(Non)-Approximability for the multi-criteria $TSP(1,2)$

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Abstract. The approximability of multi-criteria combinatorial problems motivated a lot of articles. However, the non-approximability of this problems has never been investigated up to our knowledge. We propose a way to get some results of this kind which works for several problems and we put it into practice on a multi-criteria version of the traveling salesman problem with distances one and two ($TSP(1,2)$). Following the article of Angel et al. (FCT 2003) who presented an approximation algorithm for the bi-criteria $TSP(1,2)$, we extend and improve the result to any number k of criteria.

Keywords: non-approximability in multi-criteria optimization ; design and analysis of algorithms

1 Introduction

Multi-criteria optimization refers to problems with two or more objective functions which are normally in conflict. Vilfredo Pareto stated in 1896 a concept (known today as "Pareto optimality") that constitutes the origin of research in this area. According to this concept, the solution to a multi-criteria optimization problem is normally not a single value, but instead a set of values (the so-called *Pareto curve*). From a computational point of view, this Pareto curve is problematic. Approximating it with a performance guarantee, i.e. computing an $\varepsilon-$*approximate Pareto curve*, motivated a lot of papers (see [1, 6, 9] among others). Up to our knowledge, non-approximability in the specific context of multi-criteria optimization has surprisingly never been investigated. Of course, some straightforward results can be stated if we remark that a multi-criteria problem generalizes a mono-criterion problem. Consequently, we aim to state some negative results which are specific to this area. In multi-criteria optimization, one tries to approximate a set of solutions (the Pareto curve) with another set of solutions (the ε-approximate Pareto curve) and the more the ε-approximate Pareto curve contains solutions, the more accurate the approximation can be. Then, the best approximation ratio that could be achieved can be related to the size of the approximate Pareto curve. As a first attempt, we propose a way to get some negative results which works for several multi-criteria problems and we put it into practice on a special case of the multi-criteria traveling salesman problem.

The traveling salesman problem is one of the most studied problem in the operations research community, see for instance [4]. The case where distances are either one or two (denoted by $TSP(1,2)$) was investigated by Papadimitriou and Yannakakis [7] who gave some positive and negative approximation results (see also [2]). Interestingly, this problem finds an application in a frequency assignment problem [3]. In this article, we deal with a generalization of the $TSP(1,2)$ where the distance is a vector of length k instead of a scalar: the k-criteria $TSP(1,2)$. Previously, Angel et al. [1] proposed a *local search* algorithm (called BLS) for the bi-criteria $TSP(1,2)$ which, with only two solutions generated in $\mathcal{O}(n^3)$, was able to approximate the whole Pareto curve within a ratio of 3/2.

A question arises concerning the ability to improve the approximation ratio with an approximate Pareto curve containing two (or more) solutions. Conversely, given a fixed number of solutions, how accurate an approximate Pareto curve can be? More generally, given a multi-criteria problem, how many solutions are necessary to approximate the Pareto curve within a level of approximation? A second question arises concerning the ability to generalize BLS to any number of criteria. Indeed, a large part of the literature on multi-criteria optimization is devoted to bi-criteria problems and an algorithm which works for any number of criteria would be interesting.

The paper is organized as follows: In Section 2, we recall some definitions on exact and approximate Pareto curves. Section 3 is devoted to a method to derive some negatives results in the specific context of multi-criteria optimization. We use it for the k-criteria $TSP(1,2)$ but it works for several other problems. In Section 4, we study the approximability of the k-criteria $TSP(1,2)$. Instead of generalizing BLS, we adapt the classical *nearest neighbor* heuristic which is more manageable. This multi-criteria nearest neighbor heuristic works for any k and produces a 3/2-approximate Pareto curve when $k \in \{1,2\}$ and a $2k/(k+1)$-approximate Pareto curve when $k \geq 3$. This result extends for several reasons the one of Angel et al.. First, the new algorithm works for any $k \geq 2$, second the time complexity is decreased when $k = 2$.
In this article, proofs not presented in the text are given in Appendix.

2 Generalities

The Traveling Salesman Problem (TSP) is about to find in a complete graph $G = (V,E)$ a Hamiltonian cycle whose total distance is minimal. For the k-criteria TSP, each edge e has a *distance* $\boldsymbol{d}(e) = (\boldsymbol{d}_1(e), \ldots, \boldsymbol{d}_k(e))$ which is a vector of length k (instead of a scalar). The *total distance* of a tour T is also a vector $\boldsymbol{D}(T)$ where $\boldsymbol{D}_j(T) = \sum_{e \in T} \boldsymbol{d}_j(e)$ and $j = 1, \ldots, k$. In fact, a tour is evaluated with k objective functions. Given this, the goal of the optimization problem could be the following: Generating a feasible solution which simultaneously minimizes each coordinate. Unfortunately, such an ideal solution rarely exists since objective functions are normally in conflict. However a set of solutions representing all best possible trade-offs always exists (the so-called Pareto curve). Formally, a Pareto curve is a set of feasible solutions, each of them optimal in the sense of Pareto, which *dominates* all the other solutions. A tour T dominates another one T' (usually denoted by $T \leq T'$) iff $\boldsymbol{D}_j(T) \leq \boldsymbol{D}_j(T')$ for $j = 1, \ldots, k$ and, for at least one coordinate j', one has $\boldsymbol{D}_{j'}(T) < \boldsymbol{D}_{j'}(T')$. A solution is optimal in the sense of Pareto if no solution dominates it.

