The hypocoloring problem: complexity and approximability results when the chromatic number is small

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Abstract. We consider a weighted version of the subcoloring problem that we call the hypocoloring problem: given a weighted graph $G = (V, E; w)$ where $w(v) \geq 0$, the goal consists in finding a partition $\mathcal{S} = (S_1, \ldots, S_k)$ of the node set of G into hypostable sets and minimizing $\sum_{i=1}^{k} w(S_i)$ where an hypostable S is a subset of nodes which generates a collection of node disjoint cliques K. The weight of S is defined as $\max\{\sum_{v \in K} w(v) | \ K \in S\}$. Properties of hypocolorings are stated; complexity and approximability results are presented in some graph classes. The associated decision problem is shown to be **NP-complete** for bipartite graphs and triangle-free planar graphs with maximum degree 3. Polynomial algorithms are given for graphs with maximum degree 2 and for trees with maximum degree Δ.

1 Introduction

Chromatic scheduling is the domain of scheduling problems which can be formulated in terms of graph coloring or more precisely of generalized graph coloring (i.e., coloring with a few additional requirements). These generalizations appear in [16, 11, 9, 5] and are called *conditional coloring* of G with respect to a graph theoretical property $\mathcal{P}$; the conditional (or $\mathcal{P}$) chromatic number $\chi_{\mathcal{P}}(G)$ is the minimum integer k such that there is a partition of the nodes into k sets such that the subgraph induced by each set has the property $\mathcal{P}$. Note that $\chi(G)$ corresponds to the case $\mathcal{P}(V') = true$ iff the subgraph induced by V' does not contain an induced P_2 (i.e., chain of length 1). An important application of conditional coloring is the circuit manufacturing problem and is defined by $\mathcal{P}(V') = true$ iff the subgraph induced by V' is planar (see [18] for a survey). To our knowledge the weighted case has not been studied specifically until now.

In particular the concept of weighted coloring has been introduced in [10] to generalize classical coloring models and to handle situations where operations occur with possibly different processing times in some types of batch scheduling problems.

Our generalized weighted coloring model can be described in terms of conditional coloring where property $\mathcal{P}$ is defined by $\mathcal{P}(V') = true$ iff the subgraph induced by V' does not contain an induced P_3. This induces the so-called *subcoloring* problem that has been studied in $[1, 13, 7]$. An alternate definition consists of finding a partition $\mathcal{S} = (S_1, \ldots, S_k)$ of the node set into *hypostable* sets minimizing k. We shall say that a subset $S = \{K_j : j \in J\}$ of nodes is a hypostable set in G if it induces a collection of node-disjoint cliques (with no edges between them). However, since we study a weighted model, the weight of a hypostable set $S = \{K_j : j \in J\}$ will be $w(S) = \max\{w(K_j) : j \in J\}$ and our problem, called MIN HYPOCOLORING, consists of finding a hypocoloring $(S_1, \ldots, S_k)$ of the nodes of G, i.e., a partition of the node set into hypostable sets such that:

$$opt = \sum_{i=1}^{k} w(S_i) \text{ is minimum} \tag{1}$$

In terms of batch scheduling, there exist many situations where operations have to be assigned to batches (of compatible operations) that are processed sequentially ([6]). Examples in satellite communication and in production have also been modelled as special cases of the above batch scheduling problem (see [19, 6]). In current model, all operations in a batch are assigned to different processors and processed simultaneously. The processing time of a batch S is limited by the largest processing time of the operations in S. If the processing times may take different values, it may be worthwhile to assign two (or more) incompatible operations v with small processing times $w(v)$ to the same batch; they will be processed consecutively on the same processor. This will not increase the processing time $w(S)$ of the batch S as long as the sum of processing times of these operations do not exceed the longest processing time $w(v)$ in S. In order to allow this possibility in our model, a natural way to define weight $w(K_j)$ is $w(K_j) = \sum_{v \in K_j} w(v)$. Since K_j corresponds to incompatible operations (assigned to the same processor), the processing time of all operations in K_j will be the sum of all processing times.

