Weighted coloring on planar, bipartite and split graphs: complexity and improved approximation

D. de Werra[1], M. Demange[2], B. Escoffier[3], J. Monnot[3], V. Th. Paschos[3]

[1] Ecole Polytechnique Fédérale de Lausanne, Switzerland, dewerra@ima.epfl.ch
[2] ESSEC, Dept. SID, France, demange@essec.fr
[3] Université Paris Dauphine, LAMSADE, CNRS UMR 7024, 75016 Paris, France
{escoffier,monnot,paschos}@lamsade.dauphine.fr

Abstract. We study complexity and approximation of MIN WEIGHTED NODE COLORING in planar, bipartite and split graphs. We show that this problem is **NP**-complete in planar graphs, even if they are triangle-free and their maximum degree is bounded above by 4. Then, we prove that MIN WEIGHTED NODE COLORING is **NP**-complete in P_8-free bipartite graphs, but polynomial for P_5-free bipartite graphs. We next focus ourselves on approximability in general bipartite graphs and improve earlier approximation results by giving approximation ratios matching inapproximability bounds. We next deal with MIN WEIGHTED EDGE COLORING in bipartite graphs. We show that this problem remains strongly **NP**-complete, even in the case where the input-graph is both cubic and planar. Furthermore, we provide an inapproximability bound of $7/6 - \varepsilon$, for any $\varepsilon > 0$ and we give an approximation algorithm with the same ratio. Finally, we show that MIN WEIGHTED NODE COLORING in split graphs can be solved by a polynomial time approximation scheme.

1 Introduction

We give in this paper some complexity results as well as some improved approximation results for MIN WEIGHTED NODE COLORING, originally studied in Guan and Zhu [7] and more recently in [4]. A k-coloring of $G = (V, E)$ is a partition $\mathcal{S} = (S_1, \ldots, S_k)$ of the node set V of G into stable sets S_i. In this case, the objective is to determine a node coloring minimizing k. A natural generalization of this problem is obtained by assigning a strictly positive integer weight $w(v)$ for any node $v \in V$, and defining the weight of stable set S of G as $w(S) = \max\{w(v) : v \in S\}$. Then, the objective is to determine $\mathcal{S} = (S_1, \ldots, S_k)$ a node coloring of G minimizing the quantity $\sum_{i=1}^{k} w(S_i)$. This problem is easily shown **NP**-hard; it suffices to consider $w(v) = 1$, $\forall v \in V$ and MIN WEIGHTED NODE COLORING becomes the classical node coloring problem. Other versions of weighted colorings have been studied in Hassin and Monnot [8].

Consider an instance I of an **NP**-hard optimization problem Π and a polynomial time algorithm $\mathtt{A}$ computing feasible solutions for Π. Denote by $m_{\mathtt{A}}(I, S)$ the value of a Π-solution S computed by $\mathtt{A}$ on I and by $\mathrm{opt}(I)$, the value of an optimal Π-solution for I. The quality of $\mathtt{A}$ is expressed by the ratio (called approximation ratio in what follows) $\rho_{\mathtt{A}}(I) = m_{\mathtt{A}}(I,S)/\text{opt}(I)$, and the quantity $\rho_{\mathtt{A}} = \inf\{r : \rho_{\mathtt{A}}(I) < r, I \text{ instance of } \Pi\}$. A very favourable situation for polynomial approximation occurs when an algorithm achieves ratios bounded above by $1+\varepsilon$, for any $\varepsilon > 0$. We call such algorithms *polynomial time approximation schemes*. The complexity of such schemes may be polynomial or exponential in $1/\varepsilon$ (they are always polynomial in the sizes of the instances). A polynomial time approximation scheme with complexity polynomial also in $1/\varepsilon$ is called *fully polynomial time approximation scheme*.

This paper extends results on MIN WEIGHTED NODE COLORING, the study of which has started in Demange et al. [4]. We first deal with planar graphs and we show that, for this family, the problem studied is **NP**-complete, even if we restrict to triangle-free planar graphs with node-degree not exceeding 4.

We then deal with particular families of bipartite graphs. The **NP**-completeness of MIN WEIGHTED NODE COLORING has been established in [4] for general bipartite graphs. We show here that this remains true even if we restrict to planar bipartite graphs or to P_{21}-free bipartite graphs (for definitions graph-theoretical notions used in this paper, the interested reader is referred to Berge [1]). It is interesting to observe that these results are obtained as corollaries of a kind of generic reduction from the precoloring extension problem shown to be **NP**-complete in Bodlaender et al. [2], Hujter and Tuza [10, 11], Kratochvil [13]. Then, we slightly improve the last result to P_8-free bipartite graphs and show that the problem becomes polynomial in P_5-free bipartite graphs. Observe that in [4], we have proved that MIN WEIGHTED NODE COLORING is polynomial for P_4-free graphs and **NP**-complete for P_5-free graphs.