From a computational point of view, Pareto curves are problematic [6, 9]. Two of the main reasons are:

- the size of a Pareto curve which is often exponential with respect to the size of the corresponding problem,
- a multi-criteria optimization problem often generalizes a mono-criterion problem which is itself hard.

As a consequence, one tries to get a relaxation of this Pareto curve, i.e. an *ε-approximate Pareto curve* [6, 9]. An ε-approximate Pareto curve P_ε is a set of solutions such that for every solution s of the instance, there is an s' in P_ε which satisfies $\boldsymbol{D}_j(s') \leq \varepsilon \boldsymbol{D}_j(s)$ for $j = 1, \ldots, k$.

In [6], Papadimitriou and Yannakakis prove that every multi-criteria problem has an ε-approximate Pareto curve that is polynomial in the size of the input, and $1/\varepsilon$, but exponential in the number k of criteria. The design of polynomial time algorithms which generate approximate Pareto curves with performance guarantee motivated a lot of recent papers. In this article we study the k-criteria $TSP(1, 2)$. In this problem, each edge e of the graph has a distance vector $\boldsymbol{d}(e)$ of length k and $\boldsymbol{d}_j(e) \in \{1, 2\}$ for all j between 1 and k.

3 Non-approximability related to the number of generated solutions

Up to our knowledge, non-approximability of combinatorial problems with multiple objectives has never been investigated. As a first attempt, we propose a way to get some negative results which works for several multi-criteria problems and we put it into practice on the k-criteria $TSP(1, 2)$.

Usually, non-approximability results for mono-criterion problems bring thresholds of performance guarantee under which no polynomial time algorithm is likely to exist. Given a result of this kind for a mono-criterion problem Π, we directly get a negative result for a multi-criteria version of Π. Indeed, the multi-criteria version of Π generalizes Π. For example, hardness of inherent difficulty of the mono-criterion $TSP(1, 2)$ has been studied in [2, 7] and the best known lower bound is $5381/5380 - \epsilon$ (for all $\epsilon > 0$). Consequently, for all $\epsilon > 0$, no polynomial time algorithm can generate a $(5381/5380 - \epsilon)$-approximate Pareto curve unless $P = NP$. However, the structure of the problem, namely the fact that several criteria are involved, is not taken into account.

In multi-criteria optimization, one tries to approximate a set of solutions (the Pareto curve) with another set of solutions (the ε-approximate Pareto curve) and the more the ε-approximate Pareto curve contains solutions, the more accurate the approximation can be. As a consequence, the best approximation ratio that could be achieved can be related to the size of the approximate Pareto curve. Formally, ε is a function of $|P_\varepsilon|$. If we consider instances for which the whole (or a large part of the) Pareto curve P is known and if we suppose that we approximate it with a set $P' \subset P$ such that $|P'| = x$ then the best approximation ratio ε such that P' is an ε-approximate Pareto curve is related to x. Indeed, there must be a solution in P' which approximates at least two (or more) solutions in P.

In the following, we explicitly give a family of instances of the k-criteria $TSP(1,2)$ for which we known a lot of different Pareto optimal tours covering a large spectrum of the possible values.

Lemma 1. *For any $r \geq 1$, for any $n \geq 2k+1$, there exists an instance $I_{n,r}$ of the k-criteria $TSP(1,2)$ with nr vertices such that there are $\binom{r+k-1}{r}$ Pareto optimal tours (denoted by $T_{c_1,\ldots,c_{k-1}}$ where c_i for $1 \leq i \leq k-1$ are $k-1$ indexes in $\{0,\ldots,r\}$) satisfying:*

(i) $\forall i = 1,\ldots,k-1$, $c_i \in \{0,\ldots,r\}$ *and* $\sum_{i=1}^{k-1} c_i \leq r$.
(ii) $\forall i = 1,\ldots,k-1$, $\boldsymbol{D}_i(T_{c_1,\ldots,c_{k-1}}) = 2rn - c_i n$ *and* $\boldsymbol{D}_k(T_{c_1,\ldots,c_{k-1}}) = rn + n(\sum_{i=1}^{k-1} c_i)$.

Proof. We first consider an instance I_n with $n \geq 2k+1$ vertices where distances belong to $\{(1,2,\ldots,2), (2,1,2,\ldots,2),\ldots,(2,\ldots,2,1)\}$. Moreover, we suppose that for any $i = 1,\ldots,k$, the subgraph induced by the edges where the distance has a 1 only on coordinate i is Hamiltonian (T_i denotes this tour). For any $n \geq 2k+1$, using an old result (see [5]), we know that K_n is Hamiltonian cycles decomposable into k disjoint tours and then, such an instance exists. Finally, the instance $I_{n,r}$ is built by the following way: We duplicate $I_n = (K_n, d)$ r times (v_i^c denotes the vertex v_i of the c-th copy) and between two copies with $c_1 < c_2$, we set $\boldsymbol{d}([v_i^{c_1}, v_j^{c_2}]) = \boldsymbol{d}([v_i, v_j])$ if $i \neq j$ and $\boldsymbol{d}([v_i^{c_1}, v_i^{c_2}]) = (1,2,\ldots,2)$. Let $c_1,\ldots,c_{k-1}$ be integers satisfying (i), we build the tour $T_{c_1,\ldots,c_{k-1}}$ by applying the following process: On the c_1 first copies, we take the tour T_1, on the c_2 second copies, we take the tour T_2 and so on. Finally, for the $r - \sum_{i=1}^{k-1} c_i$ last copies, we take T_k. For any $1 \leq l_1 < l_2 \leq r$, and any tours T, T', we patch the tour T on copy l_1 with the tour T' on copy l_2 by replacing edges $[v_i^{l_1}, v_j^{l_1}] \in T$, $[v_j^{l_2}, v_m^{l_2}] \in T'$ by edges $[v_i^{l_1}, v_j^{l_2}], [v_m^{l_2}, v_j^{l_1}]$. Observe that the resulting tour has a weight $\boldsymbol{D}(T') + \boldsymbol{D}(T)$. So, by applying r times this process, we can obtain a tour $T_{c_1,\ldots,c_{k-1}}$ satisfying (ii). Moreover, the number of tours is equal to the number of choices of $k-1$ elements among $r + (k-1)$. □

