Differential approximation results for the traveling salesman problem with distances 1 and 2
(Extended Abstract)

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Abstract. We prove that both minimum and maximum traveling salesman problems on complete graphs with edge-distances 1 and 2 are approximable within 3/4. Based upon this result, we improve the standard approximation ratio known for maximum traveling salesman with distances 1 and 2 from 3/4 to 7/8. Finally, we prove that, for any $\epsilon > 0$, it is **NP**-hard to approximate both problems within better than $5379/5380 + \epsilon$.

1 Introduction

Given a complete graph on n vertices, denoted by K_n, with edge distances either 1 or 2 the minimum traveling salesman problem (min_TSP12) consists in minimizing the cost of a Hamiltonian cycle, the cost of such a cycle being the sum of the distances on its edges (in other words, in finding a Hamiltonian cycle containing a maximum number of 1-edges). The maximum traveling salesman problem (max_TSP) consists in maximizing the cost of a Hamiltonian cycle (in other words, in finding a Hamiltonian cycle containing a maximum number of 2-edges). A generalization of TSP12, denoted by TSPab, is the one where the edge-distances are either a, or b, $a < b$. Both min_ and max_TSP12, and TSPab are **NP**-hard.

Given an instance I of an **NP** optimization (**NPO**) problem Π and a polynomial time approximation algorithm $\mathtt{A}$ feasibly solving Π, we will denote by $\omega(I)$, $\lambda_{\mathtt{A}}(I)$ and $\beta(I)$ the values of the worst solution of I, of the approximated one (provided by $\mathtt{A}$ when running on I), and the optimal one for I, respectively. Commonly ([9]), the quality of an approximation algorithm for an **NP**-hard minimization (resp., maximization) problem Π is expressed by the ratio (called standard in what follows) $\rho_{\mathtt{A}}(I) = \lambda(I)/\beta(I)$, and the quantity $\rho_{\mathtt{A}} = \inf\{r : \rho_{\mathtt{A}}(I) < r, I \text{ instance of } \Pi\}$ (resp., $\rho_{\mathtt{A}} = \sup\{r : \rho_{\mathtt{A}}(I) > r, I \text{ instance of } \Pi\}$) constitutes the approximation ratio of $\mathtt{A}$ for Π. Another approximation-quality criterion used by many well-known researchers ([2, 1, 3, 4, 14, 15]) is what in [7, 6] we call *differential-approximation ratio*. It measures how the value of an approximate solution is placed in the interval between $\omega(I)$ and $\beta(I)$. More formally, the differential-approximation ratio of an algorithm $\mathtt{A}$ is defined as $\delta_{\mathtt{A}}(I) =$

$|\omega(I)-\lambda(I)|/|\omega(I)-\beta(I)|$. The quantity $\delta_{\mathtt{A}} = \sup\{r : \delta_{\mathtt{A}}(I) > r, I$ instance of $\Pi\}$ is the differential approximation ratio of A for Π. Another type of ratio, very close to the differential one, has been defined and used in [5]. There, instead of $\omega(I)$, the authors used a value z_R, called *reference-value*, smaller than $\omega(I)$. The ratio introduced in [5] is defined as $d_{\mathtt{A}}(I) = |\beta(I) - \lambda_{\mathtt{A}}(I)|/|\beta(I) - z_R|$. The quantity $|\beta(I) - \lambda_{\mathtt{A}}(I)|$ is called deviation of A, while $|\beta(I) - z_R|$ is called absolute deviation. For reasons of economy, we will call $d_{\mathtt{A}}(I)$ *deviation ratio*. For a given problem, setting $z_R = \omega(I)$, $d_{\mathtt{A}}(I) = 1 - \delta_{\mathtt{A}}(I)$ and both ratios have, as it has already mentioned above, a natural interpretation as the estimation of the relative position of the approximate value in the interval worst solution-value – optimal value. In [2], the term "trivial solution" is used to denote the solution realizing the worst among the feasible solution-values of an instance. Moreover, all the examples in [2] carry over **NP**-hard problems for which worst solution can be trivially computed. This is for example the case of maximum independent set where, given a graph, the worst solution is the empty set, or of minimum vertex cover, where the worst solution is the vertex-set of the input-graph, or even of the minimum graph-coloring where one can trivially color the vertices of the input-graph using a distinct color per vertex. On the contrary, for TSP things are very different. Let us take for example min_TSP. Here, given a graph K_n, the worst solution for K_n is a maximum total-distance Hamiltonian cycle, i.e., the optimal solution of max_TSP in K_n. The computation of such a solution is very far from being trivial since max_TSP is **NP**-hard. Obviously, the same holds when one considers max_TSP and tries to compute a worst solution for its instance, as well as for optimum satisfiability, for minimum maximal independent set and for many other well-known **NP**-hard problems. In order to remove ambiguities about the concept of the worst-value solution of an instance I of an **NPO** problem Π, we will defined it as the optimal solution $\mathrm{opt}(\Pi')$ of an **NPO** problem Π' having the same set of instances and feasibility constraints as Π verifying

