

# The minimum reload $s$ - $t$ path/trail/walk problems

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**Abstract.** This paper deals with problems on non-oriented edge-colored graphs. The aim is to find a route between two given vertices  $s$  and  $t$ . This route can be a walk, a trail or a path. Each time a vertex is crossed by a walk there is an associated non-negative *reload* cost  $r_{i,j}$ , where  $i$  and  $j$  denote, respectively, the colors of successive edges in this walk. The goal is to find a route whose total reload cost is minimum. Polynomial algorithms and proofs of **NP**-hardness are given for particular cases: when the triangle inequality is satisfied or not, when reload costs are symmetric (i.e.  $r_{i,j} = r_{j,i}$ ) or asymmetric. We also investigate bounded degree graphs and planar graphs.

## 1 Introduction

In the last few years a great number of applications have been modelled as problems in edge-colored graphs. More recently, we find interesting applications involving edge-colored graphs and connection (or reload) costs arising at each vertex, according to the pair of colors of the edges used by a walk through that vertex [8, 5, 6, 4]. Despite their apparent importance in the telecommunications and transportation industry, reload costs have not been extensively studied in the literature.

In [8, 4], each color is viewed as a subnetwork and is used to model a cargo transportation network which uses different means of transportation or data transmission costs arising in large communication networks. In all these models, the transportation or communication costs between the subnetworks usually dominate the costs within individual subnetworks. Some applications in satellite networks are also discussed in [5] where the various subnetworks may represent different products offered by the commercial satellite service providers. In [5], terrestrial satellite dishes are required to first capture the radio signals and then special electric-to-fiber converters are required to transform the electric signals from the satellite dishes to optical pulses that can be sent over optical

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fibers. These interface costs are referred to as reload costs and depend on the technologies being connected. As another example, suppose at a road network with many tolls one may be interested in paying as little as possible. Each different road between points of toll will be associated to a different color. In addition, suppose we pay a fee (reload cost) each time we change from one road to another. The objective in this case will be to find a path between two different places minimizing the total toll cost (we will call it, the minimum toll cost  $s$ - $t$  path problem).

In our case, we will be particularly concerned with problems involving reload paths, trails and walks with both symmetric and asymmetric costs between a fixed source  $s$  and destination  $t$  (with obvious applications in transportation and communication networks as mentioned above).

### 1.1 The model

Let  $I_c = \{1, 2, \dots, c\}$  be a given set of colors with  $c \geq 2$ . In this work,  $G^c$  denotes a simple connected non-oriented edge-colored graph containing two particular vertices  $s$  and  $t$ . Each edge has a color of  $I_c$ .

We recall here some standard graph terminology: the vertex and edge sets of  $G^c$  are denoted by  $V(G^c)$  and  $E(G^c)$ , respectively. The order of  $G^c$  is the number  $n$  of its vertices and the size of  $G^c$  is the number  $m$  of its edges. For a given color  $i$ ,  $E^i(G^c)$  denotes the set of edges of  $G^c$  colored by  $i$ . We denote by  $N_{G^c}(x)$  the set of all neighbors of  $x$  in  $G^c$ , and by  $N_{G^c}^i(x)$ , the set of vertices of  $G^c$ , linked to  $x$  with edges colored by  $i$ . The degree of  $x$  in  $G^c$  is  $d_{G^c}(x) = |N_{G^c}(x)|$  and the maximum degree of  $G^c$ , denoted by  $\Delta(G^c)$ , is  $\Delta(G^c) = \max\{d_{G^c}(x) : x \in V(G^c)\}$ .

A non-oriented edge between two vertices  $x$  and  $y$  is denoted by  $[x, y]$  while its color is denoted by  $c([x, y])$ . Given a graph  $G = (V, E)$ , a *walk*  $\rho$  from  $s$  to  $t$  in  $G$  (called  $s$ - $t$  walk) is a sequence  $\rho = (v_0, e_0, v_1, e_1, \dots, e_k, v_{k+1})$  where  $v_0 = s$ ,  $v_{k+1} = t$  and  $e_i = [v_i, v_{i+1}]$  for  $i = 0, \dots, k$ . A *trail* from  $s$  to  $t$  in  $G$  (called  $s$ - $t$  trail) is a walk  $\rho = (v_0, e_0, v_1, e_1, \dots, e_k, v_{k+1})$  from  $s$  to  $t$  where  $e_i \neq e_j$  for  $i \neq j$ . A *path* from  $s$  to  $t$  in  $G$  (called  $s$ - $t$  path) is a trail  $\rho = (v_0, e_0, v_1, e_1, \dots, e_k, v_{k+1})$  from  $s$  to  $t$  where  $v_i \neq v_j$  for  $i \neq j$ .