Theorem 1. *For any $k \geq 2$, any algorithm $\mathcal{A}$ producing a ρ-approximate Pareto curve with at most x solutions for the k-criteria $TSP(1,2)$ satisfies:*

$$\rho \geq 1 + max_{i=2,\ldots,k}\{\frac{1}{(2i-1)r(i,x)-1}\} \text{ where } r(i,x) = min\{r|\ x \leq \binom{r+i-1}{r} - 1\}.$$

Proof. Let $\rho = (1+\varepsilon)$ and let $r(k,x) = r$ be the smallest integer such that $x \leq \binom{r+k-1}{r} - 1$ and consider the instance $I_{n,r}$ of Lemma 1. Since $x \leq \binom{r+k-1}{r} - 1$, there exists two distinct tours $T_{c_1,\ldots,c_{k-1}}$ and $T_{c'_1,\ldots,c'_{k-1}}$ and a tour T produced by $\mathcal{A}$ such that:

$$\boldsymbol{D}(T) \leq (1+\varepsilon)\boldsymbol{D}(T_{c_1,\ldots,c_{k-1}}) \text{ and } \boldsymbol{D}(T) \leq (1+\varepsilon)\boldsymbol{D}(T_{c'_1,\ldots,c'_{k-1}}) \tag{1}$$

Let $l_i = \max\{c_i, c'_i\}$ for $i = 1, \ldots, k-1$ and $l_k = \min\{\sum_{i=1}^{k-1} c_i, \sum_{i=1}^{k-1} c'_i\}$. By construction, we have $l_k \leq \sum_{i=1}^{k-1} l_i - 1$. Moreover, the total distance of T can be written $\boldsymbol{D}_i(T) = 2rn - q_i$ for $i = 1, \ldots, k-1$ and $\boldsymbol{D}_k(T) = rn + \sum_{i=1}^{k-1} q_i$ for some value of q_i (q_i is the number of edges of T where the distance has a 2 on coordinate i and 1 on the others). Thus, using inequalities (1), we deduce that, for $i = 1, \ldots, k-1$, we have $2nr - q_i \leq (1+\varepsilon)(2rn - l_i n)$ which is equivalent to

$$q_i \geq l_i n(1+\varepsilon) - 2rn\varepsilon. \tag{2}$$

We also have $rn + \sum_{i=1}^{k-1} q_i \leq (1+\varepsilon)(rn + l_k n)$ which is equivalent to

$$\sum_{i=1}^{k-1} q_i \leq \varepsilon rn + l_k n(1+\varepsilon). \tag{3}$$

Adding inequalities (2) for $i = 1, \ldots, k-1$ and by using inequality (3) and $l_k \leq \sum_{i=1}^{k-1} l_i - 1$, we deduce:

$$\varepsilon \geq \frac{1}{(2k-1)r(k,x) - 1}. \tag{4}$$

Finally, since a ρ-approximation for the k-criteria $TSP(1,2)$ is also a ρ-approximation for the i-criteria $TSP(1,2)$ with $i = 2, \ldots, k-1$ (for the $k-i$ last coordinates, we get a factor 2), we can apply $k-1$ times the inequality (4) and the result follows. □

The following table gives some (truncated) numerical values of the best approximation ratio that it is possible to achieve:

$k \backslash x$	1	2	3	4	5	6	7	8	9
2	1.500	1.200	1.125	1.090	1.071	1.058	1.050	1.043	1.038
3	1.500	1.250	1.125	1.111	1.111	1.071	1.071	1.071	1.071
4	1.500	1.250	1.166	1.111	1.111	1.076	1.076	1.076	1.076

The method presented in this section can be applied to several other multicriteria problems. For instance, it works with problems where all feasible solutions have the same size ($|V|$ for a Hamiltonian cycle, $|V|-1$ for a spanning tree, etc).

4 Nearest neighbor heuristic for the k-criteria $TSP(1,2)$

Angel et al. present in [1] a *local search* algorithm (called BLS) for the bi-criteria $TSP(1,2)$. This algorithm returns in time $\mathcal{O}(n^3)$ a 3/2-approximate Pareto curve. Since BLS works only for the bi-criteria $TSP(1,2)$, an algorithm which works for any number of criteria would be interesting.

A generalization of BLS may exist but it is certainly done with difficulty. Since BLS uses the $2-opt$ neighborhood, two neighboring solutions differ on two edges. Defining an order on each couple of possible distance vector is necessary to decide, among two neighboring solutions, which one is the best. When k grows, such an order is hard to handle.