Then, we focus ourselves on approximability of MIN WEIGHTED NODE COLORING in (general) bipartite graphs. As proved in [4], this problem is approximable in such graphs within approximation ratio $4/3$; in the same paper a lower bound of $8/7-\varepsilon$, for any $\varepsilon > 0$, was also provided. Here we improve the approximation ratio of [4] by matching the $8/7$-lower bound of [4] with a same upper bound; in other words, we show here that MIN WEIGHTED NODE COLORING in bipartite graphs is approximable within approximation ratio bounded above by $8/7$.

We next deal with MIN WEIGHTED EDGE COLORING in bipartite graphs. In this problem we consider an edge-weighted graph G and try to determine a partition of the edges of G into matchings in such a way that the sum of the weights of these matchings is minimum (analogously to the node-model, the weight of a matching is the maximum of the weights of its edges). In [4], it is shown that MIN WEIGHTED EDGE COLORING is **NP**-complete for cubic bipartite graphs. Here, we slightly strengthen this result showing that this problem remains strongly **NP**-complete, even in cubic and planar bipartite graphs. Furthermore, we strengthen the inapproximability bound provided in [4], by reducing it from $8/7-\varepsilon$ to $7/6-\varepsilon$, for any $\varepsilon > 0$. Also, we match it with an upper bound of the same value, improving so the $5/3$-approximation ratio provided in [4].

Finally, we deal with approximation of MIN WEIGHTED NODE COLORING in split graphs. As proved in [4], MIN WEIGHTED NODE COLORING is strongly **NP**-complete in such graphs, even if the nodes of the input graph receive only one of two distinct weights. It followed that this problem cannot be solved by fully polynomial time approximation schemes, but no approximation study was addressed there. In this paper we show that MIN WEIGHTED NODE COLORING in split graphs can be solved by a polynomial time approximation scheme.

In the remainder of the paper we shall assume for any weighted node or edge coloring $\mathcal{S} = (S_1, \ldots, S_\ell)$ considered, we will have $w(S_1) \geqslant \ldots \geqslant w(S_\ell)$.

2 Weighted node coloring in triangle-free planar graphs

The node coloring problem in planar graphs has been shown **NP**-complete by Garey and Johnson [5], even if the maximum degree does not exceed 4. On the other hand, this problem becomes easy in triangle-free planar graphs, (see Grotzsch [6]). Here, we show that the weighted node coloring problem is **NP**-complete in triangle-free planar graphs with maximum degree 4 by using a reduction from 3-SAT PLANAR, proved to be **NP**-complete in Lichtenstein [14]. This problem is defined as follows: Given a collection $\mathcal{C} = (C_1, \ldots, C_m)$ of clauses over the set $X = \{x_1, \ldots, x_n\}$ of Boolean variables such that each clause C_j has at most three literals (and at least two), is there a truth assignment f satisfying $\mathcal{C}$? Moreover, the bipartite graph $BP = (L, R; E)$ is planar where $|L| = n$, $|R| = m$ and $[x_i, c_j] \in E$ iff the variable x_i (or $\overline{x_i}$) appears in the clause C_j.

Theorem 1. MIN WEIGHTED NODE COLORING *is* ***NP****-complete in triangle-free planar graphs with a maximum degree 4.*

Proof. Let $BP = (L, R; E)$ be the bipartite graph representing an instance $(X, \mathcal{C})$ of 3-SAT PLANAR where $L = \{x_1, \ldots, x_n\}$, $R = \{c_1, \ldots, c_m\}$. We construct an instance $I = (G, w)$ of MIN WEIGHTED NODE COLORING by using two gadgets: The gadgets clause $F(C_j)$ are given in Figure 1 for clause C_j of size 3 and in Figure 2 for clause C_j of size 2. The nodes c_j^k are those that will be linked to the rest of the graph.

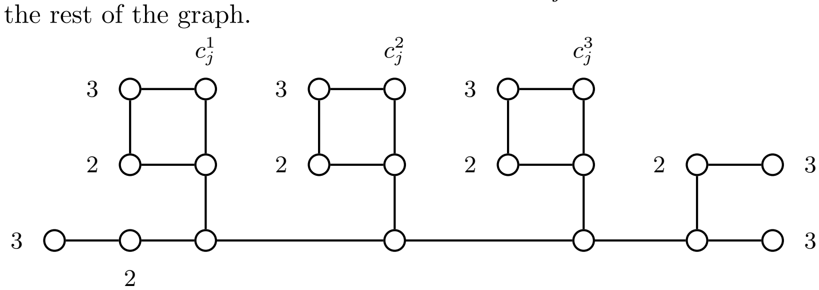

Fig. 1. Graph $F(C_j)$ representing a clause C_j of size 3.