We are also given a  $c \times c$  matrix  $R = [r_{i,j}]$  (for  $i, j \in I_c$ ) whose entries define the *reload cost* (or *connection cost*) when changing color  $i$  for color  $j$ . It is assumed that each entry  $r_{i,j}$  of  $R$  is a non-negative integer (i.e.  $r_{i,j} \in \mathbb{N}$ ). Here, we will both consider symmetric and asymmetric matrices. We say that a matrix  $R$  satisfies the *triangle inequality*, if and only if, for all edges  $e_i, e_j, e_k \in E(G^c)$  which are adjacent to a common vertex, we have  $r_{c(e_i), c(e_j)} \leq r_{c(e_i), c(e_k)} + r_{c(e_k), c(e_j)}$  (see [8, 4] for similar definitions). Given a path/trail/walk  $\rho = (v_1, e_1, v_2, e_2, \dots, e_k, v_{k+1})$  between vertices  $s$  and  $t$ , we define the reload cost of  $\rho$  as:

$$r(\rho) = \sum_{j=1}^{k-1} r_{c(e_j), c(e_{j+1})} \quad (1)$$

Given a (non-colored) digraph  $D = (V, \overrightarrow{E})$ , an arc from  $x$  to  $y$  is denoted by  $(x, y)$ . The length of the path, trail or walk  $\rho_c$  in  $G^c$  (resp.  $D$ ), denoted by  $|\rho_c|$ , is the number of its edges (resp. arcs).

An instance of the *minimum reload  $s$ - $t$  path/trail/walk* consists of a simple connected edge-colored graph  $G^c$ , a pair  $s, t \in V(G^c)$  and a  $c \times c$  matrix  $R = [r_{i,j}]$  associating a non-negative cost to each pair of colors. The objective is to find a path/trail/walk  $\rho$  between  $s$  and  $t$  with minimum reload cost. For instance, in the minimum toll cost  $s$ - $t$  path problem,  $r_{i,j} = r_j$  for  $i, j \in I_c$  with  $i \neq j$  and  $r_{i,i} = 0$  where every  $r_j$  is a non-negative integer. Finally, notice that if  $c = 1$  (i.e. there is only one color in  $G^c$ ), these 3 problems are equivalent to finding an  $s$ - $t$  path of minimum length in  $G^c$ . Thus, we will assume  $c \geq 2$ .

## 1.2 Related work

To the best of our knowledge, reload cost optimization has been mainly studied in the context of spanning trees [8, 5, 6, 4], but also very recently for some variants of paths, tours and flow problems [2]. In [8], the authors consider the problem of finding a spanning tree of minimum diameter with respect to the reload costs and they propose inapproximability results for graphs of maximum degree 5 and polynomial results for graphs of maximum degree 3. In [4], the author discusses inapproximability results for the same problem when restricted to graphs with maximum degree 4. In [5, 6], the authors give several formulations with computational results to solve the reload cost spanning tree problem.

In [2], the authors consider several models for reload cost paths, tours and flow problems, which have several applications in telecommunication, transportation, and energy distribution. In particular, they study the following model: given a directed arc-colored graph  $D^c = (V, \overrightarrow{E})$  where each arc  $e \in \overrightarrow{E}$  has a non-negative cost  $w(e)$  and a color  $c(e) \in I_c$ , and given a non-negative integer reload cost matrix  $R = [r_{i,j}]$  for  $i, j \in I_c$ , they want to find an oriented  $s$ - $t$  trail  $\rho = (s, e_1, v_1, e_2, \dots, e_k, t)$  of  $D^c$  minimizing  $\sum_{i=1}^k w(e_i) + \sum_{i=1}^{k-1} r_{c(e_i), c(e_{i+1})}$ . They prove that this problem, called here *the minimum reload+weight directed  $s$ - $t$  trail problem*, is solvable in polynomial time.

The minimum reload  $s$ - $t$  path (resp., trail) problem is also related to the problem of deciding whether a simple connected edge-colored graph  $G^c$  has a properly edge-colored  $s$ - $t$  path (resp.,  $s$ - $t$  trail). A path or trail in  $G^c$  is called properly edge-colored if each pair of successive edges differ in color. The properly edge-colored  $s$ - $t$  path (or trail) problem and some of its variants, including the determination of a longest properly edge-colored  $s$ - $t$  path (or trail) for a particular class of graphs, and the determination of two or more properly edge-colored  $s$ - $t$  paths (or trails), have been considered in [1]. For instance, if we set for the reload cost  $r_{i,i} = 1$  and  $r_{i,j} = 0$  for  $i, j \in I_c$  with  $i \neq j$ , then there exists an  $s$ - $t$  path (resp.,  $s$ - $t$  trail) with reload cost 0 in  $G^c$ , if and only if,  $G^c$  has a properly edge-colored  $s$ - $t$  path (resp., trail).

### 1.3 Contribution and organization of the paper

In Section 2, we discuss the case of finding a minimum reload  $s$ - $t$  walk, either with symmetric or asymmetric reload cost matrix. In Section 3 we deal with paths and trails when reload costs are symmetric. We prove that the minimum reload  $s$ - $t$  trail problem can be solved in polynomial time for every  $c \geq 2$ . In addition, we show that the minimum reload  $s$ - $t$  path problem is polynomially solvable either if  $c = 2$  and the triangle inequality holds (here  $R$  is not necessarily a symmetric matrix) or if  $G^c$  has a maximum degree 3. However it is **NP**-hard when  $c \geq 3$ , even for graphs of maximum degree 4 and reload cost matrix satisfying the triangle inequality. We conclude the section by showing that, if  $c \geq 4$  and the triangle inequality is satisfied, the minimum symmetric reload  $s$ - $t$  path problem remains **NP**-hard even for planar graphs with maximum degree 4. In Section 4 we deal with asymmetric reload costs. For a reload cost matrix satisfying the triangle inequality, we construct a polynomial time procedure for the minimum reload  $s$ - $t$  trail problem and we prove that the minimum asymmetric reload  $s$ - $t$  trail problem is **NP**-hard even for graphs with 3 colors and maximum degree equal to 3. Table 1 summarizes the main results given in the paper.