In this section, we present a different algorithm which is more manageable. It works for any number of criteria and its time complexity is better than BLS's one for the bi-criteria $TSP(1,2)$. We propose a *nearest neighbor* heuristic which computes in $\mathcal{O}(n^2k!)$ time a $\frac{2k}{k+1}$-approximate Pareto curve when $k \geq 3$ and a 3/2-approximate Pareto curve when $k \in \{1,2\}$. Let us observe here that the dependence of the time complexity on $k!$ is not surprising since the size of the approximate ε-Pareto curve is not necessarily polynomial on the number of the optimization criteria [6].

Traditionally, the nearest neighbor heuristic [8] consists in starting from a randomly chosen node and greedily insert non-visited vertices, chosen as the closest ones from the last inserted vertex. Adapting this heuristic to the k-criteria $TSP(1,2)$ gives rise to two questions : How can we translate the notion of closeness when multiple objectives are considered? How many solutions must be generated to get an approximation of the Pareto curve? In the following, we propose a way which simultaneously brings an answer to both questions. Given the problem, the total distance of a Pareto optimal tour T^* is enclosed in a k-dimensional cost space. The way to generate a tour T which approximates T^*, and also the notion of closeness, depend on where $\boldsymbol{D}(T^*)$ is located in the cost space. The idea is to partition the cost space into a fixed number of parts. Then, with each part we associate an appropriate notion of closeness. Given a part and its proper notion of closeness, we can generate with the nearest neighbor rule a tour which approximates any Pareto optimal solution whose total distance is in the part. For any instance of the k-criteria $TSP(1,2)$, we propose to divide the cost space into $k!$ parts as follows: Each part is identified by a permutation of $\{1,\ldots,k\}$. Given a permutation L of $\{1,\ldots,k\}$, a tour T is in the part identified by L if $\boldsymbol{D}_{L(1)}(T) \leq \ldots \leq \boldsymbol{D}_{L(k)}(T)$. For the notion of closeness, we introduce a preference relation over all possible distance vectors which looks like a lexicographic order. This preference relation which depends on L (denoted by $\prec_L$) is defined by using $k+1$ sets $S_1,\ldots,S_{k+1}$:

$$S_q = \{\boldsymbol{a} \in \{1,2\}^k \mid \forall j \leq k+1-q \quad \boldsymbol{a}_{L(j)} = 1\}, \text{ for } 1 \leq q \leq k$$
$$S_{k+1} = \{1,2\}^k.$$

Definition 1. *For any edge e, we say that e is S_q-*preferred *(for $\prec_L$) if $\boldsymbol{d}(e) \in S_q \backslash S_{q-1}$ (where $S_0 = \emptyset$). For two edges e and e' such that e is S_q-preferred and e' is $S_{q'}$-preferred, we say that $\boldsymbol{d}(e)$ is preferred (resp., weakly preferred) to $\boldsymbol{d}(e')$ and we note $\boldsymbol{d}(e) \prec_L \boldsymbol{d}(e')$ (resp., $\boldsymbol{d}(e) \preccurlyeq_L \boldsymbol{d}(e')$) iff $q < q'$ (resp., $q \leq q'$).*

An example where $k=3$ and L is the identity permutation is given in Figure 1.

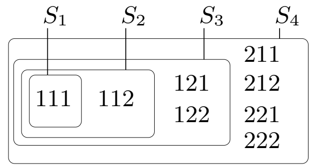

Fig. 1. One has $111 \prec_L 112 \prec_L 121 \preccurlyeq_L 122 \prec_L 211 \preccurlyeq_L 212 \preccurlyeq_L 221 \preccurlyeq_L 222$.

The algorithm that we propose for the k-criteria $TSP(1,2)$ is given in Table 1. Called kNN for k-criteria Nearest Neighbor, it is composed of $k!$ steps. At each step, a permutation L of $\{1, 2, \ldots, k\}$ is determined. With L, we build a preference relation $\prec_L$ and finally, a solution is generated with the nearest neighbor rule.

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kNN: k-criteria Nearest Neighbor
 P := ∅;
 For all permutations L of {1, 2, ..., k} do
      Take arbitrarily v ∈ V ;
      W := {v} ; u := v ;
      While W ≠ V do
           Take r ∈ V\W s.t. r is the closest vertex to u by ≼_L ;
           W := W ∪ {r} ;
           p(u) := r ; u := r ;
      End While ;
      p(r) := v ;
     P := P ∪ {p};
 End do ;
 Return P ;
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Table 1. For $v \in V$ and p a tour, $p(v)$ denotes the node which immediately follows v in p.

Theorem 2. kNN *runs in polynomial time. It returns a* $2k/(k+1)$*-approximate Pareto curve for the* k*-criteria* $TSP(1,2)$ *when* $k \geq 3$ *and a* $3/2$*-approximate Pareto curve when* $k \in \{1, 2\}$.

The proof of the theorem requires some notations and intermediate lemmata. In the following, we consider two particular tours p and p^*. We assume that p is the tour generated by kNN with the preference relation $\prec_L$ and that p^* is a Pareto optimal tour satisfying

$$\boldsymbol{D}_{L(1)}(p^*) \leq \boldsymbol{D}_{L(2)}(p^*) \leq \ldots \leq \boldsymbol{D}_{L(k)}(p^*). \tag{5}$$

The set of all possible distance vectors $\{1,2\}^k$ is denoted by Ω. For all $j \leq k$, we introduce $U_j = \{\boldsymbol{a} \in \Omega \mid \boldsymbol{a}_j = 1\}$ and $\overline{U}_j = \{\boldsymbol{a} \in \Omega \mid \boldsymbol{a}_j = 2\}$. For $\boldsymbol{a} \in \Omega$, we note $X_{\boldsymbol{a}} = \{v \in V \mid \boldsymbol{d}([v, p(v)]) = \boldsymbol{a}\}$ and $X^*_{\boldsymbol{a}} = \{v \in V \mid \boldsymbol{d}([v, p^*(v)]) = \boldsymbol{a}\}$. Finally, $x_{\boldsymbol{a}}$ (resp. $x^*_{\boldsymbol{a}}$) denotes the cardinality of $X_{\boldsymbol{a}}$ (resp. $X^*_{\boldsymbol{a}}$).