The gadgets variable $H(x_i)$ is given in Figure 3 for variable x_i. Assume that x_i appears p_1 times positively and p_2 times negatively in $(X, \mathcal{C})$, then in $H(x_i)$

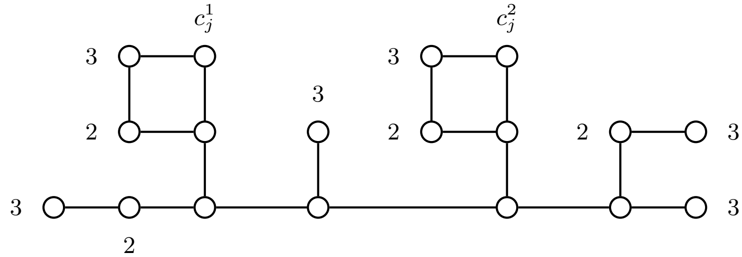

Fig. 2. Graph $F(C_j)$ representing a clause C_j of size 2.

there are $2p = 2(p_1 + p_2)$ special nodes $x_i^k, \overline{x_i^k}$, $k = 1, \ldots, p$. These nodes form a path which meets nodes x_i^k, $\overline{x_i^k}$ alternately.

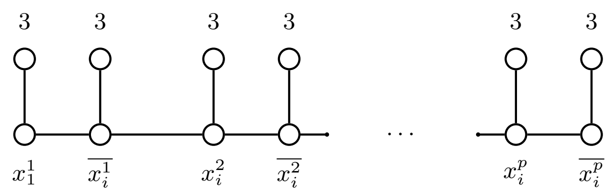

Fig. 3. Graph $H(x_i)$ representing variable x_i

The weight of nodes which are not given in Figures 1, 2 and 3 are 1. These gadgets are linked together by the following process. If variable x_i appears positively (resp. negatively) in clause c_j, we link one of the variables $\overline{x_i^k}$ (resp. x_i^k), with a different k for each C_j, to one of the three nodes c_j^l of gadget $F(C_j)$. This can be done in a way which preserves the planarity of the graph. Observe that G is triangle-free and planar with maximum degree 4. Moreover, we assume that G is not bipartite (otherwise, we add a disjoint cycle Γ with $|\Gamma| = 7$ and $\forall v \in V(\Gamma), w(v) = 1$).
It is then not difficult to check that $(X, \mathcal{C})$ is satisfiable iff $opt(I) \leqslant 6$.

3 Weighted node coloring in bipartite graphs

3.1 Complexity results

The **NP**-completeness of MIN WEIGHTED NODE COLORING in bipartite graphs has been proved in [4]. Here, we show that some more restrictive versions are also **NP**-complete, namely bipartite planar graphs and P_8-free bipartite graphs, i.e. bipartite graphs which do not contain induced paths of length 8 or more. We use a generic reduction from the precoloring extension node coloring problem (in short PrExt node coloring). This latter problem studied in [2, 10, 13, 11], can be described as follows. Given a positive integer k, a graph $G = (V, E)$ and k pairwise disjoint subsets $V_1, \dots, V_k$ of V, we want to decide if there exists a node coloring $\mathcal{S} = (S_1, \dots, S_k)$ of G such that $V_i \subseteq S_i$, for all $i \leqslant k$. Moreover, we restrict to some class of graphs $\mathcal{G}$: we assume that $\mathcal{G}$ is closed when we add a pending edge with a new node (i.e., if $G = (V, E) \in \mathcal{G}$ and $x \in V$, $y \notin V$, then $G + [x, y] \in \mathcal{G}$).

Theorem 2. *Let $\mathcal{G}$ be a class of graphs which is closed when we add a pending edge with a new node. If* PrExt node coloring *is **NP**-complete for graphs in $\mathcal{G}$, then* min weighted node coloring *is **NP**-complete for graphs in $\mathcal{G}$.*

Proof. Let $\mathcal{G}$ be such a class of graphs. We shall reduce PrExt node coloring in $\mathcal{G}$ graphs to weighted node coloring in $\mathcal{G}$ graphs. Let $G = (V, E) \in \mathcal{G}$ and k pairwise disjoint subsets $V_1, \dots, V_k$ of V. We build instance $I = (G', w)$ of weighted node coloring using several gadgets T_i, for $i = 1, \dots, k$. The construction of T_i is given by induction as follows: T_1 is simply a root v_1 with weight $w(v_1) = 2^{k-1}$. Given $T_1, \dots, T_{i-1}$, T_i is a tree with a root v_i of weight $w(v_i) = 2^{k-i}$ that we link to tree T_p via edge $[v_i, v_p]$ for each $p = 1, \dots, i-1$.

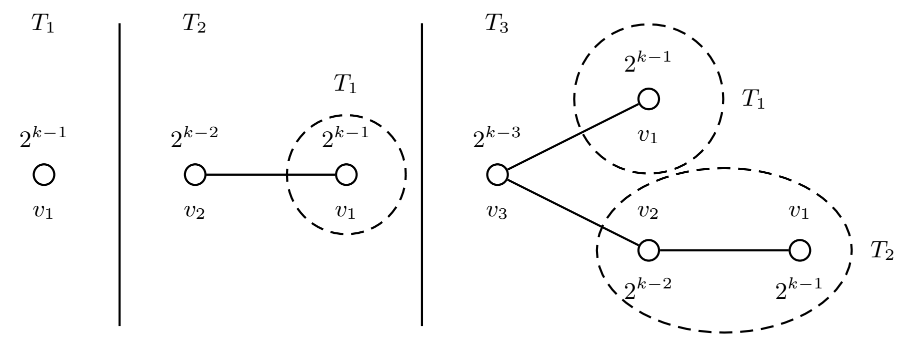

Fig. 4. Gadgets for T_1, T_2 and T_3.