**Table 1.** Summary of main results.

	Polynomial time problems	NP-hard problems
walk	all cases	
trail	(sym. R)	$(\text{asym. R}) \wedge (\Delta(G^c) = 3) \wedge (c = 3)$
	$(\text{asym. R}) \wedge (\text{triangle ineq.})$	
path	$(c = 2) \wedge (\text{triangle ineq.})$	$(\text{sym. R}) \wedge (\Delta(G^c) = 4) \wedge (c \geq 3) \wedge (\text{triangle ineq.})$
	$(\text{sym. R}) \wedge (\Delta(G^c) \leq 3)$	$(\text{sym. R}) \wedge (G^c \text{ is planar}) \wedge (\Delta(G^c) = 4) \wedge (c \geq 4) \wedge (\text{triangle ineq.})$

## 2 Walks with reload costs

Choosing a walk instead of a path can help in reducing the reload costs. For instance, Figure 1 illustrates two different  $s$ - $t$  walks,  $\rho_1 = (s, e_1, v_1, e_2, v_2, e_2, v_1, e_3, t)$  and  $\rho_2 = (s, e_1, v_1, e_3, t)$ , with reload costs  $r_{i,j} = 1$  for  $i, j \in \{1, 2, 3\}$  except for  $r_{1,3} = r_{3,1} = 4$ . The reload cost of  $\rho_2$  is  $r(\rho_2) = r_{1,3} = 4$  whereas the reload cost of  $\rho_1$  is  $r(\rho_1) = r_{1,2} + r_{2,2} + r_{2,3} = 3$ . Notice that the minimum reload cost of an  $s$ - $t$  walk is a lower bound on the minimum reload cost of an  $s$ - $t$  trail which is a lower bound on the minimum reload cost of an  $s$ - $t$  path since a path is a trail which is a walk.

We already know that the minimum reload  $s$ - $t$  walk problem is polynomial since we have:

*Remark 1.* There is a polynomial reduction from the minimum reload  $s$ - $t$  walk problem to the minimum reload+weight directed  $s$ - $t$  trail problem (see Subsection 1.2 for a description of this problem).

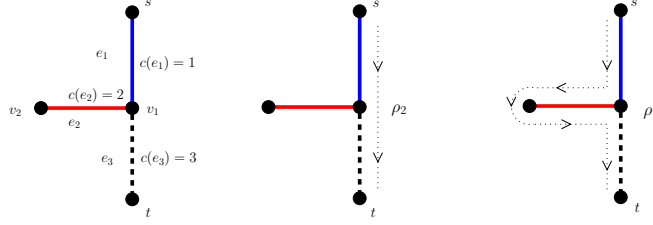


Fig. 1. Two different reload  $s$ - $t$  walks.

Thus, using the result of [2], our result follows. Here, we propose another polynomial method to solve the minimum reload  $s$ - $t$  walk problem. Notice that the construction used differs from the one given in [2] for solving the minimum reload+weight directed  $s$ - $t$  trail problem.

Let  $G^c$  with  $V(G^c) = \{s, t\} \cup \{v_1, \dots, v_n\}$  be a simple edge-colored connected graph with colors in  $I_c$ . We reduce the minimum  $s$ - $t$  walk problem to the computation of a shortest  $s_0$ - $t_0$  path in an auxiliary digraph  $H = (V', \vec{E}')$  whose arcs are weighted by  $w$ . The digraph  $H$  contains  $|I_c|$  directed subgraphs  $H_\ell$  for  $\ell \in I_c$ . The vertex set of each subgraph  $H_\ell$  is  $\{v_1^\ell, \dots, v_n^\ell\}$ . There is an arc from  $v_i^\ell$  to  $v_k^{\ell'}$  if and only if, there is a walk  $(v_j, e_1, v_i, e_2, v_k)$  in  $G^c$  such that  $c(e_1) = \ell$  and  $c(e_2) = \ell'$ . This construction can be done within polynomial time. An example is given in Figure 2.

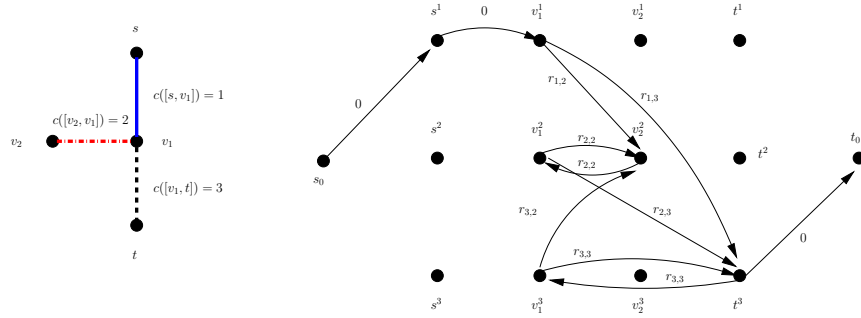


Fig. 2. Transformation of  $G^c$  into a digraph  $H$ .

