

Approximation results for the weighted P_4 partition problems

Jérôme Monnot¹, Sophie Toulouse²

¹ Université Paris Dauphine, LAMSADE, CNRS UMR 7024, 75016 Paris, France
monnot@lamsade.dauphine.fr

² Université Paris 13, Institut Galilée LIPN, CNRS UMR 7030, 93430 Villetaneuse, France, sophie.toulouse@lipn.univ-paris13.fr

Abstract. We present several new standard and differential approximation results for P_4 -partition problem by using the algorithm proposed in Hassin and Rubinfeld (Information Processing Letters, 63: 63-67, 1997), for both minimization and maximization versions of the problem. However, the main point of this paper is the robustness of this algorithm, since it provides good solutions, whatever version of the problem we deal with, whatever the approximation framework within which we estimate its approximate solutions.

Keywords: graph partition, 3-length chain, approximation algorithms, performance ratio, standard approximation, differential approximation.

1 Introduction

Consider an instance I of an NP-hard optimization problem Π and a polynomial time algorithm A computing feasible solutions for Π . Denote by $\text{apx}_\Pi(I)$ the value of a solution computed by A on I , by $\text{opt}_\Pi(I)$, the value of an optimal solution and by $\text{wor}_\Pi(I)$ the value of a worst solution (that corresponds to the optimal value when reversing the optimization goal). The quality of A is expressed by the way of approximation ratios that somehow compare the approximate value to the optimal one. So far, two measures stand out from the literature: the *standard* ratio [2] (the most widely used) and the *differential* ratio [3–5, 8]. The standard ratio is defined by $\rho_\Pi(I, A) = \frac{\text{apx}_\Pi(I)}{\text{opt}_\Pi(I)}$ if Π is a maximization problem, by $\rho_\Pi(I, A) = \frac{\text{opt}_\Pi(I)}{\text{apx}_\Pi(I)}$ otherwise, whereas the differential ratio is defined by $\delta_\Pi(I, A) = \frac{\text{wor}_\Pi(I) - \text{apx}_\Pi(I)}{\text{wor}_\Pi(I) - \text{opt}_\Pi(I)}$. Instead of dividing the approximate value by the optimum value, this latter measure divides the

distance from a worst solution to the approximate value by the instance diameter. Within the worst case analysis framework and given a universal constant $\varepsilon \leq 1$ (*resp.*, $\varepsilon \geq 1$), an algorithm **A** is said to be an ε -standard approximation for a maximization (*resp.* a minimization) problem Π if $\rho_{\mathbf{A}\Pi}(I) \geq \varepsilon \forall I$ (*resp.*, $\rho_{\mathbf{A}\Pi}(I) \leq \varepsilon \forall I$). According to differential ratio, **A** is said to be an ε -differential approximation for Π if $\delta_{\mathbf{A}\Pi}(I) \geq \varepsilon, \forall I$, for a universal constant $\varepsilon \leq 1$. Alternatively, any solution being a convex combination of the two values $\text{wor}_{\Pi}(I)$ and $\text{opt}_{\Pi}(I)$, an approximate value $\text{apx}_{\Pi}(I)$ will be an ε -differential approximation if for any instance I , $\text{apx}_{\Pi}(I) \geq \varepsilon \times \text{opt}_{\Pi}(I) + (1 - \varepsilon) \times \text{wor}_{\Pi}(I)$ (for the maximization case). Within the worst case analysis framework according to both standard and differential ratios, we focus on a special problem, the weighted P_4 -partition problem (P_4P in short). Furthermore, we study the performance of a single algorithm on various versions of this problem. Precisely, we show that this algorithm is efficient for P_4P , proving approximation ratios for both standard and differential measures, for both maximization and minimization versions of the problem.