Figure 4 illustrates the gadgets T_1, T_2, T_3. Now, $I = (G', w)$ where $G' = (V', E')$ is constructed in the following way: G' contains G. For all $i = 1, \dots, k$, we replace each node $v \in V_i$ by a copy of the gadget T_i where we identify v with root v_i. For all $v \in V \setminus (\cup_{i=1}^{k} V_i)$ we set $w(v) = 1$. Note that, by hypothesis, $G' \in \mathcal{G}$.
One can verify that the precoloring of G (given by $V_1, \dots, V_k$) can be extended to a proper node coloring of G using at most k colors iff $opt(I) \leqslant 2^k - 1$.

Using the results of Kratochvil [13] on the **NP**-completeness of PrExt node coloring in bipartite planar graphs for $k = 3$ and P_{13}-free bipartite graphs for $k = 5$, we deduce:

Corollary 1. *In bipartite planar graphs,* MIN WEIGHTED NODE COLORING *is strongly **NP**-complete and it is not* $\frac{8}{7} - \varepsilon$*-approximable unless **P**=**NP**.*

Corollary 2. *In* P_{21}*-free bipartite graphs,* MIN WEIGHTED NODE COLORING *is strongly **NP**-complete and it is not* $\frac{32}{31} - \varepsilon$*-approximable unless **P**=**NP**.*

In Hujter and Tuza [11], it is shown that PREXT NODE COLORING is **NP**-complete in P_6-free bipartite chordal graphs for unbounded k. Unfortunately, we cannot use this result in Theorem 2 since the resulting graph has an induced path with arbitrarily large length. However, we can adapt their reduction.

Theorem 3. MIN WEIGHTED NODE COLORING *is **NP**-complete in* P_8*-free bipartite graphs.*

Proof. We shall reduce 3-SAT-3, proved to be **NP**-complete in Papadimitriou [16] to our problem. Given a collection $\mathcal{C} = (C_1, \ldots, C_m)$ of clauses over the set $X = \{x_1, \ldots, x_n\}$ of Boolean variables such that each clause C_j has at most three literals and each variable has at most 3 occurrences (2 positive and one negative), we construct an instance $I = (BP, w)$ in the following way: we start from $BP_1 = (L_1, R_1; E_1)$, a complete bipartite graph $K_{n,m}$ where $L_1 = \{x_1, \ldots, x_n\}$ and $R_1 = \{c_1, \ldots, c_m\}$. Moreover, each node of BP_1 has weight 1. There is also another bipartite graph BP_2 isomorphic to $K_{2n,2n}$ where a perfect matching has been deleted. More formally, $BP_2 = (L_2, R_2; E_2)$ where $L_2 = \{l_1, \ldots, l_{2n}\}$, $R_2 = \{r_1, \ldots, r_{2n}\}$ and $[l_i, r_j] \in E_2$ iff $i \neq j$. Finally, $w(l_i) = w(r_i) = 2^{2n-i}$ for $i = 1, \ldots, 2n$. Indeed, sets $\{l_{2i-1}, r_{2i-1}\}$ and $\{l_{2i}, r_{2i}\}$ will correspond to literal x_i and $\overline{x_i}$ respectively. Between BP_1 and BP_2, there is a set E_3 of edges. $[x_i, r_j] \notin E_3$ iff $j = 2i - 1$ or $j = 2i$ and $[l_i, c_j] \notin E_3$ iff $i = 2k - 1$ and x_k is in C_j or $i = 2k$ and $\overline{x_k}$ is in C_j. Note that BP is a P_8-free bipartite graph.
One can verify that $(X, \mathcal{C})$ is satisfiable iff $opt(I) \leqslant 2^{2n} - 1$.

We end this section by stating that MIN WEIGHTED NODE COLORING is polynomial for P_5-free bipartite graphs, i.e., without induced chain on 5 nodes. There are several characterizations of P_5-free bipartite graphs, see for example, Hammer et al. [9], Chung et al. [3] and Hujter and Tuza [10]. In particular, BP is a P_5-free bipartite graph iff BP is bipartite and each connected component of BP is $2K_2$-free, i.e., its complement is C_4-free. In this case, we can show that any optimal weighted node coloring $\mathcal{S}^* = (S_1^*, \ldots, S_\ell^*)$ uses at most 3 colors (so, $\ell \leqslant 3$) and when $\ell = 3$, then for any connected component $BP_i = (L_i, R_i; E_i)$ of P_5-free bipartite graph we have $S_1^{*,i} \cap L_i \neq \emptyset$ and $S_1^{*,i} \cap R_i \neq \emptyset$, $S_2^{*,i} \subset R_i$ (resp., $S_2^{*,i} \subset L_i$) and $S_3^{*,i} \subset L_i$ (resp., $S_3^{*,i} \subset R_i$) where $(S_1^{*,i}, S_2^{*,i}, S_3^{*,i})$ is the restriction of $\mathcal{S}^*$ to the subgraph BP_i. Thus, applying an exhaustive search on $k_1 = w(S_2^*)$ and a dichotomy search $k_2 = w(S_3^*)$ we can find an optimal solution within $O(n|w|log|w|)$ time where $|w| = |\{w(v) : v \in V\}|$. Hence, we can state:

Theorem 4. MIN WEIGHTED NODE COLORING *is polynomial in* P_5*-free bipartite graphs and can be solved within time* $O(n|w|log|w|)$*.*

3.2 Approximation

In Demange et al. [4], a $\frac{4}{3}$-approximation is given for MIN WEIGHTED NODE COLORING and it is proved that a $(\frac{8}{7}-\varepsilon)$-approximation is not possible, for any $\varepsilon > 0$, unless **P**=**NP**, even if we consider arbitrarily large values of $opt(I)$. Using Corollary 1, we deduce that this lower bound also holds if we consider bipartite planar graphs. Here, we give a $\frac{8}{7}$-approximation in bipartite graphs.

BIPARTITECOLOR

1 Sort the nodes in non-increasing weight order (i.e., $w(v_1) \geqslant \ldots \geqslant w(v_n)$);
2 For $i = 1$ to n do
  2.1 Set $V_i = \{v_1, \ldots, v_i\}$;
  2.2 Compute $\mathcal{S}_i^* = (S_1^i, S_2^i)$ (S_2^i may be empty) an optimal weighted node 2-coloring in the subgraph $BP[V_i]$ induced by V_i ;
  2.3 Define node coloring $\mathcal{S}^i = (S_1^i, S_2^i, L \setminus V_i, R \setminus V_i)$ ($L \setminus V_i$ or/and $R \setminus V_i$ may be empty);
3 Output $\mathcal{S} = argmin\{val(\mathcal{S}^i) : \ i = 1, \ldots, n\}$;

The step 2.2 consists of computing the (unique) 2-coloration $(S_{1,j}^*, S_{2,j}^*)$ (with $w(S_{1,j}^*) \geqslant w(S_{2,j}^*)$) of each connected component $BP_j, j = 1 \ldots p$ of $BP[V_i]$ (with $S_{2,j}^* = \emptyset$ if BP_j is an isolated node). Then it merges the most expensive sets, i.e. it computes $S_1^i = \cup_{j=1}^p S_{i,j}^*$ for $i = 1, 2$. It is easy to observe that $\mathcal{S}_i^* = (S_1^i, S_2^i)$ is the best weighted node coloring of $BP[V_i]$ among the colorings using at most 2 colors; such a coloring can be found in $O(m)$ time where $m = |E|$.

Theorem 5. *BIPARTITECOLOR polynomially solves in time $O(nm)$* MIN WEIGHTED NODE COLORING *in bipartite-graphs and it is a $\frac{8}{7}$-approximation.*

Proof. Let $I = (BP, w)$ be a weighted bipartite-graph where $BP = (L, R; E)$ and $\mathcal{S}^* = (S_1^*, ..., S_l^*)$ be an optimal node coloring of I with $w(S_1^*) \geq ... \geq w(S_l^*)$. If $l < 3$, then BIPARTITECOLOR finds an optimal weighted node coloring which is $\mathcal{S}^n$. Now, assume $l \geqslant 3$ and let $i_j = \min\{k : \ v_k \in S_j^*\}$. We have $i_1 = 1$ and $opt(I) \geqslant w(v_{i_1}) + w(v_{i_2}) + w(v_{i_3})$.
Let us examine several steps of this algorithm. When $i = i_2 - 1$, the algorithm produces a node 3-coloring $\mathcal{S}^{i_2-1} = (S_{i_2-1}^1, L \setminus S_{i_2-1}^1, R \setminus S_{i_2-1}^1)$. Indeed, by construction $V_{i_2-1} \subseteq S_1^*$ is an independent set, and then, $\mathcal{S}_{i_2-1}^*$ is defined by $S_1^{i_2-1} = V_{i_2-1}, S_2^{i_2-1} = \emptyset$ and then $val(\mathcal{S}^{i_2-1}) \leqslant w(v_{i_1}) + 2w(v_{i_2})$. When $i = i_3 - 1$, the algorithm produces on $BP[V_{i_3-1}]$ a node 2-coloring $\mathcal{S}_{i_3-1}^*$ with a cost $val(\mathcal{S}_{i_3-1}^*) \leqslant w(v_{i_1}) + w(v_{i_2})$ since the coloring $(S_1^* \cap V_{i_3-1}, S_2^* \cap V_{i_3-1})$ is a feasible node 2-coloring of $BP[V_{i_3-1}]$ with cost $w(v_{i_1}) + w(v_{i_2})$. Thus, $val(\mathcal{S}^{i_3-1}) \leqslant w(v_{i_1}) + w(v_{i_2}) + 2w(v_{i_3})$. Finally, when $i = n$, the node 2-coloring $\mathcal{S}^n$ satisfies $val(\mathcal{S}^n) \leqslant 2w(v_{i_1})$

The convex combination of these 3 values with coefficients $\frac{1}{7} \times val(\mathcal{S}^n)$, $\frac{4}{7} \times val(\mathcal{S}^{i_3-1})$ and $\frac{2}{7} \times val(\mathcal{S}^{i_2-1})$ gives the expected result.