Formally, the digraph  $H$  is built as follows:

- $V' = \{s_0, t_0\} \cup \{s^\ell, v_1^\ell, \dots, v_n^\ell, t^\ell : \ell \in I_c\}$
- For any pair of edges  $[v_j, v_i] \in E^\ell(G^c)$  and  $[v_i, v_k] \in E^{\ell'}(G^c)$ , with  $\ell, \ell' \in I_c$  and  $v_i \in V(G^c) \setminus \{s, t\}$  (possibly with  $v_j = v_k$ ), add arcs  $(v_i^\ell, v_k^{\ell'})$  and  $(v_i^{\ell'}, v_j^\ell)$  to  $\vec{E}'$ . Now, update  $\vec{E}'$ , by deleting all incoming (resp., outgoing) arcs to  $s^\ell$

- (resp., to  $t^\ell$ ) for every  $\ell \in I_c$ . Moreover, add arc  $(s_0, s^\ell)$  to  $\overrightarrow{E}^j$  (resp.,  $(t^\ell, t_0)$  to  $\overrightarrow{E}^j$ ), if and only if, there exists  $[s, v_i] \in E^\ell(G^c)$  (resp.,  $[v_i, t] \in E^\ell(G^c)$ ).
- If  $v_i \neq s, t$  and  $v_j \neq s, t$ , then  $w(v_i^{\ell'}, v_j^\ell) = r_{\ell', \ell}$  for arc  $(v_i^{\ell'}, v_j^\ell) \in \overrightarrow{E}^j$ . If  $v_i \in \{s, t\}$  or  $v_j \in \{s, t\}$ , then  $w(v_i^{\ell'}, v_j^\ell) = 0$  for arc  $(v_i^{\ell'}, v_j^\ell) \in \overrightarrow{E}^j$ . Finally,  $w(s_0, s^\ell) = 0$  for arc  $(s_0, s^\ell) \in \overrightarrow{E}^j$  and  $w(t^\ell, t_0) = 0$  for arc  $(t^\ell, t_0) \in \overrightarrow{E}^j$ .

**Theorem 1.** *For any simple connected edge-colored graph  $G^c$  and any pair  $s, t$  of vertices of  $G^c$ , the minimum reload  $s$ - $t$  walk problem can be solved in polynomial time.*

### 3 Paths and trails with symmetric reload costs

Let  $R$  be a symmetric matrix with non-negative integer reload costs. Here, we prove that the minimum reload  $s$ - $t$  trail problem can be solved in polynomial time for every  $c \geq 2$ . In addition, we show that the minimum reload  $s$ - $t$  path problem can be solved in polynomial time either if  $c = 2$  and the triangle inequality holds (here  $R$  is not necessarily a symmetric matrix) or if  $G^c$  has a maximum degree 3. However the problem is **NP**-hard when  $c \geq 3$  for graphs satisfying the triangle inequality and with maximum degree equal to 4. We conclude the section by showing that, if  $c \geq 4$  and the triangle inequality is satisfied, the minimum reload  $s$ - $t$  path problem remains **NP**-hard even for planar graphs with maximum degree 4.

In the sequel, we show how to turn the minimum reload  $s$ - $t$  trail problem into a minimum perfect matching problem in a weighted non-colored graph  $G$  defined as follows.

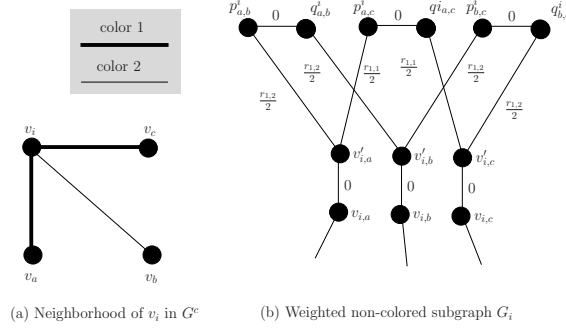
Given two vertices  $s$  and  $t$  in  $V(G^c) = \{v_1, \dots, v_n\}$ , set  $W = V(G^c) \setminus \{s, t\}$ . Now, for each  $v_i \in W$ , we define a subgraph  $G_i$  with vertex and edge sets as illustrated in Figure 3. Formally:

- $V(G_i) = \{v_{i,j}, v'_{i,j} : v_j \in N_{G^c}(v_i)\} \cup \{p_{j,k}^i, q_{j,k}^i : j < k \text{ and } v_j, v_k \in N_{G^c}(v_i)\}$
- $E(G_i) = \{\{v_{i,j}, v'_{i,j}\} : v_j \in N_{G^c}(v_i)\} \cup \{\{v'_{i,j}, p_{j,k}^i\}, \{p_{j,k}^i, q_{j,k}^i\}, \{q_{j,k}^i, v'_{i,k}\} : j < k \text{ and } v_j, v_k \in N_{G^c}(v_i)\}$

The non-colored graph  $G = (V', E')$  edge weighted by  $w$  is constructed as follows:

- $V' = \{s', t'\} \cup (\bigcup_{v_i \in W} V(G_i))$ , and
- $E' = \{\{v_{i,j}, v_{x,y}\} : j = x \text{ and } i = y\} \cup \{\{s', v_{i,j}\} : v_j = s \text{ and } [v_i, v_j] \in E(G^c)\} \cup \{\{v_{i,j}, t'\} : v_j = t \text{ and } [v_i, v_j] \in E(G^c)\}$
- $w(\{v'_{i,j}, p_{j,k}^i\}) = \frac{1}{2}r_{c([v_i, v_j]), c([v_i, v_k])}$ ,  $w(\{v'_{i,k}, q_{j,k}^i\}) = \frac{1}{2}r_{c([v_i, v_k]), c([v_i, v_j])}$  and all remaining edges have a weight 0.