In the weighted P_4 -partition problem, we are given a complete graph K_{4n} together with a distance function $d : E \rightarrow \mathbb{R}^+$ on its edges. A P_4 is an induced path of length 3 (or, equivalently, an induced path on 4 vertices); for any instance $I = (K_{4n}, d)$, the cost of a path P is given by the sum of its edges weight, and the goal is to find a partition $T^* = \{P_1^*, \dots, P_n^*\}$ of n vertex-disjoint chains of length 3 (that we call a P_4 -partition) such that $d(T^*) = \sum_{i=1}^n d(P_i^*)$ is optimum, *i.e.*, of maximum distance if the goal is to maximize ($\text{MAX}P_4P$), of minimum distance otherwise ($\text{MIN}P_4P$). For the minimization version, we will more often assume that the distance function satisfies the triangular inequality, *i.e.*, $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z$; $\text{MINMETRIC}P_4P$ will refer to this restriction. Finally, we also deal with a special case of metric instances where the distance function is worth either 1 or 2; the corresponding problems will be denoted by $\text{MAX}P_4P_{1,2}$ and $\text{MIN}P_4P_{1,2}$. All these problems are known to be **NP**-hard, [7, 14]; nevertheless, $\text{MAX}P_4P$ is standard-approximable within ratio $3/4$, Hassin and Rubinfeld, [9], whereas (to our knowledge), no approximation rate has been established for $\text{MIN}P_4P$ yet. Concerning $\text{MIN}P_kP$ (the general version), it cannot be approximated within $2^{p(n)}$ for any polynomial p and for any k ; this is due

to the fact that deciding whether a graph admits a P_k -partition of its vertex set is **NP**-complete, [7, 13, 14]. The P_k -partition problem consists in, given a simple graph $G = (V, E)$, deciding if there exists a collection \mathcal{P} of vertex-disjoint k -length paths $\{P_1, \dots, P_\ell\}$ such that any vertex from V belongs to exactly one path P_i from \mathcal{P} . Finally, note that $\text{MINMETRIC}P_kP$ is very close to the k -vehicle routing problem when restricting the route of each vehicle to at most k intermediate stops, [1, 6].

In the first section, we study the relationship between TSP and P_kP under differential ratio, showing how a differential approximation for TSP allows a differential approximation for P_kP (where P_kP seeks to determine a partition of a vertex set into k -length paths of optimum weight). In the second section, that contains the main result of this paper, we propose a complete analysis, from both a standard and a differential point of view, of an algorithm proposed by Hassin and Rubinstein [9]. We prove that, with respect to the standard ratio, this algorithm provides new approximation rates for $\text{METRIC}P_4P$, that is to say: the approximate solution respectively achieves a $3/2$ -, a $7/6$ - and a $9/10$ -standard approximation for $\text{MINMETRIC}P_4P$, $\text{MIN}P_4P_{1,2}$ and $\text{MAX}P_4P_{1,2}$. Under differential ratio, the approximate solution is a $1/2$ -approximation for general P_4P , a $2/3$ -approximation for $P_4P_{a,b}$. The gap between differential and standard ratios for maximization version of a problem come from the fact that within differential framework, the approximate value has to be located into $[\text{wor}(I), \text{opt}(I)]$, instead of $[0, \text{opt}(I)]$ within the standard one. That is the point of differential measure: the reference it does to $\text{wor}(I)$ makes it more precise and makes robust the approximation level (since minimizing and maximizing are equivalent and more generally, differential ratio is stable under affine transformation of the objective function). In addition to the new approximation bounds that we provide, the main result of this paper is the robustness of the algorithm we study, since this latter provides good solutions, whatever version of the problem we deal with, whatever the approximation framework within which we estimate its approximate solutions.