The MIN HYPOCOLORING problem may also be used for representing some machine scheduling problems: for instance, we are given a collection of jobs v with processing times $w(v)$ in a flexible manufacturing system; we link the nodes representing two jobs, if they share a certain number of tools; thus, it will be interesting to assign these jobs to the same machine on which the appropriate tools will be installed. A batch will consist of an assignment of jobs to some machines; in such an assignment, we try to assign to a same machine jobs sharing the same tools. Since there exists only a limited number of tools of each type, we will try to assign to different machines jobs that do not need the same tools. Hence a batch will be represented by a hypostable set in the graph of compatibilities (common tools) and the processing time of a batch will be the maximum load of a machine (maximum of the sums of processing times of jobs assigned to the same machine). We will focus on this model of weighted hypocoloring which is motivated in a natural way by the batch scheduling context.

In this paper, the *neighborhood* of node v will be denoted by $N(v)$, the *degree* of v by $d(v)$ or $d_{G_i}(v)$ when the particular graph G_i in which it is considered has to be emphasized, the *maximum degree* by $\Delta(G)$ or Δ and the *subgraph* *of* G *induced* by S by $G[S]$. The size of hypocoloring $\mathcal{S} = (S_1, \ldots, S_k)$ will be denoted by $|\mathcal{S}| = k$ and, finally, the number of different values of weights w by $|w|$. For graph-theoretical terms not defined here, the reader is referred to [3]. Moreover, we always assume that $\mathcal{S}$ is sorted by non-increasing weights (i.e., $w(S_1) \geq \ldots \geq w(S_k)$) and, without additional specification, we assume that $w(v) > 0, \forall v \in V$.

2 Elementary properties

We will derive here some properties which are based on the fact that hypocolorings are in some sense extensions of node colorings;

Lemma 1. *Any optimal hypocoloring* $\mathcal{S}$ *satisfies* $|\mathcal{S}| \leq \Delta(G) + 1$.

Proof. Let $\mathcal{S} = (S_1, \ldots, S_k)$ be an optimal hypocoloring and let $v \in S_k$. If $k > \Delta(G) + 1$ then there exists color $c \leq \Delta(G) + 1$ such that $N(v) \cap S_c = \emptyset$. So, we can recolor v with color c without increasing the value of $\mathcal{S}$.

This bound is not the best possible; by analogy with the theorem of Brooks [8], we could try to get a bound of $\Delta(G)$ instead of $\Delta(G) + 1$.

Proposition 1. *There exists an optimal hypocoloring* $\mathcal{S}$ *satisfying the following:*

(i) $\forall i \leq k,\ \forall v \in S_i,\ d_{G_{i,v}}(v) \geq i - 1$ *where* $G_{i,v} = G[S_1 \cup \ldots \cup S_{i-1} \cup \{v\}]$.
(ii) $\forall i \leq k$, S_i *contains no* $K_{\Delta(G)+3-i}$.
(iii) $|\mathcal{S}| \leq \Delta(G)$.

Proof. Let $\mathcal{S} = (S_1, \ldots, S_k)$ be an optimal hypocoloring. For (i): If $d_{G_{i,v}}(v) < i - 1$, then we can recolor node v with some color missing in $\{1, \ldots, i-1\}$. For (ii): Assume that S_i contains a $K_{\Delta(G)+3-i}$ and let $v \in K_{\Delta(G)+3-i}$. We deduce that $d_{G_{i,v}}(v) \leq i - 2$ which gives a contradiction with (i). For (iii): Let $v \in S_k$. From Lemma 1, we can assume $k \leq \Delta(G) + 1$ and $\exists u \in N(v) \cap S_{\Delta(G)}$. Moreover, using (i) with node u, we have $N(u) \cap S_{\Delta(G)} = \emptyset$. So we can recolor v with color $\Delta(G)$ (at this stage, the solution may have a greater value). By repeating this as long as $S_k \neq \emptyset$, we obtain another optimal hypocoloring.

Note that it is always possible to find in polynomial time a hypocoloring which verifies Proposition 1. We can also obtain a bound of the number of different colors used in any optimal coloring $\mathcal{S}^*$ using a relation between $|w|$ and the chromatic number $\chi(G)$.