$$\mathrm{opt}(\Pi') = \begin{cases} \max & \mathrm{opt}(\Pi) = \min \\ \min & \mathrm{opt}(\Pi) = \max \end{cases}$$

In general, no apparent links exist between standard and differential approximations in the case of minimization problems, in the sense that there is no evident transfer of a positive, or negative, result from the one framework to the other. Hence a "good" differential-approximation result signifies nothing for the behavior of the approximation algorithm studied when dealing with the standard framework and vice-versa. When dealing with maximization problems, we show in [11] that the approximation of a maximization **NPO** problem Π within differential-approximation ratio δ, implies its approximation within standard-approximation ratio δ.

The best known standard-approximation ratio known for min_TSP12 is $7/6$ (presented in [12]), while the best known standard inapproximability bound is $5381/5380 - \epsilon$, for any $\epsilon > 0$ ([8]). On the other hand,the best known standard-ratio max_TSP is $3/4$ ([13]). To our knowledge, no better result is known in standard approximation for max_TSP12. Furthermore, no special study of TSPab

has been performed until now (a trivial standard-approximation ratio or b/a of a/b is in any case very easily deduced for min_ or max_TSPab).

Here we show that min_ and max_TSP12 and min_ and max_TSPab are all equi-approximable within 3/4 for the differential approximation. We also prove that all these problems are inapproximable within better (more than) $3475/3476 + \epsilon$, for any $\epsilon > 0$. Finally, we improve the *standard*-approximation ratio of max_TSP12 from 3/4 ([13]) to 7/8.

In what follows, we will denote by $V = \{v_1, \ldots, v_n\}$ the vertex-set of K_n, by E its edge-set and, for $v_iv_j \in E$, we denote by $d(v_i, v_j)$ the distance of the edge $v_iv_j \in E$; we consider that the distance-vector is symmetric and integer. Given a feasible TSP-solution $T(K_n)$ of K_n (both min_ and max_TSP have the same set of feasible solutions), we denote by $d(T(K_n))$ its (objective) value. Given a graph G, we denote by $V(G)$ its vertex-set. Finally, given any set C of edges, we denote by $d(C)$ the total distance of C, i.e., the quantity $\sum_{v_iv_j \in C} d(i,j)$.

2 Differential-approximation preserving reductions for TSP12

Theorem 1. *min_ TSP12, max_ TSP12, min_ TSPab and max_ TSPab are all equi-approximable for the differential approximation.*

Proof (sketch). In order to prove the theorem we will prove the following stronger quoted proposition.

> *Consider any instance $I = (K_n, \boldsymbol{d})$ (where $\boldsymbol{d}$ denotes the edge-distance vector of K_n). Then, any legal transformation $\boldsymbol{d} \mapsto \gamma.\boldsymbol{d} + \eta.\mathbf{1}$ of $\boldsymbol{d}$ ($\gamma, \eta \in \mathbb{Q}$) produces differentially equi-approximable TSP-problems.*

Suppose that TSP can be approximately solved within differential-approximation ratio δ and remark that both the initial and the transformed instance have the same set of feasible solutions. By the transformation considered, the value $d(T(K_n))$ of any tour $T(K_n)$ is affinely transformed into $\gamma d(T(K_n)) + \eta n$. Since differential-approximation ratio is stable under affine transformation, the equi-approximability of the original and of the transformed problem is immediately deduced, concluding so the proof of the quoted proposition.