2 From Traveling salesman problem to P_kP

A common technic to obtain an approximate solution for $MAXP_kP$ from a Hamiltonian cycle is called the *deleting and turning around* method, see [9, 10, 6]. This method consists, starting from a tour, in producing k solutions of $MAXP_kP$ by turning around the cycle and deleting 1 edge upon k ; then it picks the best solution. Obviously, the quality of the output T' depends on the quality of the initial tour. Hence, it is proved in [9, 10], that any ε -standard approximation of $MAXTSP$ provides a $\frac{k-1}{k}\varepsilon$ -standard approximation for $MAXP_kP$. Under a differential point of view, things are less optimistic, since already for $k = 4$ there exists an instance family $(I_n)_{n \geq 1}$ verifying $\text{apx}(I_n) = \frac{1}{2}\text{opt}_{MAXP_4P}(I_n) + \frac{1}{2}\text{wor}_{MAXP_4P}(I_n)$. For $n \geq 1$, the instance is $I_n = (K_{8n}, d)$ where the vertex set can be partitioned into two sets $L = \{\ell_1, \dots, \ell_{4n}\}$ and $R = \{r_1, \dots, r_{4n}\}$; the associated distance function d equals 0 on L^2 , 2 on R^2 , and 1 on $L \times R$. We have for any $n \geq 1$:

Property 1. $\text{apx}(I_n) = 6n$, $\text{opt}_{MAXP_4P}(I_n) = 8n$ and $\text{wor}_{MAXP_4P}(I_n) = 4n$.

Nevertheless, using this method one can obtain:

Lemma 1. *From an ε -differential approximation of $MAXTSP$, one can polynomially compute a $\frac{\varepsilon}{k}$ -differential approximation of $MAXP_kP$*

Thus, from [8, 11], we deduce a $\frac{2}{3k}$ -differential approximation for $MAXP_kP$.

Finally, observe that even if we consider $MINMETRICP_4P$, we are not able to obtain a result as good as standard approximation for MAX_4P . Consider the instances $I'_n = (K_{8n}, d')$ built from I_n where we modify the distance d by $d'(\ell_i, \ell_j) = d'(r_i, r_j) = 1$ and $d'(\ell_i, r_j) = n^2 + 1$ for all i, j . We thus have $\text{opt}_{MINMETRICTSP}(I'_n) = 2n^2 + 8n$ and $\text{opt}_{MINMETRICP_4P}(I'_n) = 6n$.

3 Approximating P_4P by the way of optimal matchings

Here starts the analysis, both on a standard and a differential point of view, of an algorithm proposed by Hassin and Rubinstein [9];

the authors show that the approximate solution is a $3/4$ -standard approximation for MAXP_4P . First, dealing with the standard ratio, we prove that this algorithm provides a $3/2$ -approximation for $\text{MINMETRICP}_4\text{P}$ and respectively a $7/6$ and a $9/10$ -approximation for $\text{MINP}_4\text{P}_{1,2}$ and $\text{MAXP}_4\text{P}_{1,2}$ when restricting us to $1, 2$ -valuated graphs. As corollary of a general result, we also obtain an alternate proof of the result of [9].

We then prove that, with respect to the differential measure, the approximate solution achieves a $1/2$ -approximation in general graphs, for both maximization and minimization versions of the problem; this latter ratio is raised up to $2/3$ in bi-valuated graphs.

3.1 Description of the algorithm

The algorithm proposed in [9] runs in two stages: first, it computes an optimum weight perfect matching $M_{T'}$ on (K_{4n}, d) ; then, it builds a second optimum weight perfect matching $R_{T'}$ on edges $M_{T'}$ in order to complete the solution (note that “optimum weight” signifies “*maximum weight*” if the goal of P_4P is to maximize, “*minimum weight*” if the goal is to minimize). Precisely, we define the instance (K_{2n}, d') (to any edge $e_v \in M_{T'}$ corresponds a vertex v in K_{2n}), where the distance function d' is defined as follows: for any edge $[v_1, v_2]$, $d'(v_1, v_2)$ is set to the weight of the heaviest edge linking e_{v_1} and e_{v_2} , that is, if v_1 represents $e_{v_1} = [x_1, y_1]$ and v_2 represents $e_{v_2} = [x_2, y_2]$, then $d'(v_1, v_2)$ is set to $\max \{d(x_1, x_2), d(x_1, y_2), d(y_1, x_2), d(y_1, y_2)\}$ (when dealing with the minimum version of the problem, set the weight to the lightest). On (K_{2n}, d') , we build an optimum weight matching $R_{T'}$ that is transposed to (K_{4n}, d) by selecting the edge that realizes the same cost. Since the computation of an optimum weight perfect matching is polynomial, the whole algorithm runs in polynomial time for both versions.