After  $G$  is constructed, we have to find a minimum weighted perfect matching  $M^*$  in  $G$ . A matching is a set of edges pairwise non adjacent. A matching  $M$  is called perfect if  $M$  touches all vertices of  $G$ . The weight of matching  $M$



**Fig. 3.** Reduction of the minimum reload  $s$ - $t$  trail to a minimum perfect matching.

is  $w(M) = \sum_{e \in M} w(e)$  (computing a minimum weighted perfect matching is polynomial, see [7] for a good reference on general matchings). Notice that  $G$  has a perfect matching since  $G^c$  is assumed to be connected. We can prove that perfect matchings in  $G$  will be associated to reload  $s$ - $t$  trails in  $G^c$  and vice-versa. Formally:

**Theorem 2.** *For any simple connected edge-colored graph  $G^c$  and any pair  $s, t$  of vertices of  $G^c$ , the minimum reload  $s$ - $t$  trail problem can be solved in polynomial time.*

**Corollary 1.** *For any simple connected edge-colored graph  $G^c$  of maximum degree 3 and any pair  $s, t$  of vertices of  $G^c$ , the minimum symmetric reload  $s$ - $t$  path problem can be solved in polynomial time.*

Now, we deal with graphs  $G^c$  colored with two colors. We show that the minimum reload  $s$ - $t$  path problem is polynomial if the reload cost matrix  $R$  satisfies the triangle inequality. In the sequel, notice that  $R$  is not assumed to be symmetric.

**Theorem 3.** *For any simple connected edge-colored graph  $G^c$  with  $c = 2$  colors and such that the associated matrix  $R$  of reload costs satisfies the triangle inequality and for any pair  $s, t$  of vertices of  $G^c$ , the minimum symmetric reload  $s$ - $t$  path problem can be solved in polynomial time.*

A possible application of Theorem 3 is the following. Consider a  $(2, 2)$ -matrix  $R$  satisfying  $r_{1,1} = r_{2,2} = 0$ . It is easy to see that the involved reload cost matrix  $R$  satisfies the triangle inequality, and then one can apply Theorem 3 (on the other hand, this restriction becomes **NP**-hard for a  $(3, 3)$ -matrix with  $r_{i,i} = 0$ , see the proof of item (i) of Corollary 2). We also deduce that the minimum toll cost  $s$ - $t$  path problem (see Section 1) is polynomial for two colors since it is a subproblem of the case considered above. Notice that, the minimum toll cost  $s$ - $t$  path problem for  $r_j = 1 \forall j \in I_c$ , is equivalent to minimizing the number of color flips in an  $s$ - $t$  path. Actually, the minimum toll cost  $s$ - $t$  path problem is polynomially solvable (without constraints on the number of colors).

**Theorem 4.** *For any simple connected edge-colored graph  $G^c$  and any pair  $s, t$  of vertices of  $G^c$ , the minimum toll cost  $s$ - $t$  path problem can be solved in polynomial time.*

Now, we show that the previous restrictions on  $G^c$  are almost the best ones to obtain polynomial cases for the minimum reload  $s$ - $t$  path problem.

**Theorem 5.** *The minimum symmetric reload  $s$ - $t$  path problem is **NP**-hard if  $c \geq 3$ , the triangle inequality holds and the maximum degree of  $G^c$  is equal to 4.*

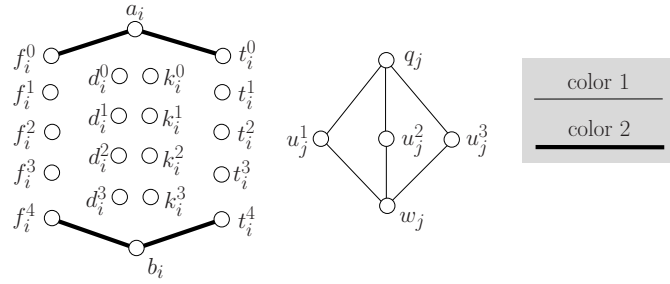
*Proof.* We are given a set  $\mathcal{C}$  of CNF clauses defined over a set  $\mathcal{X}$  of boolean variables. An instance of the  $(3, B2)$ -SAT problem, called 2-balanced 3-SAT, is such that each clause has exactly 3 literals, each of them appearing exactly 4 times in the clauses twice negated and twice unnegated. Deciding whether an instance of  $(3, B2)$ -SAT is satisfiable is **NP**-complete [3]. We are going to reduce  $(3, B2)$ -SAT to the existence of an  $s$ - $t$  path with reload cost at most  $L$ . Let  $\mathcal{I}$  be an instance of  $(3, B2)$ -SAT. We say that  $C_j$  is the  $h$ -th clause of  $x_i$  iff  $x_i$  appears in  $C_j$  and  $x_i$  appears in exactly  $h - 1$  other clauses  $C_{j'}$  with  $j' < j$ . We say that  $x_i$  is the  $\ell$ -th variable of  $C_j$ , if and only if,  $x_i$  and exactly  $\ell - 1$  other variables  $x_{i'}$  with  $i' < i$  appear in  $C_j$ . We build  $G^c$  - instance of the  $s$ - $t$  path with reload cost at most  $L$  - as follows. We have  $I_c = \{1, 2, 3\}$  and  $L = 11|\mathcal{X}| + 3|\mathcal{C}|$ . The matrix  $R$  is defined as  $r_{1,2} = r_{2,1} = M$  where  $M > L$ . The other entries of  $R$  are set to 1.