In order to prove that min_TSP12 and max_TSP12 are equi-approximable it suffices to apply the proposition above with $\gamma = -1$ and $\eta = 3$. On the other hand, in order to prove that min_ or max_TSP12 reduces to min_ or max_TSPab, we apply the quoted proposition with $\gamma = 1/(b-a)$ and $\eta = (b-2a)/(b-a)$, while for the converse reduction we apply the quoted proposition with $\gamma = b - a$ and $\eta = 2a - b$. Since the reductions presented are transitive and composable, the equi-approximability of the pairs (min_TSP12, max_TSP12) and (TSP12, TSPab) proves the theorem.

3 Approximating min_TSP12

Let us first recall that, given a graph G, a 2-matching is a set M of edges of G such that if $V(M)$ is the set of the endpoints of M, the vertices of the graph $(V(M), M)$ have degree at most 2; in other words, the graph $(V(M), M)$ is a collection of cycles and simple paths. A 2-matching is optimal if it is the largest over all the 2-matchings of G. It is called perfect if any vertex of the graph $(V(M), M)$ has degree equal to 2, i.e., if it constitutes a partition of $V(M)$ into cycles. Remark that determining a maximum 2-matching in a graph G is equivalent to determining a minimum total-distance vertex-partition into cycles into $G \cup \bar{G}$ (the complement of G), where the edges of G are considered of distance 1 and the ones of $\bar{G}$ of distance 2.

As it is shown in [10], *an optimal triangle-free 2-matching can be computed in polynomial time*. As it is mentioned just above, this becomes to compute a triangle-free minimum-distance collection of cycles in a complete graph K_n with edge-distances 1 and 2. Let us denote by M such a collection. Starting from M, we will progressively patch its cycles in order to finally obtain a unique Hamiltonian cycle in K_n.

In what follows, for reasons of paper length's constraints, lemmata 4 and 5 are presented without their proofs which can be found in [11].

3.1 Preprocessing M

Definition 1. *Let C_1 and C_2 be two vertex-disjoint cycles. Then:*

- *a* 2-exchange *is any exchange of two edges $v_1u_1 \in C_1$, $v_2u_2 \in C_2$ by the edges v_1v_2 and u_1u_2;*
- *a* 2-patching *of C_1 and C_2 is any cycle C resulting from a 2-exchange on C_1 and C_2, i.e., $C = (C_1 \cup C_2) \setminus \{v_1u_1, v_2u_2\} \cup \{v_1v_2, u_1u_2\}$, for any pair $(v_1u_1, v_2u_2) \in C_1 \times C_2$.*

A matching minimal with respect to the 2-exchange operation will be called *2-minimal*.

Definition 2. *A 2-matching $M = (C_1, C_2, \ldots, C_{|M|})$ is 2-minimal if it verifies, $\forall (C_i, C_j) \in M \times M$, $C_i \neq C_j$, $\forall v_1u_1 \in C_i$, $\forall v_2u_2 \in C_j$, $d(v_1, v_2) + d(u_1, u_2) > d(u_1v_1) + d(u_2v_2)$.*

In other words, a 2-matching M is 2-minimal if any 2-patching of its cycles produces a 2-matching of total distance strictly greater than the one of M. Obviously, *starting from a 2-matching M transformation of M into a 2-minimal one can be performed in polynomial time*. Moreover, suppose that there exist two distinct cycles C and C' of M, both containing 2-edges and denote by $uv \in C$ and $u'v' \in C'$ two such edges. Then, $d(uu') + d(vv') \geqslant 4$, while $d(uv) + d(u'v') = 4$, a contradiction. So, the following proposition holds.

Proposition 1. *In any 2-minimal 2-matching, at most one of its cycles contains 2-edges.*

Remark 1. If the size of a 2-minimal triangle-free 2-matching M is 1, then, since a Hamiltonian tour is a particular case of triangle-free 2-matching, M is an optimal min_TSP12-solution. Hence, in what follows we will suppose 2-matchings of size at least 2.

Assume now a 2-minimal triangle-free 2-matching $M = (C_1, \ldots, C_p, C_0)$, verifying remark 1, where by C_0 is denoted the unique cycle of M' (if any) containing 2-edges. Construct a graph $H = V_H, E_H)$ where $V_H = \{w_1, \ldots, w_p\}$ and contains a vertex per cycle of M' and, for $i \neq j$, $w_i w_j \in E_H$ iff $\exists (u, v) \in C_i \times C_j$ such that $d(u, v) = 1$. Consider a maximum matching M_H, $|M_H| = q$, of H. With any edge $w_{i^s} w_{j^s}$ of M_H we associate the pair (C_{i^s}, C_{j^s}) of the corresponding cycles of M. So, M can be described (up to renaming its cycles) as

$$M = \bigcup_{s=1}^{q} \{C_1^s, C_2^s\} \bigcup_{t=1}^{r=p-2q} \{C_t\} \bigcup \{C_0\} \tag{1}$$

where for $s = 1, \ldots, q$, $\exists e^s \in V(C_1^s) \times V(C_2^s)$ such that $d(e^s) = 1$.