If n is the number of vertices then by construction we have $\sum_{\boldsymbol{a}\in\Omega} x_{\boldsymbol{a}} = \sum_{\boldsymbol{a}\in\Omega} x^*_{\boldsymbol{a}} = n$, $\boldsymbol{D}_j(p) = 2n - \sum_{\boldsymbol{a}\in U_j} x_{\boldsymbol{a}}$ and $\boldsymbol{D}_j(p^*) = 2n - \sum_{\boldsymbol{a}\in U_j} x^*_{\boldsymbol{a}}$.

Lemma 2. *The following holds for any $q \leq k$:*

$$2 \sum_{\boldsymbol{a}\in \bigcap_{j=1}^{k+1-q} U_{L(j)}} x_{\boldsymbol{a}} \geq \sum_{\boldsymbol{a}\in \bigcap_{j=1}^{k+1-q} U_{L(j)}} x^*_{\boldsymbol{a}}.$$

Proof. We define $F_q = \{v \in V \mid \boldsymbol{d}([v, p(v)]) \in S_q\}$ and $F^*_q = \{v \in V \mid \boldsymbol{d}([v, p^*(v)]) \in S_q\}$. Then, we have to prove that $2|F_q| \geq |F^*_q|$. The key result is to see that $p^*[F^*_q \backslash F_q] \subseteq F_q$ where $p^*[W] = \bigcup_{v\in W}\{p^*(v)\}$. Take a vertex v in $F^*_q \backslash F_q$ (see Figure 2). Then, $\boldsymbol{d}([v, p^*(v)]) \in S_q$, $\boldsymbol{d}([v, p(v)]) \in S_{q'}$ and $q' > q$. During the computation of p, suppose that v is the current node and that $p^*(v)$ is not already visited. We get a contradiction (the nearest neighbor rule is violated) since $p(v)$ immediately follows v in p and $\boldsymbol{d}([v, p^*(v)]) \prec_L \boldsymbol{d}([v, p(v)])$. Now, suppose $p^*(v)$ was already visited. It directly precedes $p \circ p^*(v)$ in p and then $\boldsymbol{d}([p*(v), p \circ p^*(v)]) \preccurlyeq_L \boldsymbol{d}([v, p^*(v)])$. As a consequence, $\boldsymbol{d}([p^*(v), p \circ p^*(v)]) \in S_{q''}$ such that $q'' \leq q$ and $p^*(v) \in F_q$ since $S_{q''} \subseteq S_q$.

Since $|p^*[F^*_q \backslash F_q]| = |F^*_q \backslash F_q|$, $|F^*_q| = |F^*_q \backslash F_q| + |F^*_q \cap F_q|$ and $|F_q| \geq |F^*_q \cap F_q|$, we deduce $|F^*_q| = |p^*[F^*_q \backslash F_q]| + |F^*_q \cap F_q| \leq 2|F_q|$. Finally, since $\bigcap_{j=1}^{k+1-q} U_{L(j)} = S_q$, $|F_q| = \sum_{\boldsymbol{a}\in S_q} x_{\boldsymbol{a}}$ and $|F^*_q| = \sum_{\boldsymbol{a}\in S_q} x^*_{\boldsymbol{a}}$, the result follows. □

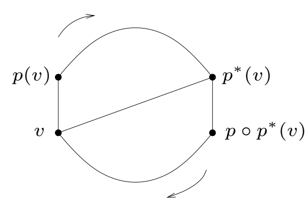

Fig. 2. The tour p generated by KNN. The edge $[v, p^*(v)]$ belongs to p^*.

The following inequality is equivalent to (5):

$$\sum_{\boldsymbol{a}\in U_{L(1)}} x^*_{\boldsymbol{a}} \geq \sum_{\boldsymbol{a}\in U_{L(2)}} x^*_{\boldsymbol{a}} \geq \ldots \geq \sum_{\boldsymbol{a}\in U_{L(k)}} x^*_{\boldsymbol{a}}.$$

We easily deduce that for any couple j_1, j_2 such that $j_1 < j_2$ we have:

$$\sum_{\boldsymbol{a}\in U_{L(j_2)} \backslash U_{L(j_1)}} x^*_{\boldsymbol{a}} \leq \sum_{\boldsymbol{a}\in U_{L(j_1)} \backslash U_{L(j_2)}} x^*_{\boldsymbol{a}}. \tag{6}$$

Let b_1, b_2, j and m be such that $b_1 \in \{1,2\}$, $b_2 \in \{1,2\}$, $1 \leq j \leq k$ and $1 \leq m < j$. Let $R(b_1, j, m, b_2)$ be the set of all $\boldsymbol{a} \in \Omega$ such that $\boldsymbol{a}_{L(j)} = b_1$ and there exists exactly m distinct coordinates of $\boldsymbol{a}$ among $\{\boldsymbol{a}_{L(1)}, \boldsymbol{a}_{L(2)}, \ldots, \boldsymbol{a}_{L(j-1)}\}$ which are equal to b_2. Remark that $R(b_1, j, m, b_2) = R(b_1, j, j-1-m, \overline{b_2})$ where $\overline{b_2} = 3 - b_2$.