3.2 General P_4P within the standard framework

For any solution T , we denote respectively by M_T and R_T the set of the final edges and the set of middle edges of its chains. Moreover, for any edge $e \in T$, we denote by $P_T(e)$ the P_4 from T that contains e , by $C_T(e)$ the cycle of length 4 containing $P_T(e)$ ($C_T(e)$ is somehow

the closure of $P_T(e)$). Finally, we define $\overline{R_T} = \cup_{e \in T} (C_T(e) \setminus P_T(e))$; note that $R_T \cup \overline{R_T}$ is a perfect matching.

Lemma 2. *For any instance $I = (K_{4n}, d)$, if T is a feasible solution and T^* an optimal solution, then there exist 3 pairwise disjoint edge sets A , B and C verifying:*

- (i) $A \cup B = T^*$ and $C \subseteq \overline{R_{T^*}}$.
- (ii) $A \cup C$ and $B \cup (\overline{R_{T^*}} \setminus C)$ are both perfect matchings on I .
- (iii) $A \cup C \cup M_T$ is a 2-matching on I of which cycles are of length a multiple of 4.

Proof. Let $T^* = M_{T^*} \cup R_{T^*}$ be an optimal solution; we apply the following process:

-
- 1 Set $A = M_{T^*}$, $B = R_{T^*}$, $C = \emptyset$ and $G' = (V, A \cup M_T)$;
 - 2 While there exists a connected component V' of G' having an edge $e = [x, y] \in R_{T^*}$ with $x \in V'$ and $y \notin V'$, do:
 - 2.1 $A \leftarrow (A \setminus P_{T^*}(e)) \cup \{e\}$, $B \leftarrow (B \cup P_{T^*}(e)) \setminus \{e\}$, $C \leftarrow C \cup (C_{T^*}(e) \setminus P_{T^*}(e))$;
 - 2.2 $G' \leftarrow (V, A \cup M_T)$;
 - 3 output A , B and C ;
-

At the initialization stage, the connected components of the partial graph induced by $(A \cup C \cup M_T)$ are either cycles alternating edges from $(A \cup C)$ and M_T , or isolated edges from $M_{T^*} \cap M_T$. During step 2, at each iteration, the process merges together two cycles, or a cycle and an isolated edge, or two isolated edges, into a single cycle; an illustration of the process is proposed in Figure 1. Note that all along the process, the sets A , B and C remain pairwise disjoint.

For (i): Immediate from definition of the process.

For (ii): Sets C and $\overline{R_{T^*}} \setminus C$ are both matchings by construction; let us now prove that A and B also are matchings. Two edges $e \neq e'$ from A (*resp.*, B) are adjacent one to each other if and only if they belong to the same chain P_{T^*} . However, according to steps 1 and 2.1,

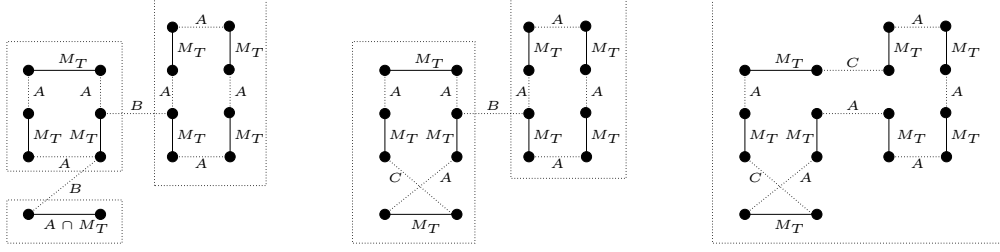


Fig. 1. The construction of sets A and C .

for any chain P_{T^*} , either $P_{T^*} \cap R_{T^*}$ or $P_{T^*} \cap M_{T^*}$ may belong to A (resp., B), that contradicts the fact that e and e' could be adjacent.

