation ratio which can be achieved, we present a way to get negative results which do not rely on **NP**-hardness. We applied the method to the k-criteria $TSP(1,2)$ but it also works with problems where all feasible solutions have the same size.

The approximability of the k-criteria $TSP(1,2)$ is also investigated with multi-criteria versions of the classical local search and the nearest neighbor heuristics. However, as the number of criteria grows, and even though the number of solutions is large ($k!$), the approximation ratio of KNN tends to 2. Then, it would be interesting to reduce the gap between positive and negative results. Following this direction, Manthey et al. [MANTHEY 06] recently considered randomized algorithms for several particular cases of the multicriteria traveling salesman problem.

1.6. Bibliographie

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Chapitre 2

Index