Proposition 2. *Any optimal hypocoloring* $\mathcal{S}$ *satisfies:* $|\mathcal{S}| \leq 1 + |w|(\chi(G) - 1)$.

Proof. The proof is by induction on $|w|$. Let $\mathcal{S} = (S_1, ..., S_k)$ be an optimal hypocoloring and let $t = \max\{i : w(S_i) \geq \max_{v \in V} w(v)\}$. Remark that $t \leq \chi(G)$ (otherwise, an optimal coloring gives a better solution); moreover, if $t = \chi(G)$, then $t = |\mathcal{S}|$ (for the same reason). So, if $t = |\mathcal{S}|$ then we have $|\mathcal{S}| = t \leq \chi(G) \leq 1 + |w|(\chi(G) - 1)$. Now, assume $|\mathcal{S}| > t$ (we deduce $t \leq \chi(G) - 1$); then $(S_{t+1}, \ldots, S_k)$ is an optimal hypocoloring on $G' = G[S_{t+1} \cup \ldots \cup S_k]$ and using an inductive hypothesis, we deduce $|\mathcal{S}| - t \leq 1 + (|w| - 1)(\chi(G') - 1) \leq 1 + (|w| - 1)(\chi(G) - 1)$ and the result follows.

3 Complexity results

We will now show that MIN HYPOCOLORING is close to MIN COLORING in some cases; more precisely, we prove that MIN HYPOCOLORING is at least as hard to approximate as MIN COLORING. Let Ψ be a class of graphs.

Theorem 1. *There exists a approximation preserving reduction from* MIN COLORING *restricted to Ψ-graphs to* MIN HYPOCOLORING *restricted to Ψ-graphs.*

Thus, using results of [12], we deduce that MIN HYPOCOLORING is not approximable within n^ε for any $\varepsilon > 0$, unless **ZPP=NP**. It also follows that MIN HYPOCOLORING is **NP-hard** for graphs with $\Delta(G) \geq 4$, even if these graphs do not contain triangles, or for planar graphs, etc. Moreover, when $\Delta(G) = 3$ and $|w| = 1$, the previous proof and Brooks theorem [8] show that this case is polynomial. On the other hand, we now prove that when G is a triangle-free graph with $\Delta(G) = 3$ and $|w| = 2$, then MIN HYPOCOLORING becomes difficult.

Theorem 2. MIN HYPOCOLORING *is **strongly NP-hard** even for triangle-free graphs with $\Delta(G) = 3$.*

Proof. We shall reduce 1-IN 3SAT, proved to be **NP-complete** in [20] to our problem. This problem is defined as follows: Given a collection $\mathcal{C} = (C_1, \ldots, C_m)$ of m clauses over the set $X = \{x_1, \ldots, x_n\}$ of n Boolean variables such that each clause C_j has exactly three literals (i.e., $C_j = x \vee y \vee z$), is there a truth assignment f satisfying $\mathcal{C}$ such that each clause in $\mathcal{C}$ has exactly one true literal?

From instance $I = (\mathcal{C}, X)$ of 1-IN 3SAT, we construct an instance $I' = (G, w)$ of MIN HYPOCOLORING such that the answer of I is yes iff $opt(I) \leq 3$. We use gadget clauses and gadget variables. From the clause $C_i = x \vee y \vee z$, we build the graph F_i in Figure 1 and the weights are given by $w(v_1(F_i)) = 2$ and $\forall v \in V(F_i) \setminus \{v_1(F_i)\},\ w(v) = 1$.