The graph  $G^c$  has a source vertex  $s$  and a sink vertex  $t$ . In addition, for each  $x_i \in \mathcal{X}$  (resp.  $C_j \in \mathcal{C}$ ) we build a gadget as depicted on the left (resp. right) of Figure 4. The gadget of a variable  $x_i$  consists of a left part (vertices  $f_i^0$  to  $f_i^4$  and  $d_i^0$  to  $d_i^3$ ), a right part (vertices  $t_i^0$  to  $t_i^4$  and  $k_i^0$  to  $k_i^3$ ), an entrance  $a_i$  and an exit  $b_i$ . The left (resp. right) part corresponds to the case where  $x_i$  is set to FALSE (resp. TRUE). The gadget of a clause  $C_j$  consists of an entrance  $q_j$ , an exit  $w_j$  and three vertices  $u_j^1$ ,  $u_j^2$ , and  $u_j^3$  which correspond to the first, second and third variable of  $C_j$  respectively. We link the gadgets by adding the following edges (see Figure 5):

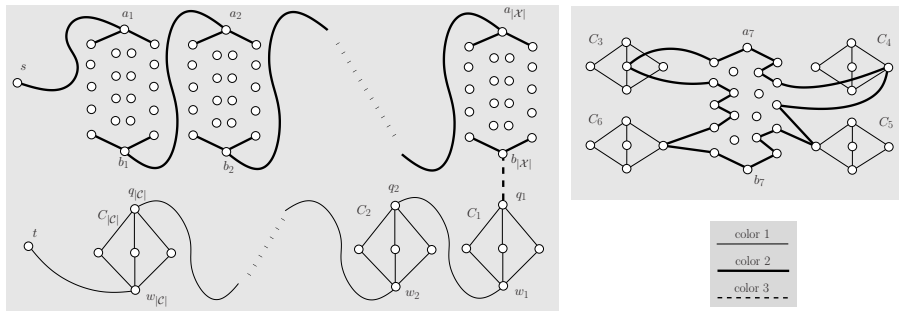
- $[s, a_1], [b_1, a_2], [b_2, a_3], \dots, [b_{|\mathcal{X}|-1}, a_{|\mathcal{X}|}]$  with color 2 (bold) ;
- $[b_{|\mathcal{X}|}, q_1]$  with color 3 (dashed) ;
- $[w_1, q_2], [w_2, q_3], \dots, [w_{|\mathcal{C}|-1}, q_{|\mathcal{C}|}], [w_{|\mathcal{C}|}, t]$  with color 1 (thin).

For each pair  $x_i, C_j$  such that  $x_i$  is the  $\ell$ -th variable of  $C_j$  and  $C_j$  is the  $h$ -th clause of  $x_i$  we proceed as follows. If  $x_i$  appears negated in  $C_j$  then add  $[t_i^{h-1}, u_j^\ell]$ ,  $[t_i^h, u_j^\ell]$ ,  $[f_i^{h-1}, d_i^{h-1}]$  and  $[f_i^h, d_i^{h-1}]$  with color 2 (bold). If  $x_i$  appears unnegated in  $C_j$  then add  $[f_i^{h-1}, u_j^\ell]$ ,  $[f_i^h, u_j^\ell]$ ,  $[t_i^{h-1}, k_i^{h-1}]$  and  $[t_i^h, k_i^{h-1}]$  with color 2 (bold). It is not difficult to see that each vertex's degree is at most 4. Moreover the triangle inequality holds.

Since  $r_{1,2} > L$  and  $r_{2,1} > L$ , any  $s$ - $t$  path  $\rho_c$  with reload cost at most  $L$  starts at  $s$ , enters the gadget of  $x_1$  and visits the variable-gadgets in turn. When  $b_{|\mathcal{X}|}$  is reached,  $\rho_c$  uses  $[b_{|\mathcal{X}|}, q_1]$  and visits the clause-gadgets in turn. Finally  $t$  is reached from  $w_{|\mathcal{C}|}$ . Then exactly 11 (resp., 3) vertices are visited when passing through a variable-gadget (resp., a clause-gadget).



