Abstract

The cardinality of a maximum minimal dominating set of a graph is called its upper domination number. The problem of computing this number is generally $\text{NP}$-hard but can be solved in polynomial time in some restricted graph classes. In this work, we consider the complexity and approximability of the weighted version of the problem in two special graph classes: planar bipartite, split. We also provide an inapproximability result for unweighted version of this problem in regular graphs.

Keywords: Maximum weighted minimal dominating set (WUDS); $\text{NP}$-hard; inapproximability; planar bipartite; split graphs; UDS in regular graphs.

1 Introduction

Given a simple graph $G = (V, E)$, a dominating set $D \subseteq V$ is a subset of vertices such that $\forall v \notin D, \exists u \in D$ with $[u, v] \in E$. $D$ is said to be a minimal
**dominating set** of\( G \) if \( D \) is minimal for inclusion, i.e., \( \forall v \in D, D - v \) is not a dominating set. For \( x \in V \), \( N_G(x) \) denotes the neighborhood of \( x \) while \( N_G[x] \) denotes the closed neighborhood of \( x \), that is \( N_G[x] = N_G(x) \cup \{x\} \). For \( A \subseteq V \), \( N_G(A) = \bigcup_{x \in A} N_G(x) \) and \( N_G[A] = \bigcup_{x \in A} N_G[x] \). For a dominating set \( D \subseteq V \) and a vertex \( x \in D \), let \( s_D(x) = N_G(x) - N_G[D - x] \) and \( s_D[x] = N_G[x] - N_G[D - x] \). The set \( s_D(x) \) (resp., \( s_D[x] \)) corresponds to the private vertices of \( V - D \) (resp., \( V - D \cup \{x\} \)) that are only dominated by \( x \). In other words, \( s_D(x) = \{ y \notin D : N_G(y) \cap D = \{x\} \} \). It is well known that \( D \) is a minimal dominating set of \( G \) iff \( N_G[D] = V \) and \( \forall x \in D, s_D[x] \neq \emptyset \). Here, we consider simple graph \( G = (V, E) \) where each node \( v \in V \) is weighted by \( w(v) \geq 0 \) a non negative integer. For \( U \subseteq V \), \( w(U) = \sum_{v \in U} w(v) \).

**Definition 1.1** The **Weighted Upper dominating set problem** (WUDS in short) consists in finding, given a simple node weighted graph \( G = (V, E, w) \), a minimal dominating set \( U \) of \( G \) with maximum weight \( w(U) \). The weight of any optimal solution of \( G \) will be denoted by \( \Gamma_w(G) \) and it will be called the weighted upper domination number of \( G \).

2 Related papers

To our best knowledge, the complexity of computing the weighted upper domination number has never been studied in the literature, while several results appear for the unweighted case (corresponding to \( w(v) = 1 \) for every \( v \in V \)).

In this case, the size of any maximum minimal dominating set of \( G \) is usually denoted by \( \Gamma(G) \) and it is known in the literature as the **upper domination number** of \( G \). The complexity of computing \( \Gamma(G) \) has already been studied for the main classes of graphs. For instance, it was shown to be \( \text{NP}-hard \) for general graphs in [6] and \( \text{W}[2]-hard \) in [3], for graphs of maximum degree 3 and planar graphs in [4], co-bipartite graphs in [1]. A dichotomy result for monogenic classes of graphs (i.e., classes defined by a single forbidden induced subgraph) is also given in [1], where it is proved that if the only forbidden induced subgraph is a \( P_4 \) or a \( 2K_2 \) (or any induced subgraph of these graphs), then computing \( \Gamma(G) \) can be solved in polynomial time; otherwise, it is \( \text{NP}-hard \). The first boundary property for upper domination number very recently is given in [2]. The upper domination number is one of the most inapproximable \( \text{NPO} \) problems because it has been proved in [4] that for any \( \varepsilon > 0 \), the upper domination number, even restricted to cobipartite graphs, is not \( n^{1-\varepsilon} \)-approximable, where \( n \) is the number of vertices of the input graph, unless \( \text{P=NP} \). \( \Gamma(G) \) is also related to \( \alpha(G) \), the size of any maximum independent set of \( G \) because we always get \( \alpha(G) \leq \Gamma(G) \). The inequality is tight.
Fig. 1. A weighted path graph $P$ on four vertices such that $\alpha_w(P) = 3$ and $\Gamma_w(P) = 4$.