4 Weighted edge coloring in bipartite graphs

The weighted edge coloring problem on a graph G can be viewed as the weighted node coloring problem on $L(G)$ where $L(G)$ is the line graph of G. Here, for simplicity, we refer to the edge model.

4.1 Complexity results

Demange et al. [4] have proved that MIN WEIGHTED EDGE COLORING in bipartite cubic graphs is strongly **NP**-complete and a lower bound of $\frac{8}{7}$ is given for the approximation. Here, we slightly improve these complexity results.

Theorem 6. *In bipartite cubic planar graphs,* MIN WEIGHTED EDGE COLORING *is strongly* ***NP****-complete and it is not* $\frac{7}{6} - \varepsilon$*-approximable unless* ***P=NP****.*

Proof. We shall reduce PREXT EDGE COLORING in bipartite cubic planar graphs to our problem. Given a bipartite cubic planar graph BP and 3 pairwise disjoint matchings E_i, the question of PREXT EDGE COLORING is to determine if it is possible to extend the edge precoloring E_1, E_2, E_3 to a proper 3-edge coloring of G. Very recently, this problem has been shown **NP**-complete in Marx [15].
Let $BP = (V, E)$ and E_1, E_2, E_3 be an instance of PREXT EDGE COLORING; we construct an instance $I = (BP', w)$ of weighted edge coloring as follows. Each edge in E_1 receives weight 3. Each edge $[x, y] \in E_2$ is replaced by a gadget F_2 described in Figure 4.1, where we identify x and y to v_0 and v_9 respectively. Each edge in E_3 is replaced by a gadget F_3 which is the same as gadget F_2 except that we have exchanged weights 1 and 2. The other edges of G receive weight 1. Remark that BP' is still a bipartite cubic planar graph.

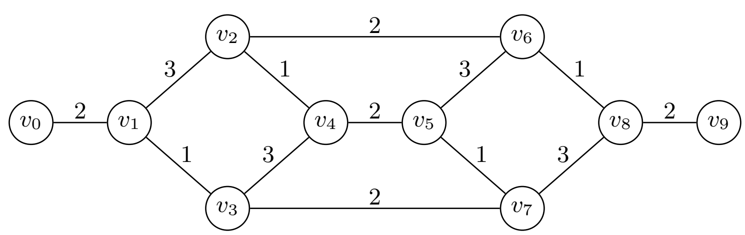

Fig. 5. Gadget F_2 for $e \in E_2$.

We can verify that the answer of PREXT EDGE COLORING instance is yes if and only if there exists an edge coloring $\mathcal{S}$ of I with cost $val(\mathcal{S}) \leqslant 6$.

4.2 Approximation

In Demange et al. [4], a $\frac{5}{3}$-approximation is given for MIN WEIGHTED EDGE COLORING in bipartite graphs with maximum degree 3. Here, we give a $\frac{7}{6}$-approximation. We need some notations: If $BP = (V, E)$ is a bipartite graph with node set $V = \{v_1, \ldots, v_n\}$, we always assume that its edges $E = \{e_1, \ldots, e_m\}$ are sorted in non-increasing weight order (i.e., $w(e_1) \geqslant \ldots \geqslant w(e_m)$). If V' is a subset of nodes and E' a subset of edges, $BP[V']$ and $BP[E']$ denote the subgraph of BP induced by V' and the partial graph of BP induced by E' respectively. For any $i \leqslant m$, we set $E_i = \{e_1, \ldots, e_i\}$ and $\overline{E_i} = E \setminus E_i$. Finally, V_i denotes the set of nodes of BP incident to an edge in E_i (so, it is the subset of non-isolated nodes of $BP[E_i]$).

`BIPARTITEEDGECOLOR`

1 For $i = m$ downto 1 do
  - 1.1 Apply algorithm `SOL1` on $BP[E_i]$;
  - 1.2 If `SOL1`$(BP[E_i]) \neq \emptyset$, complete in a greedy way all the colorings produced by `SOL1` on the edges of $\overline{E_i}$. Let $\mathcal{S}_{1,i}$ be a best one among these edge colorings of BP;
  - 1.3 For $j = i$ downto 1 do
    - 1.3.1 Apply algorithm `SOL2` on $BP[E_j]$;
    - 1.3.2 If `SOL2`$(BP[E_j]) \neq \emptyset$, complete in a greedy way all the colorings produced by `SOL2` on the edges of $\overline{E_j}$. Let $\mathcal{S}_{2,j,i}$ be a best one among these edge colorings of BP;
    - 1.3.3 Apply algorithm `SOL3` on $BP[E_j]$;
    - 1.3.4 If `SOL3`$(BP[E_j]) \neq \emptyset$, complete in a greedy way all the colorings produced by `SOL3` on the edges of $\overline{E_j}$. Let $\mathcal{S}_{3,j,i}$ be a best one among these edge colorings of BP

2 Output $\mathcal{S} = argmin\{val(\mathcal{S}_{1,i}), val(\mathcal{S}_{k,j,i}) : \ k = 2, 3, \ j = 1, \ldots, i, \ i = 1, \ldots, m\}$.