Consider M as expressed in expression (1), denote by V_s the set of the four vertices of C_1^s and C_2^s adjacent to the endpoints of e^s, and construct the bipartite graph $B = (V_B^1 \cup V_B^2, E_B)$ where $V_B^1 = \{w_1, \ldots, w_r\}$ (i.e., we associate a vertex with a cycle C_t, $t = 1, \ldots, r$), $V_B^2 = \{w^1 1, \ldots, w^q\}$ (i.e., we associate a vertex with a pair (C_1^s, C_2^s), $s = 1, \ldots q$) and, $\forall (t, s)$, $w_t w^s \in E_B$ iff $\exists u \in C_t$, $\exists v \in V_s$ such that $d(u, v) = 1$. Compute a maximum matching M_B, $|M_B| = q'$ in B. With any edge $w_t w^s \in M_B$ we associate the triple (C_1^s, C_2^s, C_t). So, M can be described (up to renaming its cycles) as

$$M = \bigcup_{s=1}^{q'} \{C_1^s, C_2^s, C_3^s\} \bigcup_{s=q'+1}^{q} \{C_1^s, C_2^s\} \bigcup_{t=1}^{r'=r-q'} \{C_t\} \bigcup \{C_0\} \tag{2}$$

where for $s = 1, \ldots, q'$, $\exists f^s \in V_s \times V(C_3^s)$ such that $d(f^s) = 1$. In what follows we will reason with respect to M as it has been expressed in expression (2).

3.2 Computation and evaluation of the approximate solution and a lower bound for the optimal tour

In the sequel, call s.d.e.p. a set of vertex-disjoint elementary paths, denote by PREPROCESS the algorithm of the achievement of M (expression (2)) following from the discussion of section 3.1 and consider the following algorithm.

```
BEGIN (*TSP12*)
      compute a 2-minimal triangle-free 2-matching M in Kn;
      M ← PREPROCESS(M);
      D ← ∅;
(1)   FOR s ← 1 TO q' DO
          let g1^s be the edge of C1^s adjacent to both e^s and f^s;
          choose in C2^s an edge g2^s adjacent to e^s;
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        choose in C_3^s an edge g_3^s adjacent to f^s;
        D ← D ∪ C_1^s ∪ C_2^s ∪ C_3^s \ {g_1^s, g_2^s, g_3^s} ∪ {e^s, f^s};
    OD
(2) FOR s ← q' + 1 TO q DO
        choose in C_1^s an edge g_1^s adjacent to e^s;
        choose in C_2^s an edge g_2^s adjacent to e^s;
        D ← D ∪ C_1^s ∪ C_2^s \ {g_1^s, g_2^s} ∪ {e^s};
    OD
(3) FOR t ← 1 TO r' DO
        choose any edge g_t in C_t;
        D ← D ∪ C_t \ {g_t};
    OD
(4) IF there exists in C_0 an 1-edge e
      THEN choose an edge g_0 of C_0 adjacent to e;
      ELSE choose any edge g_0 of C_0;
      D ← D ∪ C_0 \ {g_0};
    FI
(5) complete D in order to obtain a Hamiltonian tour T(K_n);
    OUTPUT T(K_n);
END (*TSP12*)
```

Both achievement of a 2-minimal triangle free 2-matching and the PREPROCESS of it, can be performed in polynomial time. Moreover, steps (1) to (4) are also executed in polynomial time. Finally, step (5) can be performed by arbitrarily ordering (mod$|D|$) the chains of the s.d.e.p. D and then, for $i = 1, \ldots, |D|$, adding in D the edge linking the "last" vertex of chain i to the "first" vertex of chain $i + 1$. Consequently, the whole algorithm TSP12 is polynomial.