for bipartite graphs [7] and chordal graphs [8]. In particular, computing $\Gamma(G)$ can be done in linear time in these classes of graphs. The main reason is that, for these graphs, we can (polynomially) convert any minimal dominating set into a stable set of same size. Dealing with the weighted upper domination number, the tightness of this equality is less clear although for any node-weighted graph $G = (V, E, w)$, we have $\alpha_w(G) \leq \Gamma_w(G)$. Actually, already for an induced path $P$, we may have $\alpha_w(P) < \Gamma_w(P)$, see Figure 1. In [3], the following bounds between independent number and upper dominating number are given for any connected graph $G$ with $n$ vertices: $\Gamma(G) \leq \frac{n}{2} + \frac{\chi(G)}{2}$. We cannot generalize this bound to weighted case, even by replacing $n$ by $w(V)$.

However, we now propose simple bounds which are indeed tights. For any weighted graph $(G, w)$ with non-negative weights, we have:

$$\alpha_w(G) \leq \Gamma_w(G) \leq \chi(G) \alpha_w(G)$$

where $\chi(G)$ denotes the chromatic number of $G$. The first bound holds because any maximal independent set is a minimal dominating set and the weights are non negative. For the second bound, $\Gamma_w(G) \leq w(V) \leq \chi(G) \alpha_w(G)$. The tightness of the first bound is already known for all unweighted bipartite or chordal graphs [7,8]. For the other bound, consider $(G_p, w)$ where $G_p = (V_p, E)$ is isomorphic to $K_p \bigoplus pK_2$ which means that $V_p = \{v_i, v'_i : i = 1, \ldots, p\}$ where $\{v_i, v'_i : i = 1, \ldots, p\}$ is a clique and $\{[v_i, v'_i] : i = 1, \ldots, p\}$ is an induced matching. Finally, for $i \leq p$, $w(v_i) = 1$ and $w(v'_i) = 0$. We have $\chi(G_p) = p$, $\alpha_w(G_p) = 1$ and $\Gamma_w(G_p) = p$.

In this paper, we will prove that WUDS remains NP-hard even in very restrictive cases of bipartite and split graphs. In particular, dealing with $k$-partite graphs with $k$ constant, inequalities (1) allow us to conclude that WUDS is $k$-approximable, while we will show that for split graphs WUDS is hard to approximate with a ratio $n^{1-\epsilon}$ where $n$ is the number of vertices. We will consider the complexity of WUDS in split and in bipartite graphs because for the unweighted case, (called UDS here) is known to be solvable in polynomial-time. A split graph (resp. bipartite graph) $G = (L, R, E)$ is a simple graph where the vertex set is partitioned into $L \cup R$ such that the subgraph induced by $L$ is a clique and $R$ is a stable set (resp. $L$ and $R$ are
stable sets). A split graph is called a $p$-subregular split graph if for $\ell \in L$, $d_G(\ell) - |L| + 1 \leq p$ and for $r \in R$, $d_G(r) \leq p$. This is a subclass of chordal graphs, and then it satisfies $\Gamma(G) = \alpha(G)$ [8].