Moreover, before processing step 2, $A \cup C = M_{T^*}$ is a perfect matching; now, at each iteration of step 2, we swap two couples of edges $C_{T^*} \cap M_{T^*}$ and $C_{T^*} \setminus M_{T^*}$ (the former edges are removed from A , the latter are added to $A \cup C$); thus, the resulting edge set $A \cup C$ remains a perfect matching. Finally, since the equality $B \cup (\overline{R_{T^*}} \setminus C) = (T^* \cup \overline{R_{T^*}}) \setminus (A \cup C)$ holds, the set $B \cup (\overline{R_{T^*}} \setminus C)$ also is a perfect matching.

For (iii): At the end of process, $(A \cup C) \cap M_T = \emptyset$ and thus, $A \cup C \cup M_T$ is a perfect 2-matching. Now consider a cycle Γ of $(V, A \cup C \cup M_T)$; by definition of step 2, for any edge $e = [x, y] \in \Gamma \cap R_{T^*}$, its begin and end points both belong to $V(\Gamma)$, which means that $\Gamma \setminus M_T$ is a subset of T^* and therefore, we get $|V(\Gamma)| = 4k$.

Theorem 1. *The solution T' provided by the algorithm achieves a $\frac{3}{2}$ -standard approximation for $\text{MINMETRICP}_4\text{P}$ and this ratio is tight.*

Proof. Let T^* be an optimal solution on $I = (K_{4n}, d)$, we consider 3 pairwise disjoint sets A , B and C according to the application of Lemma 2 to T' ; according to property (iii), we can split $A \cup C$ into two sets A_1 and A_2 in such a way that $A_i \cup M_{T'}$ ($i = 1, 2$) is a P_4 -partition (see Figure 2 for an illustration). Thus, we get from the optimality of $R_{T'}$ the inequality $d(A_i) \geq d(R_{T'})$ and deduce:

$$2d(R_{T'}) \leq d(A) + d(C) \quad (1)$$

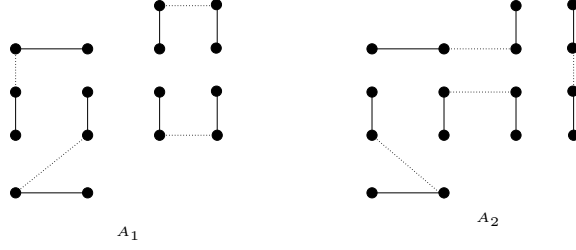


Fig. 2. The sets $A_1 \cup M_{T'}$ and $A_2 \cup M_{T'}$.

From property (ii) of Lemma 2 which states that $B \cup (\overline{R_{T^*}} \setminus C)$ is a perfect matching, and because $M_{T'}$ is optimal, we get:

$$d(M_{T'}) \leq d(B) + d(\overline{R_{T^*}} \setminus C) \quad (2)$$

Adding inequalities (1) and (2), and since I satisfies the triangular inequality, we obtain:

$$d(M_{T'}) + 2d(R_{T'}) \leq 2\text{opt}_{\text{MINMETRICP}_4\text{P}}(I) \quad (3)$$

(Note that this latter inequality is only true when minimizing.) Considering $d(M_{T'}) \leq \text{opt}_{\text{MINMETRICP}_4\text{P}}(I)$, the proof is complete. The tightness is given by the instances $I_n = (K_{8n}, d)$ described in Property 1.