The greedy steps 1.2, 1.2.2 and 1.2.4 give a solution using at most 5 colors. More generally, in [4], we have proved that, in any graph G, the greedy coloring and at least one optimal weighted node coloring use at most $\Delta(G) + 1$ colors, where $\Delta(G)$ denotes the maximum degree of G. In our case, we have $G = L(H)$, the line graph of H, and we deduce $\Delta(L(H)) + 1 \leqslant 2(\Delta(H) - 1) + 1 = 2\Delta(H) - 1$. The 3 algorithms `SOL1`, `SOL2` and `SOL3` are used on several partial graphs BP' of BP. In the following, V', E' and m' denote respectively the node set, the edge set and the number of edge of the current graph BP'. Moreover, we set $\overline{V_i'} = V' \setminus V_i'$. If $M = (M_1, \ldots, M_l)$ with $w(M_1) \geqslant \ldots \geqslant w(M_\ell)$ is an edge coloring of BP', we note $i_j = \min\{k : e_k \in M_j\}$. We assume, for reason of readability, that some colors M_j may be empty (in this case $i_j = m' + 1$). The principle of theses algorithms consist in finding a decomposition of BP' (a subgraph of BP) into two subgraphs BP_1' and BP_2' having each a maximum degree 2. When there exists such a decomposition, we can color BP_i' with at most 2 colors since BP is bipartite.

`SOL1`

1 For $j = m'$ downto 1 do

1.1 If the degree of $BP'[E'_j]$ is at most 2 then

1.1.1 Consider the graph BP'^j induced by the nodes of BP' incident to at least 2 edges of $\overline{E'_j}$ and restricted to the edges of $\overline{E'_j}$.

1.1.2 Determine if there exists a matching M^j of BP'^j such that every node of $\overline{V'_j}$ is saturated;

1.1.3 If such a matching is found, consider the decomposition $BP'_{1,j}$ and $BP'_{2,j}$ of BP' induced by $E'_j \cup M^j$ and $E' \setminus (E'_j \cup M^j)$ respectively;

1.1.4 Find an optimal 2-edge coloring (M_1^j, M_2^j) of $BP'_{1,j}$;

1.1.5 Color greedily the edges of $BP'_{2,j}$ with two colors (M_3^j, M_4^j);

1.1.6 Define $\mathcal{S}_1^j = (M_1^j, M_2^j, M_3^j, M_4^j)$ the edge coloring of BP';

2 Output $\{\mathcal{S}_1^j : \; j = 1, \ldots, m' - 1\}$;

Note that the step 1.1.2 is polynomial. Indeed, more generally, given a graph G and $V' \subseteq V$, it is polynomial to determine if there exists a matching such that each node of V' is matched. To see this, consider G' where we add to G all missing edges between nodes of $V \setminus V'$. If $|V|$ is odd, then we add a node to the clique $V \setminus V'$. It is easy to see that G' has a perfect matching if and only if G has a matching such that each node of V' is saturated.

Lemma 1. *If $\mathcal{S} = (M_1, M_2, M_3, M_4)$ is an edge coloring of BP', then we have $val(\mathcal{S}_1^{i_3-1}) \leqslant w(M_1) + w(M_2) + 2w(M_3)$.*

SOL2

1 For $k = m'$ downto 1 do

1.1 If E'_k is a matching :

1.1.1 Determine if there exists a matching M_k of $BP'[\overline{V'_k}]$ such that each node of $BP'[\overline{V'_k}]$ having a degree 3 in BP' is saturated.

1.1.2 If such a matching is found, consider the decomposition $BP'_{1,k}$ and $BP'_{2,k}$ of BP' induced by $E'_k \cup M_k$ and $E' \setminus (E'_j \cup M_k)$ respectively;

1.1.3 Color $BP'_{1,k}$ with one color M_1^k;

1.1.4 Color greedily $BP'_{2,k}$ with two colors M_2^k and M_3^k;

1.1.5 Define $\mathcal{S}_2^k = (M_1^k, M_2^k, M_3^k)$ the edge coloring of BP';

2 Output $\{\mathcal{S}_2^k : \; k = 1, \ldots, m'\}$;

Lemma 2. *If $\mathcal{S} = (M_1, M_2, M_3)$ is an edge coloring of BP', then we have $val(\mathcal{S}_2^{i_2-1}) \leqslant w(M_1) + 2w(M_2)$.*

SOL3

1 For $k = m'$ downto 1 do
 1.1 Determine if there is a matching M_k in $BP'[\overline{E'_k}]$ such that each node of degree 3 in BP' is saturated.
 1.2 If such a matching is found, consider the decomposition $BP'_{1,k}$ and $BP'_{2,k}$ of BP' induced by M_k and $E' \setminus M_k$ respectively;
 1.3 Color $BP'_{1,k}$ with one color M_3^k;
 1.4 Color greedily $BP'_{2,k}$ with two colors M_1^k and M_2^k;
 1.5 Define $\mathcal{S}_3^k = (M_1^k, M_2^k, M_3^k)$ the edge coloring of BP';
2 Output $\{\mathcal{S}_3^k : \; k = 1, \ldots, m' - 1\}$;