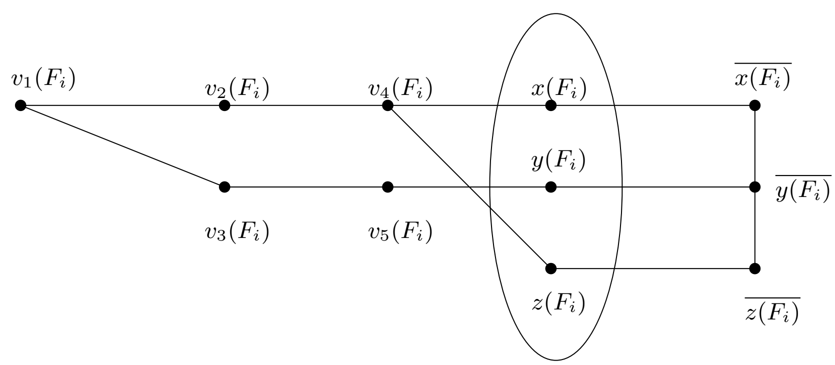

Fig. 1. Gadget clause F_i

From the variable x_j, we build the graph H_j in Figure 2 where the weights of nodes are one (i.e., $w(z) = 1,\ \forall z \in V(H_j)$). In addition, we link theses different graphs in the following way: if variable x_j appears positively in clause C_i, then we add edge $[x_i(H_j), x(F_i)]$ and otherwise, we add edge $[\overline{x_i(H_j)}, x(F_i)]$.

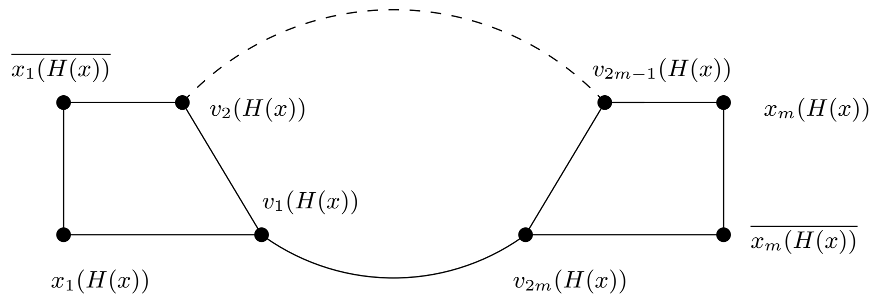

Fig. 2. Gadget variable H_j

This graph G satisfies $\Delta(G) = 3$. Let f be a truth assignment of $I = (\mathcal{C}, X)$. The hypostable sets S_1 and S_2 are given by:

$$
\begin{aligned}
S_1 = {} & \cup_{i=1}^{m} [\ \{v_1(F_i), v_4(F_i), v_5(F_i)\} \cup \{x(F_i) : f(x) = 1\} \cup \{\overline{x(F_i)} : f(x) = 0\}\ ] \\
& \cup_{j=1}^{n} \cup_{k=1}^{m} \{x_k(H_j), v_{2k}(H_j) : f(x) = 0\} \cup \{\overline{x_k(H_j)}, v_{2k-1}(H_j) : f(x) = 1\} \\
S_2 = {} & V(G) \setminus S_1
\end{aligned}
$$

It is easy to verify that $\mathcal{S} = (S_1, S_2)$ satisfies $w(S_1) = 2$ and $w(S_2) = 1$; thus $opt(I) \leq 3$. Conversely, let $\mathcal{S}$ be a hypocoloring of I' such that $val(\mathcal{S}) \leq 3$. We can observe that: (i) $\mathcal{S} = (S_1, S_2)$ with $w(S_1) = 2$ and $w(S_2) = 1$, (ii) S_2 is a stable set and $\forall i \leq m$, $|S_1 \cap \{x(F_i), y(F_i), z(F_i)\}| = 1$, (iii) $\forall j \leq n$, $H_j \cap S_1$ and $H_j \cap S_2$ are stable sets and (iv) $x_i(H_j)$ (resp. $\overline{x_i(H_j)}$) and $x(F_i)$ have two distinct colors if these nodes are linked.

So, we can exhibit a truth assignment f of I by taking $f(x) = 1$ iff $x \in S_1$.

Theorem 3. Min Hypocoloring *is* ***strongly NP-hard*** *for triangle-free planar graphs with* $\Delta(G) = 3$.