Based upon Lemma 2, one can also obtain an alternate proof of the result given in [9].

Theorem 2. *The solution T' provided by the algorithm achieves a $\frac{3}{4}$ -standard approximation for MAXP_4P .*

Proof. The inequality (3) becomes

$$d(M_{T'}) + 2d(R_{T'}) \geq \text{opt}_{\text{MAXP}_4\text{P}}(I) + d(\overline{R_{T^*}}) \quad (4)$$

Considering this time that $2 \times d(M_{T'}) \geq \text{opt}_{\text{MAXP}_4\text{P}}(I) + d(\overline{R_{T^*}})$, we deduce $\text{apx}_{\text{MAXP}_4\text{P}}(I) \geq \frac{3}{4} (\text{opt}_{\text{MAXP}_4\text{P}}(I) + d(\overline{R_{T^*}}))$.

3.3 General P_4P within the differential framework

Dealing with differential ratio, MAXP_4P MINP_4P and MINMETRICP_4P are equivalent to approximate ; this is generally true for any couple of problems that only differ by an affine transformation of their objective function.

Theorem 3. *The solution T' provided by the algorithm achieves a $\frac{1}{2}$ -differential approximation for P_4P and this ratio is tight.*

Proof. We prove the result for the maximization version. First, observe that $\overline{R_{T^*}}$ is a n -cardinality matching; hence, considering any perfect matching M of I such that $M \cup \overline{R_{T^*}}$ do form a P_4 -partition, we get:

$$d(M) + d(\overline{R_{T^*}}) \geq \text{WOR}_{\text{MAXP}_4P}(I) \quad (5)$$

Adding inequalities (4) and (5), we obtain:

$$2\text{apx}_{\text{MAXP}_4P}(I) \geq d(M_{T'}) + 2d(R_{T'}) + d(M) \geq \text{WOR}_{\text{MAXP}_4P}(I) + \text{opt}_{\text{MAXP}_4P}(I)$$

To show the tightness of this ratio, we refer to Property 1.

3.4 Bi-valuated metric P_4P with weights 1 & 2 within the standard framework

As done in [12] for MINTSP , we now focus on instances where any edge is worth either 1 or 2; indeed, such an analysis enables a keener comprehension of a given algorithm. Moreover, since the P_4 -partition problem is NP -complete, the problems $\text{MAXP}_4P_{1,2}$ and $\text{MINP}_4P_{1,2}$ are NP -hard.

Theorem 4. *The solution T' provided by the algorithm achieves a $\frac{9}{10}$ -standard approximation for $\text{MAXP}_4P_{1,2}$ and a $\frac{7}{6}$ -standard approximation for $\text{MINP}_4P_{1,2}$. These ratios are tight.*

Proof. We only prove the maximization case. Let $I = (K_{4n}, d)$ be an instance of $\text{MAXP}_4P_{1,2}$ with $d(e) \in \{1, 2\}$. We will denote by $M_{T',2}$ (resp., $R_{T',2}$) the set of $M_{T'}$ edges of weight 2, by p (resp., q) its cardinality. Trivially, cardinalities p and q verify: $p \leq 2n$, $d(M_{T'}) = 2n + p$ and $q \leq n$, $d(R_{T'}) = n + q$. Similarly, let us denote

by $M_{T^*,2} = \{e \in M_{T^*} \mid d(e) = 2\}$ with size p^* and by $R_{T^*,2} = \{e \in R_{T^*} \mid d(e) = 2\}$ with size q^* . We have $\text{apx}_{\text{MAXP}_4\text{P}_{1,2}}(I) = 3n + p + q$ and $\text{opt}_{\text{MAXP}_4\text{P}_{1,2}}(I) = 3n + p^* + q^*$. Wlog., we may assume that the following property always holds for T^* :

Property 2. For any 3-length chain $P \in T^*$, $|P \cap M_{T^*,2}| \geq |P \cap R_{T^*,2}|$.