**Theorem 2.1** Computing $\Gamma_w(G)$ is strongly NP-hard even for subcubic split graphs with bivalued weights.

**Proof.** The reduction is done from the maximum induced matching problem (MIM in short) in subcubic bipartite graphs.

<table>
<thead>
<tr>
<th>Maximum Induced Matching (MIM)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong> A graph $G = (V, E)$.</td>
</tr>
<tr>
<td><strong>Goal:</strong> Find a maximum induced matching of $G$.</td>
</tr>
</tbody>
</table>

The inputs of the decision version of MIM is a simple graph $G = (V, E)$ and an integer $k$. We want to decide if there is an induced matching of $G$ with a size at least $k$. An induced matching of a graph $G$ is a matching $M$ such that every two vertices that are incident with distinct edges in $M$ are non-adjacent. The maximum number of edges in an induced matching of $G$ is known as the strong matching number and is denoted by $\nu_s(G)$. MIM has been proved NP-hard for planar bipartite graphs even if each vertex in one partite set has degree 2 and each vertex in the other partite set has degree 3 [9].

Let $G = (L, R, E)$ and an integer $k$ be an instance of MIM where $G$ is a subcubic bipartite graph with $n$ vertices. We build an instance $G' = (L, R, E', w)$ of WUDS where $G'$ is a node weighted subcubic split graph as follows:

- $G'$ contains $G$ and we complete the set $L$ into a clique, i.e., $E' = \{[u, v] : u, v \in L\} \cup E$.
- $w(v) = n$ for $v \in L$ and $w(v) = 1$ for $v \in R$.

Clearly, this construction can be done within polynomial time and $G'$ is a 3-subcubic split graph where $w$ is bivalued.

**Claim 2.2** There is an induced matching of $G$ with a size at least $k \geq 1$ iff there is a UDS of $G'$ with weight at least $kn \geq n$.

Theorem 2.1 is the stronger hardness result that we can produce because we can prove either for 2-subregular split graphs or for onevalued weights, WUDS becomes polynomial-time solvable. In [10], it is proved that the maximum induced matching problem is not approximable to ratio $O(n^{1-\varepsilon})$ in general.
graphs (actually, the bound $n^{1/2-\varepsilon}$ is indicated in [10], but the right bound derived from the proof is $O(n^{1-\varepsilon})$). Recently, this lower bound has been also given for bipartite graphs [5]. Hence, we deduce:

**Corollary 2.3** For any $\varepsilon > 0$, WUDS is not $O(n^{1-\varepsilon})$-approximable in split graphs on $n$ vertices, even for bivalued weights, unless $NP \neq ZPP$.

It is easy to produce a weighted UDS with performance ratio $O(n)$ (in particular, in bipartite graphs) by taking any maximal independent set containing a node of maximum weight $w_{\text{max}}$.

**Theorem 2.4** Computing $\Gamma_w(G)$ is strongly $NP$-hard for planar bipartite graphs of maximum degree 4, even for trivalued weights.

**Corollary 2.5** WUDS is $APX$-complete in bipartite graphs.

### 2.1 Unweighted Upper dominating set problem in regular graphs

We end the hard cases, by studying the inapproximability of UDS the unweighted case, i.e., $w(v) = 1$ in regular graphs. In [4], the following bounds are given for any connected graph $G$ of $n$ vertices with minimum degree $\delta$ and maximum degree $\Delta$. In particular, for connected regular graphs of degree $\Delta$, inequality (2) becomes $\alpha(G) \leq \Gamma(G) \leq \frac{n}{2}$ which may lead to the conclusion that UDS is easier to solve in regular graphs. Here, we prove that for regular graphs, **Unweighted Upper dominating set problem** is not $\Delta^{1/2-\varepsilon}$-approximable, where $\Delta$ is the degree of each vertex, unless $P=NP$. Hence, it is the first inapproximability result depending on the maximum degree $\Delta$ for UDS.

$$\alpha(G) \leq \Gamma(G) \leq \max \left\{ \alpha(G), \frac{n}{2} + \frac{\alpha(G)(\Delta - \delta)}{2\Delta} - \frac{\Delta - \delta}{\Delta} \right\}$$  \hspace{1cm} (2)

**Theorem 2.6** For any constant $\varepsilon > 0$, unless $NP \subseteq ZPTIME(n^{\text{poly log} n})$, it is hard to approximate UDS on $\Delta$-regular graphs to within a factor of $\Delta^{1/2-\varepsilon}$.

### References