Lemma 1. *$d(T(K_n)) \leqslant d(M) + q + r'$.*

Proof. During steps (1) to (4) of algorithm TSP12, set D remains a s.d.e.p. At the end of step (4), D contains M minus the $3q' + 2(q - q') + r' = q' + 2q + r'$ 1-edges of the set $\cup_{s=1}^{q'} \{g_1^s, g_2^s, g_3^s\} \cup_{s=q'+1}^{q} \{g_1^s, g_2^s\} \cup_{s=1}^{r'} \{g_t\}$ minus (if $C_0 \neq \emptyset$) one 2-edge of C_0 plus the $2q' + (q - q') = q' + q$ 1-edges of the set $cup_{s=1}^{q'} \{e_1^s, f_2^s\} cup_{s=q'+1}^{q} \{e^s\}$. So D is a s.d.e.p. of size $n - (q + r') - \mathbf{1}_{C_0 \neq \emptyset}$ and of total distance $d(M) - (q + r') - 2.\mathbf{1}_{C_0 \neq \emptyset}$. Completion of D in order to obtain a tour in K_n, can be done by adding $q + r' + \mathbf{1}_{C_0 \neq \emptyset}$ new edges. Each of these new edges can be, at worst, of distance 2. We so have $d(T(K_n)) \leqslant d(M) - (q + r' + 2.\mathbf{1}_{C_0 \neq \emptyset}) + 2(q + r' + \mathbf{1}_{C_0 \neq \emptyset}) = d(M) + q = r'$, q.e.d.

On the other hand, the optimal tour being a particular triangle-free 2-matching, the following lemma holds immediately.

Lemma 2. *$\beta(K_n) \geqslant d(M)$.*

3.3 Evaluation of the worst-value solution

In what follows in this section we will bring to the fore a s.d.e.p., all of its edges being of distance 2 (called 2-s.d.e.p.). Given such a s.d.e.p. W, one can proceed as in step (5) of algorithm TSP12 (section 3.2), in order to construct a Hamiltonian tour T_w whose total distance is a lower bound for $\omega(K_n)$.

Denote by $E2$ the set of 2-edges of cycle C_0. If $q = 0$, i.e., $M_H = \emptyset$, and if $C_0 = E2$, then the tour computed by TSP12 is optimal.

Lemma 3. $((q = 0) \wedge (C_0 = E2)) \Rightarrow \delta_{\texttt{TSP12}}(K_n) = 1$.

Proof. Let $k = |V(C_0)| = d(M) - n$ and set $V(C_0) = \{a_1, \dots, a_k\}$. By the fact that M is 2-minimal, all the edges of K_n incident to these vertices have distance 2. On the other hand, between two distinct cycles in the set $\{C_1, \dots, C_{p=r'}\}$ of M, there exist only edges of distance 2. Consider the family

$$\mathcal{F} = \left\{ \{a_1\}, \dots, \{a_k\}, V(C_1), \dots V(C_p) \right\}.$$

By the remarks just above, any edge linking vertices of two distinct sets of $\mathcal{F}$ is a 2-edge. Any feasible tour of K_n (a posteriori an optimal one) integrates the $k+p$ sets of $\mathcal{F}$ by using at least $k+p$ 2-edges pairwise linking these sets. Hence, any tour uses at least $k+p$ 2-edges, so does tour $T(K_n)$ computed by algorithm TSP12, q.e.d.

So, we suppose in the sequel that $q = 0 \Rightarrow C_0 \neq C_2$. We will now prove the existence of a 2-s.d.e.p. W of size $d(M) + 4(q + r') - n$, where M is as expressed by expression (2).

Proposition 2. *Between two cycles C_a and C_b of M of size at least k, there always exists a path, alternating vertices of C_a and C_b, containing at least k 2-edges.*

Proof. Let $\{a_1, \dots, a_{k+1}\}$ and $\{b_1, \dots, b_{k+1}\}$ be $k+1$ successive vertices of two distinct cycles C_a and C_b of size at least k (eventually $a_1 = a_{k+1}$ if $|V(C_a)| = k$ and $b_1 = b_{k+1}$ if $|V(C_b)| = k$). We will show that there exists a path, alternating vertices of C_a and C_b, of size $2k-1$ and of distance at least $3k-1$. Consider paths $C = \cup_{i=1}^{k}\{a_i b_i\} \cup_{i=1}^{k-1} \{a_{i+1} b_i\}$ and $D = \cup_{i=2}^{k+1}\{a_i b_i\} \cup_{i=1}^{k-1} \{a_i b_{i+1}\}$. By the 2-minimality of M we get: $\forall i = 1, \dots, k$

$$\max\left\{d(a_i, b_i), d(a_{i+1}, b_{i+1})\right\} = 2 \Rightarrow d(a_i, b_i) + d(a_{i+1}, b_{i+1}) \geqslant 3$$

and $\forall i = 1, \dots, k-1$

$$\max\left\{d(a_i, b_{i+1}), d(a_{i+1}, b_i)\right\} = 2 \Rightarrow d(a_i, b_{i+1}) + d(a_{i+1}, b_i) \geqslant 3$$