Otherwise, T^* would contain a chain $P = \{[x, y], [y, z], [z, t]\}$ verifying $d(x, y) = d(z, t) = 1$ and $d(y, z) = 2$; thus, by swapping P and $P' = \{[y, z], [z, t], [t, x]\}$ into T^* , one does generate another optimal solution.

Property 3. $q \geq \frac{q^*}{2}$.

Let G' be the multi-graph induced by $M_{T'} \cup R_{T^*,2}$ (edges from $M_{T'} \cap R_{T^*,2}$ appear twice). This graph is constituted of chains, cycles and isolated edges: its chains alternate edges from $M_{T'}$ and $R_{T^*,2}$, with the particularity that their extremal edges all belong to $M_{T'}$; its cycles also alternate edges from $M_{T'}$ and $R_{T^*,2}$ and the 2-length cycles correspond to the edge set $R_{T^*,2} \cap M_{T'}$; finally, its isolated edges belong to $M_{T'} \setminus R_{T^*,2}$. For any cycle Γ on G' , there exists an edge $e(\Gamma) = [x, y]$ with $e(\Gamma) \in M_{T^*,2}$, $x \in V(\Gamma)$, and y is an endpoint of a chain; we denote by $e'(\Gamma)$ the edge from $\Gamma \cap R_{T^*,2}$ to which $e(\Gamma)$ is incident. The existence of $e(\Gamma)$ follows from the Property 2; moreover, vertex y cannot be incident to any edge from $R_{T^*,2}$, since otherwise $e(\Gamma) \in M_{T^*}$ would link two edges of T^* , contradiction! Thus, setting $A = R_{T^*,2}$, and replacing in A for any cycle Γ on G' the edge $e'(\Gamma)$ by the edge $e(\Gamma)$, we build an edge set A satisfying $|A| = |R_{T^*,2}| = q^*$ and $(V, M_{T'} \cup A)$ is a simple graph made of pairwise disjoint chains. Like we did while proving Theorem 1, we split A into two sets A_1 and A_2 in such a way that the partial graph induced by $M_{T'} \cup A_i$ for $i = 1, 2$ is a set of at-most-3-length chains. We arbitrarily complete A_i by the way of an edge set B_i in order to obtain a P_4 -partition $A_i \cup B_i \cup M_{T'}$. Obviously, $|A_i| + |B_i| = n$ and $d(B_1) + d(B_2) \geq |B_1| + |B_2| = (n - |A_1|) + (n - |A_2|) = 2n - q^*$. Moreover, $d(A_i \cup B_i) \leq d(R_{T'})$ due to $R_{T'}$ optimality. Since $d(A) = 2q^*$, we deduce $2n + q^* \leq d(B_1) + d(B_2) + d(A) \leq 2d(R_{T'}) = 2n + 2q$ and Property 3 is established. Thanks to this latter, we obtain:

$$\text{apx}_{\text{MAXP}_4\text{P}_{1,2}}(I) \geq 3n + p + \frac{q^*}{2} \quad (6)$$

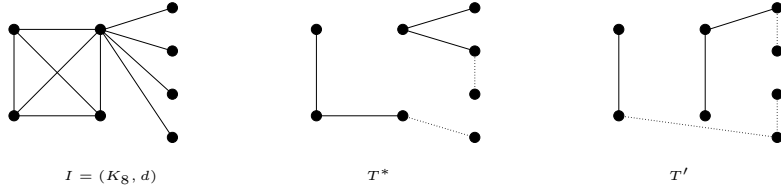


Fig. 3. Instance $I = (K_8, d)$ establishing the tightness for $\text{MAXP}_4\text{P}_{1,2}$.