Lemma 3. *If $\mathcal{S} = (M_1, M_2, M_3)$ is an edge coloring of BP', then we have $val(\mathcal{S}_3^{i_3-1}) \leqslant 2w(M_1) + w(M_3)$.*

Theorem 7. *BIPARTITEEDGECOLOR is a $\frac{7}{6}$ approximation for* MIN WEIGHTED EDGE COLORING *in bipartite graphs with maximum degree 3.*

Proof. Let $\mathcal{S}^* = (M_1^*, \ldots, M_5^*)$ with $w(M_1^*) \geqslant \ldots \geqslant w(M_5^*)$ be an optimal weighted edge coloring of BP. Denote by i_k^* the smallest index of an edge in M_k^* ($i_k^* = m+1$ if the color is empty). Consider the iteration of BIPARTITEEDGECOLOR corresponding to the cases $i = i_5^* - 1$ and $j = i_4^* - 1$. Then, applying Lemma 1, we produce on $BP' = BP[E_i]$ an edge coloring of weight at most $w(M_1^*) + w(M_2^*) + 2w(M_3^*)$. Then the greedy coloring of the edges of $\overline{E_i}$ produces a coloring $\mathcal{S}'_1$ of weight $val(\mathcal{S}'_1) \leqslant w(M_1^*) + w(M_2^*) + 2w(M_3^*) + w(M_5^*)$. Applying the same arguments on Lemma 2 and Lemma 3, we produce two solutions $\mathcal{S}'_2$ and $\mathcal{S}'_3$ respectively satisfying $val(\mathcal{S}'_2) \leqslant w(M_1^*) + 2w(M_2^*) + 2w(M_4^*)$ and $val(\mathcal{S}'_3) \leqslant 2w(M_1^*) + w(M_3^*) + 2w(M_4^*)$.

Notice that if there is an empty color produced by one of the algorithms SOLi, then the bounds are still valid. The convex combination of these 3 values with coefficients $\frac{3}{6} \times val(\mathcal{S}'_1)$, $\frac{2}{6} \times val(\mathcal{S}'_2)$ and $\frac{1}{6} \times val(\mathcal{S}'_3)$ gives the expected result.

5 Weighted node coloring in split graphs

The split graphs are a class of graphs related to bipartite graphs. Formally, $G = (K_1, V_2; E)$ is a split graph if K_1 is a clique of G with size $|K_1| = n_1$ and V_2 is an independent set with size $|V_2| = n_2$. So, a split graph can be viewed as a bipartite graph where the left set is a clique. Since split graphs forms a subclass of perfect graphs, the node coloring problem on split graphs is polynomial. On the other hand, in [4], it is proved that the weighted node coloring problem is strongly **NP**-complete in split graphs, even if the weights take only two values. Thus, we deduce that there is no fully polynomial time approximation scheme in such a class of graphs. Here, we propose a polynomial time approximation scheme using structural properties of optimal solutions. An immediate observation of split graphs is that any optimal node coloring $\mathcal{S}^* = (S_1^*, \ldots, S_\ell^*)$ satisfies $|K_1| \leqslant \ell \leqslant |K_1| + 1$ and any color S_i^* is a subset of V_2 with possibly one node of K_1. In particular, for any optimal node coloring $\mathcal{S}^* = (S_1^*, \ldots, S_\ell^*)$, there exists at most one index $i(\mathcal{S}^*)$ such that $S_{i(\mathcal{S}^*)}^* \cap K_1 = \emptyset$.

Lemma 4. *There is an optimal weighted node coloring* $\mathcal{S}^* = (S_1^*, \ldots, S_\ell^*)$ *with* $w(S_1^*) \geqslant \ldots \geqslant w(S_\ell^*)$ *and an index* $i_0 \leqslant \ell + 1$ *such that:*

- $\forall j < i_0 \; S_j^* = \{v_j\} \cup \{v \in V_2 : v \notin \cup_{k=1}^{j-1} S_k^*$ *and* $[v, v_j] \notin E\}$ *for some* $v_j \in K_1$.
- $S_{i_0}^* = V_2 \setminus (S_1^* \cup \ldots \cup S_{i_0-1}^*)$ *and* $\forall j > i_0 \; S_j^* = \{v_j\}$ *for some* $v_j \in K_1$.

Thus, applying an exhaustive search on all sets $K_1' \subseteq K_1$ with $k = |K_1'| \leqslant \lceil \frac{1}{\varepsilon} \rceil$ and on all bijections from $\{1, \ldots, k\}$ to K_1', one can find the k heaviest colors of an optimal weighted node coloring and thus, we deduce:

Theorem 8. MIN WEIGHTED NODE COLORING *admits a polynomial time approximation scheme in split graphs.*

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