Summing the terms of the expression above member-by-member, one obtains:

$$\sum_{i=1}^{k} \left(d(a_i, b_i) + d(a_{i+1}, b_{i+1})\right) + \sum_{i=1}^{k-1} \left(d(a_{i+1}, b_i) + d(a_i, b_{i+1})\right) \geqslant 6k - 3$$

$$\iff d(C) + d(D) \geqslant 6k - 3 \quad \Rightarrow \quad \max\{d(C), d(D)\} \geqslant \left\lceil \frac{6k-3}{2} \right\rceil = 3k - 1.$$

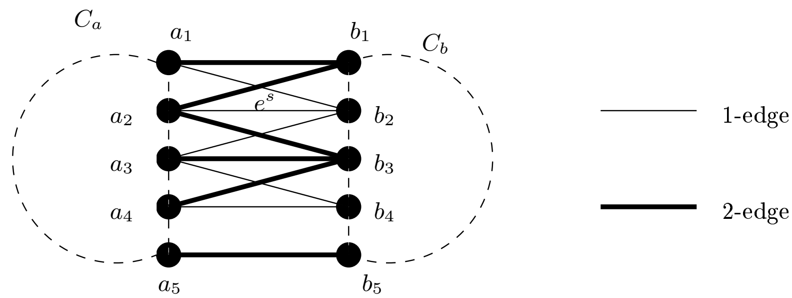

Fig. 1. An example of claim 1.

Application of proposition 2 in any pair (C_1^s, C_2^s) of M results to the following.

Claim 1. $\forall s = 1, \ldots, q$, *there exists a 2-s.d.e.p.* W^s *of size 4, alternating vertices of cycles* C_1^s *and* C_2^s*, containing a vertex of* V_s *whose degree with respect to* W^s *is 1.*

In figure 1, we show an application of claim 1. We assume $e^s = a_2b_2$; then $\{a_1, b_1\} \subset V_s$. The 2-s.d.e.p. W^s claimed is $\{b_1a_2, (a_3b_3, a_4b_4), a_5b_5\}$ and the degree of b_1 with respect to W^s is 1.

Consider now the s.d.e.p. $W_t^s = W^s \cup W'^s_t$, where W^s as in claim 1 and W'^s_t is any path of size 4 alternating vertices of C_3^s and of C_t, $s = 1, \ldots, q'$, $t = 1, \ldots, r'$. By the optimality of M_H, any edge linking vertices of C_3^s to vertices of C_t is a 2-edge. Consequently, W_t^s is a 2-s.d.e.p. and the following claim holds.

Claim 2. $\forall s = 1, \ldots, q'$, $\forall t = 1, \ldots, r'$, *there exists a 2-s.d.e.p.* W_t^s *of size 8, alternating vertices of the cycles* C_1^s *and* C_2^s*, and of the cycles* C_3^s *and* C_t*.*

For $s = q' + 1, \ldots, q$, $t = 1, \ldots r'$, consider the triple (C_1^s, C_2^s, C_t). Let $e^s = e_1^s e_2^s$, $V_s = \{u_1^s, v_1^s, u_2^s, v_2^s\}$ and consider any four vertices a_t, b_t, c_t and d_t of C_t. By the optimality of M_B, any vertex of C_t is linked to any vertex of V^s exclusively by 2-edges. Moreover, the 2-minimality of M implies that at least one of $u_1^s e_2^s$ and $e_1^s u_2^s$ is of distance 2. If we suppose $d(u_1^s, e_2^s) = 2$ (figure 2), then the path $\{e_2^s, u_1^s, a_t, v_1^s, b_t, u_2^s, c_t, v_2^s, d_t\}$ is a 2-s.e.d.p. In all, the following claim holds.