On the other hand, since $M_{T',2}$ is a matching containing a maximum number of 2-edges, we have in particular $p \geq p^*$ and deduce:

$$\text{opt}_{\text{MAXP}_4\text{P}_{1,2}}(I) \leq 3n + p + q^* \quad (7)$$

On the behalf of inequalities $p \geq q^*$, $n \geq q^*$, (6) and (7), we obtain the expected result $\text{apx}_{\text{MAXP}_4\text{P}_{1,2}}(I) \geq \frac{9}{10} \text{opt}_{\text{MAXP}_4\text{P}_{1,2}}(I)$.

The tightness comes from the instance $I = (K_8, d)$ depicted in Figure 3, where edges of distance 2 are drawn in continuous line and edges of distance 1 of T^* and T' are drawn in dotted line; other edges are not drawn. One can easily see $\text{opt}_{\text{MAXP}_4\text{P}_{1,2}}(I) = 10$ and $\text{apx}_{\text{MAXP}_4\text{P}_{1,2}}(I) = 9$.

3.5 Bi-valuated metric P_4P with weights a & b within the differential framework

As we mentioned earlier, the differential measure is stable under affine transformation; now, any instance from $\text{MAXP}_4\text{P}_{a,b}$ may be mapped into an instance of $\text{MAXP}_4\text{P}_{1,2}$ or $\text{MINP}_4\text{P}_{a,b}$ by the way of such a transformation. Thus, proving $\text{MAXP}_4\text{P}_{1,2}$ is ε -differential approximable actually establishes that $\text{MINP}_4\text{P}_{a,b}$ and $\text{MAXP}_4\text{P}_{a,b}$ are ε -differential approximable for any couple of real values $a < b$.

Theorem 5. *The solution T' provided by the algorithm achieves a $\frac{2}{3}$ -differential approximation for $\text{P}_4\text{P}_{a,b}$ and this ratio is tight.*

References

1. Arkin, E. M., Hassin, R. and Levin, A.: Approximations for minimum and min-max vehicle routing problems. J. of Algorithms, (article in press)

2. Ausiello, G., Crescenzi, P., Gambosi, G., Kann, V., Marchetti-Spaccamela, A., Protasi, M.: Complexity and Approximation (Combinatorial Optimization Problems and Their Approximability Properties). *Springer, Berlin* (1999)
3. Ausiello, G., D'Atri, A., Protasi, M.: Structure preserving reductions among convex optimization problems. *J. Comput. System Sci.* **21** (1980) 136–153
4. Bellare, M., Rogaway, P.: The complexity of approximating a nonlinear program. *Mathematical Programming* **69** (1995) 429–441
5. Demange, M., Paschos, V. Th.: On an approximation measure founded on the links between optimization and polynomial approximation theory. *Theoretical Computer Science* **158** (1996) 117–141
6. Frederickson, G. N., Hecht, M. S., Kim, C. E.: Approximation algorithms for some routing problems. *SIAM J. on Computing* **7** (1978) 178–193
7. Garey, M. R., Johnson, D. S.: Computers and intractability. a guide to the theory of NP-completeness. *CA, Freeman* (1979)
8. Hassin, R., Khuller, S.: z -approximations. *Journal of Algorithms* **41** (2001) 429–442
9. Hassin, R., Rubinfeld, S.: An Approximation Algorithm for Maximum Packing of 3-Edge Paths. *Information Processing Letters* **63** (1997) 63–67
10. Hassin, R., Rubinfeld, S.: An Approximation Algorithm for Maximum Triangle Packing. *ESA, LNCS* **3221** (2004) 403–413
11. Monnot, J.: Differential approximation results for the traveling salesman and related problems. *Information Processing Letters* **82** (2002) 229–235
12. Papadimitriou, C., Yannakakis, M.: The traveling salesman problem with distances one and two. *Mathematics of Operations Research* **18** (1993) 1–11
13. Sahni, S., Gonzalez, T.: P-complete approximation problems. *Journal of the Association for Computing Machinery* **23** (1976) 555–565
14. Steiner, G.: On the k -path partition of graphs. *Theoretical Computer Science* **290** (2003) 2147–2155