Proof. In the previous theorem, all gadgets F_i and H_j are planar and then only edges $[x_l(F_i), x_p(H_j)]$ may create some problems since they may cross each other. In this case, we apply the *crossover* technique, [14] which consists of replacing each edge crossing by a planar gadget. First, we embed the graph G' of Theorem 2 in the plane in such a way that every edge is a straight line and the crossing edge occurs only between two edges $[x_l(F_i), x_p(H_j)]$. Second, we replace each crossing edge by the gadget (L, w) indicated in Figure 3. This graph contains 8 particular nodes $x_1, x'_1, y_1, y'_1, x_2, x'_2, y_2, y'_2$. The weight of any node is 1 except for x'_1, y'_1, x'_2, y'_2 which are weighted by 2. It is easy to see that we have the fol-

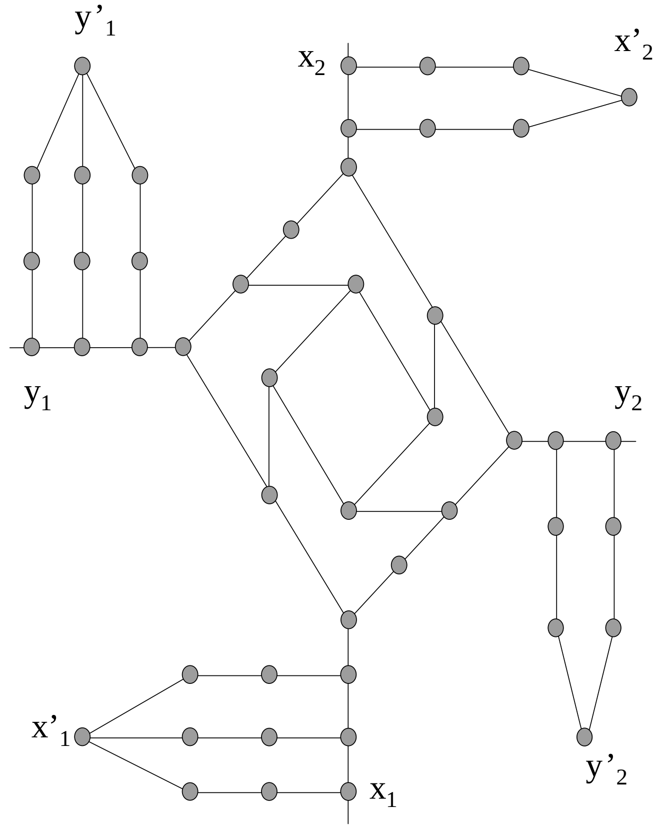

Fig. 3. Planar gadget (L, w).

lowing properties for any hypocoloring $\mathcal{S} = (S_1, S_2)$ with $val(\mathcal{S}) \leq 3$:
(i) $\{x'_1, x'_2, y'_1, y'_2\} \subseteq S_1$ and S_2 is a stable set, (ii) $\forall x = x_1, x_2, y_1, y_2$, the neighbors outside of the gadget (L, w) have not the same color as x, and (iii) x_1 and x_2 (resp., y_1 and y_2) have not the same color.

Using these properties, we deduce that there exists a hypocoloring $\mathcal{S}$ of G' with $val(\mathcal{S}) \leq 3$ if and only if there exists a hypocoloring $\mathcal{S}'$ of G'' with $val(\mathcal{S}') \leq 3$.

Now, we deal with bipartite graphs. Surprisingly, in some cases an optimal hypocoloring is just a coloring and such a coloring is difficult to compute.

Theorem 4. Min Hypocoloring *is **strongly NP-hard** for bipartite graphs with* $\Delta(G) = 39$.

Proof. We polynomially transform the pre-extension coloring problem 1-PrExt (proved to be **NP-complete** in [4]) into the hypocoloring problem in bipartite graphs; 1-PrExt can be described as follows: given a bipartite graph $G = (L, R; E)$ with $|L| \geq 3$ and $\Delta(G) = 12$ and three nodes v_1, v_2, v_3 in L, does there exist a 3-coloring (S_1, S_2, S_3) such that $v_i \in S_i$ for $i = 1, 2, 3$?