Lemma 3. *For any $j \leq k$, one has:*

$$\sum_{q=1}^{j-1} \Big(q \times \sum_{\boldsymbol{a} \in R(1,j,q,2) \cup R(2,j,q,2)} x^*_{\boldsymbol{a}} \Big) \leq (j-1) * \sum_{q=0}^{j-1} \Big(\sum_{\boldsymbol{a} \in R(2,j,q,1)} x^*_{\boldsymbol{a}} \Big).$$

Proof of Theorem 2

The proof is cut into 3 cases ($j = 1$, $j = 2$ and $j \geq 3$). In the following, we consider that L is any permutation of $\{1, \ldots, k\}$, p^* is a Pareto optimal tour satisfying (5) and p is built with the nearest neighbor rule and the preference relation $\prec_L$. Then, we have to show that:

(i) if $j = 1$ or 2 then $\boldsymbol{D}_{L(j)}(p) \leq \frac{3}{2} \boldsymbol{D}_{L(j)}(p^*)$,
(ii) if $j \geq 3$ then $\boldsymbol{D}_{L(j)}(p) \leq \frac{2j}{j+1} \boldsymbol{D}_{L(j)}(p^*)$.

Case $j = 1$. $\boldsymbol{D}_{L(1)}(p) \leq \frac{3}{2} \boldsymbol{D}_{L(1)}(p^*)$ is equivalent to the following inequality:

$$2 \sum_{\boldsymbol{a} \in U_{L(1)}} x_{\boldsymbol{a}} - \sum_{\boldsymbol{a} \in U_{L(1)}} x^*_{\boldsymbol{a}} + 2 \sum_{\boldsymbol{a} \in \overline{U}_{L(1)}} x^*_{\boldsymbol{a}} \geq 0. \tag{7}$$

$$\text{Indeed, } \boldsymbol{D}_{L(1)}(p) \leq \frac{3}{2} \boldsymbol{D}_{L(1)}(p^*) \Leftrightarrow 2\Big(2n - \sum_{\boldsymbol{a} \in U_{L(1)}} x_{\boldsymbol{a}}\Big) \leq 3\Big(2n - \sum_{\boldsymbol{a} \in U_{L(1)}} x^*_{\boldsymbol{a}}\Big)$$
$$\Leftrightarrow -2 \sum_{\boldsymbol{a} \in U_{L(1)}} x_{\boldsymbol{a}} \leq 2n - 3 \sum_{\boldsymbol{a} \in U_{L(1)}} x^*_{\boldsymbol{a}}$$

Using $n = \sum_{\boldsymbol{a} \in U_{L(1)}} x^*_{\boldsymbol{a}} + \sum_{\boldsymbol{a} \in \overline{U}_{L(1)}} x^*_{\boldsymbol{a}}$, the equivalence follows. Thus, using Lemma 2 with $q = k$ and $\sum_{\boldsymbol{a} \in \overline{U}_{L(1)}} x^*_{\boldsymbol{a}} \geq 0$ (which is true since for all $\boldsymbol{a} \in \Omega$, $x^*_{\boldsymbol{a}} \geq 0$), inequality (7) follows.

Case $j = 2$. $\boldsymbol{D}_{L(2)}(p) \leq \frac{3}{2} \boldsymbol{D}_{L(2)}(p^*)$ is equivalent to the following inequality:

$$-2 \sum_{\boldsymbol{a} \in U_{L(2)} \setminus U_{L(1)}} x_{\boldsymbol{a}} - 2 \sum_{\boldsymbol{a} \in U_{L(2)} \cap U_{L(1)}} x_{\boldsymbol{a}} \leq 2 \sum_{\boldsymbol{a} \in \overline{U}_{L(2)}} x^*_{\boldsymbol{a}} - \sum_{\boldsymbol{a} \in U_{L(2)} \setminus U_{L(1)}} x^*_{\boldsymbol{a}} - \sum_{\boldsymbol{a} \in U_{L(2)} \cap U_{L(1)}} x^*_{\boldsymbol{a}}. \tag{8}$$

$$\text{Indeed, } \boldsymbol{D}_{L(2)}(p) \leq \frac{3}{2} \boldsymbol{D}_{L(2)}(p^*) \Leftrightarrow -2 \sum_{\boldsymbol{a} \in U_{L(2)}} x_{\boldsymbol{a}} \leq 2 \sum_{\boldsymbol{a} \in \overline{U}_{L(2)}} x^*_{\boldsymbol{a}} - \sum_{\boldsymbol{a} \in U_{L(2)}} x^*_{\boldsymbol{a}}.$$

If we partition $U_{L(2)}$ into two subsets $U_{L(2)} \backslash U_{L(1)}$ and $U_{L(2)} \cap U_{L(1)}$ then the equivalence follows. By Lemma 2 with $q = k - 1$ we get:

$$2 \sum_{\boldsymbol{a} \in U_{L(1)} \cap U_{L(2)}} x_{\boldsymbol{a}} \geq \sum_{\boldsymbol{a} \in U_{L(1)} \cap U_{L(2)}} x^*_{\boldsymbol{a}}.$$