Claim 3. $\forall s = q' + 1, \ldots, q$, $\forall t = 1, \ldots, r'$, *there exists a 2-s.d.e.p.* W_t^s *of size 8, alternating vertices of the cycles* C_1^s*,* C_2^s *and* C_t*.*

Let $r' \geqslant 2$ and consider the (residual) cycles C_t, $t = 1, \ldots, r'$. All edges between these cycles are of distance 2. If we denote by a_t, b_t, c_t and d_t four vertices of C_t, the path

$$\{a_1, \ldots, a_t, \ldots, a_{r'}, b_1, \ldots, b_t, \ldots, b_{r'}, c_1, \ldots, c_t, \ldots, c_{r'}, d_1, \ldots, d_t, \ldots, d_{r'}\}$$

is a 2-s.d.e.p. of size $4r' - 1$ and the following claim holds.

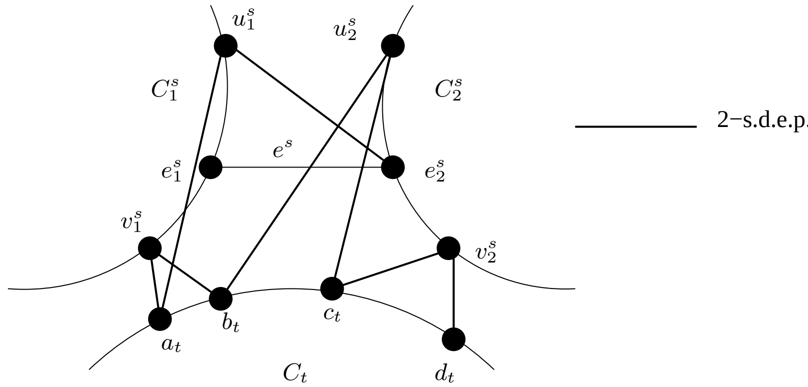

Fig. 2. The 2-s.d.e.p. W_t^s of claim 3.

Claim 4. *If $r' \geqslant 2$, then there exists a 2-s.d.e.p. $W^{r'}$ of size $4r' - 1$ alternating vertices of cycles C_t, $t = 1, \ldots, r'$.*

Lemma 4. $((q \geqslant 1) \vee ((C_0 \neq C_2) \wedge (r \neq 1))) \Rightarrow \delta_{\mathtt{TSP12}}(K_n) \geqslant 3/4$.

Lemma 4 is proved in [11]. There, we use claims 1, 2, 3 and 4 in order to bring to the fore a 2-s.d.e.p. W of total distance $4(q+r') - \mathbf{1}_{q=0} + d(M) - n + \mathbf{1}_{C_0 \neq E2} \geqslant d(M) - n + 4(q+r')$, the completion of which produces a Hamiltonian tour of distance $d(M) +' (q + r')$ at least.

Lemma 5. $((q = 0) \wedge (r = 1) \wedge (C_0 \neq C_2)) \Rightarrow \delta_{\mathtt{TSP12}}(K_n) \geqslant 3/4$.

In all, combining lemmata 1, 2, 3 and 5, the following theorem can be immediately proved.

Theorem 2. *min_ TSP12 is approximable within differential-approximation ratio 3/4.*

Theorems 1 and 2 induce the following corollary.

Corollary 1. *min_ and max_ TSP12 as well as min_ and max_ TSPab are approximable within differential-approximation ratio 3/4.*

Consider two cliques and number their vertices by $\{1, \ldots, 4\}$ and by $\{5, 6, \ldots, n+8\}$, respectively. Edges of both cliques have all distance 1. Cross-edges ij, $i = 1, 3$, $j = 5, \ldots, n+8$, are all of distance 2, while every other cross-edge is of distance 1. Unraveling of `TSP12` will produce: $T = \{1, 2, 3, 4, 5, 6, \ldots, n + 7, n + 8, 1\}$ (cycle-patching on edges $(1, 4)$ and $(5, n + 8)$), while $T_w = \{1, 5, 2, 6, 3, 7, 4, 8, 9 \ldots, n + 7, n + 8, 1\}$ (using 2-edges $(1, 5)$, $(6, 3)$, $(3, 7)$ and $(n + 8, 1)$) and $T^* = \{1, 2, n + 8, n+7, \ldots, 5, 4, 3, 1\}$ (using 1-edges $(4, 5)$ and $(2, n+8)$). In figure 3, T^* and T_w are shown for $n = 2$ ($T = \{1, \ldots, 10, 1\}$). Consequently, $\delta_{\mathtt{TSP12}}(K_{n+8}) = 3/4$ and the following proposition holds.