Let $G = (L, R; E)$ be a bipartite graph and let $\{v_1, v_2, v_3\} \subseteq L$ be a set of three nodes. We polynomially construct a new bipartite graph G' such that there exists a hypocoloring $\mathcal{S}$ of G' with $val(\mathcal{S}) \leq 7$ iff there exists a coloring (S_1, S_2, S_3) of G with $v_i \in S_i,\ i = 1, 2, 3$. In order to do that, we use the two following gadgets:

- The weighted bipartite graph $H_0 = (L_0, R_0; E_0, w)$ on 12 nodes with $l_i, l'_i \in L_0$ and $r_i, r'_i \in R_0$ for $i = 1, 2, 3$. Moreover, only edges $[l_i, r_i], [l_i, r'_i], [l'_i, r_i]$ and $[l'_i, r'_i]$ for $i = 1, 2, 3$ are missing in H_0. The weights are $w(l_i) = w(l'_i) = w(r_i) = w(r'_i) = 2^{3-i}$ for $i = 1, 2, 3$.
- The complete weighted bipartite graph $K_{3,2}$ with two specified nodes x and y (x in the left set and y in the right set). The weights are $w(x) = w(y) = 1$ and $w(v) = 2$ otherwise.

Now, $I = (G', w)$ is built in the following way: starting from G, we add a copy of H_0 and we identify nodes v_1, v_2, v_3 of G with nodes l_1, l_2, l_3 of H_0. Moreover, for each edge $e = [l, r]$ of G, we introduce a copy of $K_{3,2}$ and we identify nodes l, r with nodes x_e, y_e respectively.

Let $\mathcal{S} = (S_1, S_2, S_3)$ be a 3-coloring of G with $v_i \in S_i,\ i = 1, 2, 3$; then we extend $\mathcal{S}$ into a coloring $\mathcal{S}'$ of G' by the following process: we start with $S'_i = (S_i \setminus \{v_i\}) \cup \{l_i, l'_i, r_i, r'_i\}$. For each edge $e = [l, r]$ of G with $l \in L$ and $r \in R$, if $l \in S_j$ (resp. $r \in S_j$) with $j = 1, 2$ then we add $L_e \setminus \{x_e\}$ (resp., $R_e \setminus \{y_e\}$) to S'_i else ($j = 3$) and we add $L_e \setminus \{x_e\}$ (resp., $R_e \setminus \{y_e\}$) to S'_i where $r \in S_i$ (resp., $l \in S_i$). $\mathcal{S}'$ is a coloring of G' (thus a hypocoloring) and satisfies $val(\mathcal{S}') = w(S'_1) + w(S'_2) + w(S'_3) = 7$.

Conversely let $\mathcal{S}'$ be a hypocoloring satisfying $val(\mathcal{S}') \leq 7$. It is easy to prove that: (i) $\forall i = 1, 2, 3,\ \{l_i, l'_i, r_i, r'_i\} \subseteq S'_i$, (ii) $\mathcal{S}' = (S'_1, S'_2, S'_3)$ with $w(S'_1) = 4,\ w(S'_2) = 2,\ w(S'_3) = 1$, and (iii) the restriction of $\mathcal{S}'$ to graph G is a coloring. Thus, using these properties the result follows.

4 Approximability of some cases of hypocoloring.

We shall present here approximation algorithms for hypocolorings when coloring is easy and the chromatic number is small; formally, we denote by Ψ_k a class of graphs verifying: (i) for any G' subgraph of G, if $G \in \Psi_k$, then $G' \in \Psi_k$, (ii) $\forall G \in \Psi_k,\ \chi(G) \leq k$ and (iii) coloring on Ψ_k-graphs is polynomial. For instance, the set of forests is a Ψ_2-class. Assume that the nodes are ordered according to their non-increasing weights ($w(v_1) \geq \ldots \geq w(v_n)$) and let $G_i = G[\{v_1, \ldots, v_i\}]$. Moreover, j_0 denotes the smallest index i such that G_i contains an induced P_3 (if $G_n = G$ does not contain it, we set $j_0 = n+1$). Finally, $Colo(V')$ denotes an optimal coloring on $G[V']$ (i.e., $|colo(V')| = \chi(G[V'])$). A trivial bound of the approximability on Ψ_k-graphs is k and consists of computing an optimal coloring in the entire graph. We now propose an algorithm achieving a better constant approximation ratio.