**Fig. 4.** Gadgets for a variable  $x_i$  (left) and a clause  $C_j$  (right).



**Fig. 5.** Left: How the gadgets are linked. Right: How to link the gadget of  $x_7$  if it appears in  $C_3 = (x_1 \vee x_7 \vee x_8)$ ,  $C_4 = (\bar{x}_3 \vee x_5 \vee \bar{x}_7)$ ,  $C_5 = (\bar{x}_7 \vee \bar{x}_8 \vee x_9)$  and  $C_6 = (\bar{x}_1 \vee \bar{x}_6 \vee x_7)$ .

If a truth assignment  $\tau$  satisfies  $\mathcal{I}$  then  $G^c$  admits an  $s$ - $t$  path  $\rho_c$  with reload cost  $11|\mathcal{X}| + 3|\mathcal{C}|$ . Indeed, if  $x_i$  is FALSE (resp. TRUE) in  $\tau$  then  $\rho_c$  goes across the left (resp. right) part of  $x_i$ 's gadget. Since  $\tau$  satisfies  $\mathcal{I}$  we know that at least one literal per clause is true. If the  $\ell$ -th literal of  $C_j$  is true (choose  $\ell$  arbitrarily if it is not unique) then  $\rho_c$  passes through  $u_j^\ell$ . Conversely an  $s$ - $t$  path  $\rho_c$  with reload cost  $11|\mathcal{X}| + 3|\mathcal{C}|$  induces a truth assignment that satisfies  $\mathcal{I}$ : set  $x_i$  to FALSE (resp. TRUE) if  $\rho_c$  passes through the left (resp. right) part of  $x_i$ 's gadget.  $\square$

**Corollary 2.** *The two following statements hold:*

- (i) *In the general case, the minimum symmetric reload  $s$ - $t$  path problem is not approximable at all if  $c \geq 3$ , the triangle inequality holds and the maximum degree of  $G^c$  is equal to 4.*
- (ii) *If  $r_{i,j} \geq 1$  for every  $i, j \in I_c$ , the minimum symmetric reload  $s$ - $t$  path problem is not  $O(2^{P(n)})$ -approximable for every polynomial  $P$  if  $c \geq 3$ , the triangle inequality holds and the maximum degree of  $G^c$  is equal to 4.*

**Corollary 3.** *The minimum symmetric reload  $s$ - $t$  path problem is NP-hard if  $c \geq 4$ , the graph is planar, the triangle inequality holds and the maximum degree is equal to 4.*

## 4 Paths and trails with asymmetric reload costs

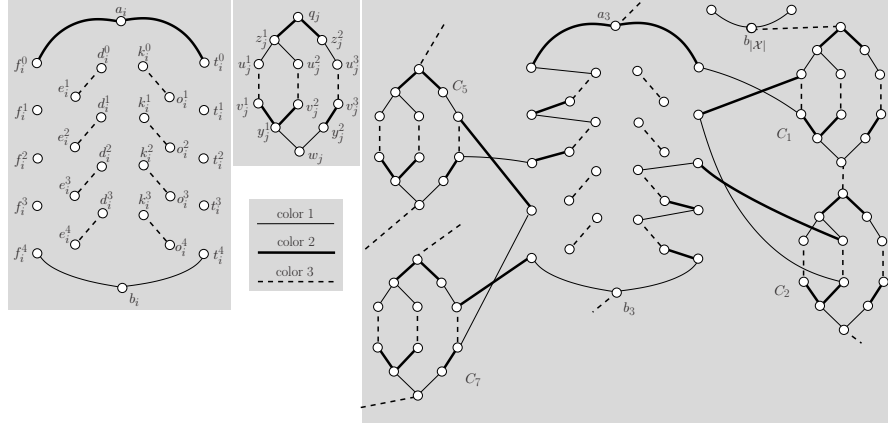
We now deal with asymmetric reload costs. We mainly prove that the minimum reload  $s$ - $t$  trail problem is NP-hard in this case.

**Theorem 6.** *The minimum asymmetric reload  $s$ - $t$  trail problem is NP-hard if  $c \geq 3$  and the maximum degree of  $G^c$  is equal to 3.*

*Proof.* This proof is similar to the one of Theorem 5, *i.e.* we reduce  $(3, B2)$ -SAT to the existence of an  $s$ - $t$  path with reload cost at most  $L$ . Hence we use the same notations and only describe how  $G^c$  is built upon  $\mathcal{I}$ . A trail must be a path in the graph of  $G^c$  since a vertex's maximum degree is 3. Hence we only deal with paths in this proof.

We have  $I_c = \{1, 2, 3\}$  and  $L = 15|\mathcal{X}| + 6|\mathcal{C}|$ . The matrix  $R$  is defined as  $r_{1,2} = r_{2,3} = r_{3,1} = M$  where  $M > L$ . The other entries of  $R$  are set to 1. The graph  $G^c$  has a source  $s$  and a sink  $t$ . In addition, for each  $x_i \in \mathcal{X}$  (resp.  $C_j \in \mathcal{C}$ ) we build a gadget as depicted on the left (resp. middle) of Figure 6. The gadget of a variable  $x_i$  consists of a left part (vertices  $f_i, d_i$  and  $e_i$ ), a right part (vertices  $t_i, k_i$  and  $o_i$ ), an entrance  $a_i$  and an exit  $b_i$ . The left (resp. right) part corresponds to the case where  $x_i$  is set to FALSE (resp. TRUE). The gadget of a clause  $C_j$  consists of a left part (vertices  $u_j^1$  and  $v_j^1$ ), a middle part (vertices  $u_j^2$  and  $v_j^2$ ), a right part (vertices  $u_j^3$  and  $v_j^3$ ), an entrance  $q_j$ , an exit  $w_j$  and four intermediate vertices  $z_j^1, z_j^2, y_j^1$  and  $y_j^2$ . The left, middle and right parts correspond to the first, second and third variable of  $C_j$  respectively.