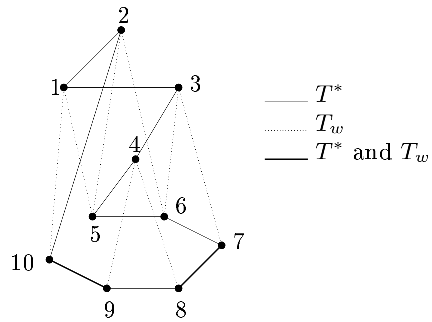

Fig. 3. Tightness of the `TSP12` approximation ratio.

Proposition 3. *Ratio 3/4 is tight for* `TSP12`

Let us note that the differential approximation ratio of the 7/6-algorithm of [12], when running on K_{n+8}, is also 3/4. The authors of [12] bring also to the fore a family of worst-case instances for their algorithm: one has k cycles of length four arranged around a cycle of length $2k$. We have performed a limited comparative study between their algorithm and ours one, for $k = 3, 4, 5, 6$ (on 24 graphs). The average differential and standard approximation ratios for the two algorithms are presented in table 1. Using corollary 1 and the facts that $\omega(K_n) \leqslant bn$ and

	k	`TSP12`	The algorithm of [12]
Differential ratio	3	0.931100364	0.846702091
	4	0.9000002	0.833333
	5	0.920289696	0.833333
	6	0.9222222	0.833333
Standard ratio	3	0.923350955	0.87013
	4	0.9094018	0.857143
	5	0.92646313	0.857143
	6	0.928178	0.857143

Table 1. A limited comparison between `TSP12` and the algorithm of [12] on some worst-case instances of the latter.

$\beta(K_n) \geqslant an$, the following holds.

Proposition 4. *min_ TSPab is approximable within* $\rho \leqslant (1 + (b - a/4a)$ *in the standard framework. This ratio tends to* ∞ *when* $b = o(a)$.

Revisit min_TSP12. Using $n \leqslant \beta(K_n) \leqslant \omega(K_n) \leqslant 2n$, one can see that approximation of min_TSP12 within $\delta = 1 - \epsilon$ implies its approximation within $\rho = 2 - (1 - \epsilon) = 1 + \epsilon$, $0 \leqslant \epsilon \leqslant 1$. Using the result of [8] and theorem 1, one gets the following differential-inapproximability result.

Theorem 3. *min_ and max_ TSPab and min_ and max_ TSP12 are inapproximable within differential-ratio greater than, or equal to,* $5379/5380 + \epsilon$, $\forall \epsilon > 0$, *unless **P=NP**.*

4 An improvement of the standard ratio for the maximum traveling salesman with distances 1 and 2

Combining expressions $\delta_{\mathrm{max\_TSP12}} \geqslant 3/4$, $\omega_{\max}(K_n) \geqslant an$ and $\beta_{\max}(K_n) \leqslant bn$, one deduces $\rho_{\mathrm{max\_TSP12}} \geqslant (3/4)+(a/4b)$. Setting $a = 1$ and $b = 2$, the following theorem immediately holds.

Theorem 4. *max_ TSP12 is polynomially approximable within standard-approximation ratio bounded below by 7/8.*

The algorithm of [12] in $\bar{K}_n$ solves max_TSP on K_n within standard-approximation ratio bounded below by 2/3.

Note that standard-approximation ratio 7/8 can be obtained by the following direct method.

```
BEGIN /max_TSP12/
      find a triangle-free 2-matching M = {C1, C2, ...};
      FOR all Ci DO delete a minimum-distance edge from Ci OD
      let MT the collection of the paths obtained;
      properly link the paths of MT to get a Hamiltonian cycle T;
      OUTPUT T;
END. /max_TSP12/
```

Let p be the number of cycles of M where 2-edges have been removed during the `FOR`-loop of algorithm `max_TSP12`. Then, $\lambda_{\mathtt{max\_TSP12}}(K_n) \geqslant d(M) - p$, $\beta(K_n) \leqslant d(M)$ and, since M is triangle-free, $d(M) \geqslant 8p$. Consequently, $\lambda_{\mathtt{max\_TSP12}}(K_n)/\beta_{\max}(K_n) \geqslant 7/8$.

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