Then, using inequality (8), we have to prove:

$$-2 \sum_{\boldsymbol{a} \in U_{L(2)} \backslash U_{L(1)}} x_{\boldsymbol{a}} \leq 2 \sum_{\boldsymbol{a} \in \overline{U}_{L(2)}} x^*_{\boldsymbol{a}} - \sum_{\boldsymbol{a} \in U_{L(2)} \backslash U_{L(1)}} x^*_{\boldsymbol{a}}.$$

By inequality (6), when $j_1 = 1$ and $j_2 = 2$, we get:

$$- \sum_{\boldsymbol{a} \in U_{L(1)} \backslash U_{L(2)}} x^*_{\boldsymbol{a}} \leq - \sum_{\boldsymbol{a} \in U_{L(2)} \backslash U_{L(1)}} x^*_{\boldsymbol{a}}$$

Thus:

$$2 \sum_{\boldsymbol{a} \in \overline{U}_{L(2)}} x^*_{\boldsymbol{a}} - \sum_{\boldsymbol{a} \in U_{L(1)} \backslash U_{L(2)}} x^*_{\boldsymbol{a}} \leq 2 \sum_{\boldsymbol{a} \in \overline{U}_{L(2)}} x^*_{\boldsymbol{a}} - \sum_{\boldsymbol{a} \in U_{L(2)} \backslash U_{L(1)}} x^*_{\boldsymbol{a}}.$$

Since $U_{L(1)} \backslash U_{L(2)} \subseteq \overline{U}_{L(2)}$, we have:

$$-2 \sum_{\boldsymbol{a} \in U_{L(2)} \backslash U_{L(1)}} x_{\boldsymbol{a}} \leq 0 \leq 2 \sum_{\boldsymbol{a} \in \overline{U}_{L(2)}} x^*_{\boldsymbol{a}} - \sum_{\boldsymbol{a} \in U_{L(1)} \backslash U_{L(2)}} x^*_{\boldsymbol{a}}.$$

Case $j \geq 3$. $\boldsymbol{D}_{L(j)}(p) \leq \frac{2j}{j+1} \boldsymbol{D}_{L(j)}(p^*)$ holds if we have the following inequality:

$$-(j+1) \sum_{\boldsymbol{a} \in U_{L(j)}} x_{\boldsymbol{a}} \leq 2(j-1) \sum_{\boldsymbol{a} \in \overline{U}_{L(j)}} x^*_{\boldsymbol{a}} - 2 \sum_{\boldsymbol{a} \in U_{L(j)}} x^*_{\boldsymbol{a}}. \tag{9}$$

$$\begin{aligned}
\boldsymbol{D}_{L(j)}(p) \leq \frac{2j}{j+1} \boldsymbol{D}_{L(j)}(p^*) &\Leftrightarrow (j+1)\Big(2n - \sum_{\boldsymbol{a} \in U_{L(j)}} x_{\boldsymbol{a}}\Big) \leq 2j\Big(2n - \sum_{\boldsymbol{a} \in U_{L(j)}} x^*_{\boldsymbol{a}}\Big) \\
&\Leftrightarrow -(j+1) \sum_{\boldsymbol{a} \in U_{L(j)}} x_{\boldsymbol{a}} \leq 2(j-1)n - 2j \sum_{\boldsymbol{a} \in U_{L(j)}} x^*_{\boldsymbol{a}} \\
&\Leftrightarrow -(j+1) \sum_{\boldsymbol{a} \in U_{L(j)}} x_{\boldsymbol{a}} \leq 2(j-1) \sum_{\boldsymbol{a} \in \overline{U}_{L(j)}} x^*_{\boldsymbol{a}} - 2 \sum_{\boldsymbol{a} \in U_{L(j)}} x^*_{\boldsymbol{a}},
\end{aligned}$$

using $n = \sum_{\boldsymbol{a} \in U_{L(j)}} x^*_{\boldsymbol{a}} + \sum_{\boldsymbol{a} \in \overline{U}_{L(j)}} x^*_{\boldsymbol{a}}$.
Let us denote by $\mathcal{A}$ and $\mathcal{B}$ the following quantities:

$$\begin{aligned}
\sum_{\boldsymbol{a} \in U_{L(j)}} x_{\boldsymbol{a}} &= \sum_{\boldsymbol{a} \in U_{L(j)} \backslash (\bigcup_{m \leq j-1} U_{L(m)})} x_{\boldsymbol{a}} + \sum_{\boldsymbol{a} \in \bigcup_{m \leq j} U_{L(m)}} x_{\boldsymbol{a}} = \mathcal{A} \\
\sum_{\boldsymbol{a} \in U_{L(j)}} x^*_{\boldsymbol{a}} &= \sum_{\boldsymbol{a} \in U_{L(j)} \backslash (\bigcup_{m \leq j-1} U_{L(m)})} x^*_{\boldsymbol{a}} + \sum_{\boldsymbol{a} \in \bigcup_{m \leq j} U_{L(m)}} x^*_{\boldsymbol{a}} = \mathcal{B}.
\end{aligned}$$

Then, inequality (9) becomes:

$$-(j+1)\mathcal{A} \leq 2(j-1) \sum_{\boldsymbol{a}\in\overline{U}_{L(j)}} x^*_{\boldsymbol{a}} - 2\mathcal{B}. \tag{10}$$