We link the gadgets by adding the following edges with color 3 (dashed):  $[s, a_1], [b_1, a_2], [b_2, a_3], \dots, [b_{|\mathcal{X}|-1}, a_{|\mathcal{X}|}]; [b_{|\mathcal{X}|}, q_1]; [w_1, q_2], [w_2, q_3], \dots, [w_{|\mathcal{C}|-1},$



**Fig. 6.** Left: Gadgets for a variable  $x_i$ . Middle: Gadget of a clause  $C_j$ . Right:  $x_3$  appears in the four clauses  $C_1 = (\bar{x}_3 \vee x_5 \vee \bar{x}_6)$ ,  $C_2 = (\bar{x}_1 \vee \bar{x}_3 \vee x_4)$ ,  $C_5 = (x_1 \vee x_2 \vee x_3)$  and  $C_7 = (\bar{x}_1 \vee x_2 \vee x_3)$ .

$q_{|C|}$ ,  $[w_{|C|}, t]$  (this construction is similar to the one described in the left part of Figure 5 except for the colors of the edges). For each pair  $x_i$ ,  $C_j$  such that  $x_i$  is the  $\ell$ -th variable of  $C_j$  and  $C_j$  is the  $h$ -th clause of  $x_i$  we proceed as follows. If  $x_i$  appears negated in  $C_j$  then add  $[t_i^{h-1}, v_j^\ell]$  with color 1 (thin),  $[t_i^h, u_j^\ell]$  with color 2 (bold),  $[f_i^{h-1}, d_i^{h-1}]$  with color 1 and  $[f_i^h, e_i^h]$  with color 2. If  $x_i$  appears unnegated in  $C_j$  then add  $[f_i^{h-1}, v_j^\ell]$  with color 1,  $[f_i^h, u_j^\ell]$  with color 2,  $[t_i^{h-1}, k_i^{h-1}]$  with color 1 and  $[t_i^h, o_i^h]$  with color 2. Now  $G^c$  is fully described. An example is given on the right of Figure 6. It is not difficult to see that each vertex's degree of  $G^c$  is at most 3.

As in the proof of Theorem 5 it is not difficult to see that a truth assignment that satisfies  $\mathcal{I}$  corresponds to an  $s$ - $t$  path with reload cost  $15|\mathcal{X}| + 6|\mathcal{C}|$  in  $G^c$  and vice-versa.  $\square$

**Corollary 4.** *The minimum asymmetric reload  $s$ - $t$  path problem is NP-hard if  $c \geq 3$  and the maximum degree of  $G^c$  is equal to 3.*

**Corollary 5.** *The two following statements hold:*

- (i) *In the general case, the minimum asymmetric reload  $s$ - $t$  trail/path problems are not approximable at all if  $c \geq 3$  and the maximum degree of  $G^c$  is equal to 3.*
- (ii) *If  $r_{i,j} \geq 1$  for every  $i, j \in I_c$ , the minimum asymmetric reload  $s$ - $t$  trail/path problems are not  $O(2^{P(n)})$ -approximable for every polynomial  $P$  if  $c \geq 3$  and the maximum degree of  $G^c$  is equal to 3.*

We know that the minimum symmetric reload  $s$ - $t$  trail problem is polynomially solvable (see Theorem 2). We now prove that this result also holds with asymmetric reload costs if the triangle inequality is satisfied.

**Theorem 7.** *For any simple connected edge-colored graph  $G^c$  and any pair  $s, t$  of vertices of  $G^c$ , the minimum asymmetric reload  $s$ - $t$  trail problem can be solved in polynomial time, if the triangle inequality holds.*

## 5 Conclusion

In this paper we give a rather complete description of the complexity of the minimum reload  $s$ - $t$  walk/trail/path problems. When  $c = 2$ , we do not know the complexity of the minimum symmetric reload  $s$ - $t$  path and the minimum asymmetric reload  $s$ - $t$  trail problems if the matrix of reload costs does not satisfy the triangle inequality. These open problems seem important to better understand the complexity of the properly edge-colored  $s$ - $t$  trail/path problems when  $G^c$  does not have a properly edge-colored  $s$ - $t$  trail/path. In this case, one could be interested in seeking an  $s$ - $t$  trail/path with a minimum number of vertices for which the adjacent edges have the same color.

Finally, notice that if we study *the min-max reload  $s$ - $t$  walk/trail/path problems*, all the results presented here also hold. In this case, we replace the reload cost of a path/trail/walk  $\rho = (v_1, e_1, v_2, e_2, \dots, e_k, v_{k+1})$  between vertices  $s$  and  $t$  defined as in (1) by  $r(\rho) = \max\{r_{c(e_j), c(e_{j+1})} : j = 1, \dots, k - 1\}$ .

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