To prove (10), we propose the following decomposition:

$$\mathcal{C} = 2(j-1) \sum_{\boldsymbol{a}\in\overline{U}_{L(j)}} x^*_{\boldsymbol{a}} - 2 \sum_{\boldsymbol{a}\in U_{L(j)}\setminus \quad {}_{m\leq j-1} U_{L(m)}} x^*_{\boldsymbol{a}} - 4 \sum_{\boldsymbol{a}\in \quad {}_{m\leq j} U_{L(m)}} x_{\boldsymbol{a}} \tag{11}$$

$$-(j+1)\mathcal{A} \leq \mathcal{C} \tag{12}$$

$$\mathcal{C} \leq 2(j-1) \sum_{\boldsymbol{a}\in\overline{U}_{L(j)}} x^*_{\boldsymbol{a}} - 2\mathcal{B} \tag{13}$$

Thus, (12) becomes:

$$\begin{aligned} -(j+1) \sum_{\boldsymbol{a}\in U_{L(j)}\setminus \quad {}_{m\leq j-1} U_{L(m)}} x_{\boldsymbol{a}} - (j-3) \sum_{\boldsymbol{a}\in \quad {}_{m\leq j} U_{L(m)}} x_{\boldsymbol{a}} \leq \\ \leq 2(j-1) \sum_{\boldsymbol{a}\in\overline{U}_{L(j)}} x^*_{\boldsymbol{a}} - 2 \sum_{\boldsymbol{a}\in U_{L(j)}\setminus \quad {}_{m\leq j-1} U_{L(m)}} x^*_{\boldsymbol{a}} \end{aligned}$$

Since the left part of this inequality is negative, we want to prove that the right part is positive:

$$0 \leq 2(j-1) \sum_{\boldsymbol{a}\in\overline{U}_{L(j)}} x^*_{\boldsymbol{a}} - 2 \sum_{\boldsymbol{a}\in U_{L(j)}\setminus \quad {}_{m\leq j-1} U_{L(m)}} x^*_{\boldsymbol{a}} \tag{14}$$

$$\sum_{\boldsymbol{a}\in U_{L(j)}\setminus \quad {}_{m\leq j-1} U_{L(m)}} x^*_{\boldsymbol{a}} \leq (j-1) \sum_{\boldsymbol{a}\in\overline{U}_{L(j)}} x^*_{\boldsymbol{a}} \tag{15}$$

We also have:

$$\sum_{\boldsymbol{a}\in U_{L(j)}\setminus \quad {}_{m\leq j-1} U_{L(m)}} x^*_{\boldsymbol{a}} = \sum_{q=1}^{j-1} \Big(\sum_{\boldsymbol{a}\in R(1,j,q,2)} x^*_{\boldsymbol{a}} \Big) \text{ and}$$

$$(j-1) \sum_{\boldsymbol{a}\in\overline{U}_{L(j)}} x^*_{\boldsymbol{a}} = (j-1) \sum_{q=0}^{j-1} \Big(\sum_{\boldsymbol{a}\in R(2,j,q,1)} x^*_{\boldsymbol{a}} \Big).$$

The first equality follows from $U_{L(j)}\setminus\bigcap_{m\leq j-1} U_{L(m)} = \bigcup_{q=1}^{j-1} R(1,j,q,2)$ since $\boldsymbol{a} \in U_{L(j)}\setminus\bigcap_{m\leq j-1} U_{L(m)}$ iff $\boldsymbol{a}_{L(j)} = 1$ and there exists exactly q indexes $\{i_1, \dots i_q\}$ such that $1 \leq q \leq j-1$ and $\boldsymbol{a}_{L(i_1)} = \boldsymbol{a}_{L(i_2)} = \dots = \boldsymbol{a}_{L(i_q)} = 2$, which is equivalent to $\boldsymbol{a} \in R(1,j,q,2)$. The second equality follows from $\overline{U}_{L(j)} = \bigcup_{q=0}^{j-1} R(2,j,q,1)$ because $\boldsymbol{a} \in \overline{U}_{L(j)}$ means $\boldsymbol{a}_{L(j)} = 2$.

As a consequence, (15) becomes:

$$\sum_{q=1}^{j-1}\Big(\sum_{\boldsymbol{a}\in R(1,j,q,2)} x^*_{\boldsymbol{a}}\Big) \le (j-1)\sum_{q=0}^{j-1}\Big(\sum_{\boldsymbol{a}\in R(2,j,q,1)} x^*_{\boldsymbol{a}}\Big).$$

With Lemma 3, we have:

$$\sum_{q=1}^{j-1}\Big(q \times \sum_{\boldsymbol{a}\in R(1,j,q,2)\cup R(2,j,q,2)} x^*_{\boldsymbol{a}}\Big) \le (j-1) * \sum_{q=0}^{j-1}\Big(\sum_{\boldsymbol{a}\in R(2,j,q,1)} x^*_{\boldsymbol{a}}\Big)$$

and (15) follows from

$$\sum_{q=1}^{j-1}\Big(q \times \sum_{\boldsymbol{a}\in R(1,j,q,2)\cup R(2,j,q,2)} x^*_{\boldsymbol{a}}\Big) \ge \sum_{q=1}^{j-1}\Big(\sum_{\boldsymbol{a}\in R(1,j,q,2)} x^*_{\boldsymbol{a}}\Big).$$

By Lemma 2 with $q = k + 1 - j$ we have:

$$2 \sum_{\boldsymbol{a}\in \bigcup_{m\le j} U_{L(m)}} x_{\boldsymbol{a}} \ge \sum_{\boldsymbol{a}\in \bigcup_{m\le j} U_{L(m)}} x^*_{\boldsymbol{a}}$$

which is exactly (13). □

There is an instance which shows that the analysis is tight but, because of the restricted number of pages, it is presented in Appendix.

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