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1 Sort the nodes of G in non-increasing weight order;
2 Compute $j_0 = min\{i : G_i \text{ contains an induced } P_3\}$ and $V_{j_0} = \{v_1, \ldots, v_{j_0}\}$;
3 For $i = 1$ to j_0 do
   3.1 $S_1^i = V_{j_0} \setminus \{v_i, \ldots, v_{j_0}\}$;
   3.2 Compute $Colo(V \setminus S_1^i)$ and define hypocoloring $\mathcal{S}^i$ by $(S_1^i, Colo(V \setminus S_1^i))$;
4 Compute $\mathcal{S} = argmin\{val(\mathcal{S}^i) : i = 1, \ldots, j_0\}$;

Theorem 5. Min Hypocoloring *is* $\frac{k^2}{2k-1}$*-approximable in* Ψ_k*-graphs.*

Proof. let $I = (G, w)$ with $G \in \Psi_k$ be an instance of Min Hypocoloring and let $\mathcal{S}^* = (S_1^*, \ldots, S_l^*)$ be an optimal hypocoloring. We can assume $j_0 \leq n$ and then $l > 1$. If $v_1 \notin S_1^*$, then $val(\mathcal{S}) \leq val(\mathcal{S}^1) \leq k\ w(v_1)$ and $opt \geq 2w(v_1)$. Now, set $i_0 = \max\{i : \{v_1, \ldots, v_i\} \subseteq S_1^*\}$ and examine solution $\mathcal{S}^{i_0}$.

If $w(S_2^{i_0}) \leq w(S_1^{i_0})\frac{k-1}{k}$, then for any $r \geq 3$, $w(S_r^{i_0}) \leq \frac{k-1}{2k-1}w(S_2^{i_0}) + \frac{k}{2k-1}w(S_2^{i_0})$ $\leq \frac{k-1}{2k-1}(w(S_1^{i_0}) + w(S_2^{i_0}))$. Summing these inequalities, we deduce:
$val(\mathcal{S}) \leq \frac{k^2}{2k-1}(w(S_1^{i_0}) + w(S_2^{i_0})) \leq \frac{k^2}{2k-1}opt$.

If $w(S_2^{i_0}) \geq w(S_1^{i_0})\frac{k-1}{k}$, then $opt \geq w(S_1^{i_0})\frac{2k-1}{k}$ and the result follows.

Using Grotzsch theorem [15], Brooks theorem [8] (with its constructive proof given by Lovasz), since we can assume without loss of generality that G does not contain any copy of $K_{\Delta(G)+1}$, and the previous theorem, we obtain:

Corollary 1. Min Hypocoloring *is* $\frac{9}{5}$*-approximable if* G *satisfies* $\Delta(G) \leq 3$ *or if* G *is triangle-free and planar; it is* $\frac{4}{3}$*-approximable if* G *is bipartite.*

We can also establish lower bounds on the approximability of these types of graphs by using the proofs of Theorem 3 and Theorem 4.

Proposition 3. *Unless* ***P=NP****,* Min Hypocoloring *is not* $(\frac{4}{3} - \varepsilon)$*-approximable, if* G *is triangle free and planar, and not* $(\frac{8}{7} - \varepsilon)$*-approximable, if* G *is bipartite, for any* $\varepsilon > 0$*.*

There is another simple approximation algorithm which works for any value of $\Delta(G)$. This algorithm uses a decomposition of G into at most $s = \lceil \frac{\Delta(G)+1}{3} \rceil$ subgraphs G_i satisfying $\Delta(G_i) \leq 2$ by applying a result of [17]. Then, for each $i = 1, \ldots, s$, we compute an optimum hypocoloring $\mathcal{S}_i^*$ on G_i by using the algorithm presented in subsection 5.2 (Proposition 6) and we color the corresponding solution with new colors. Finally, the solution $\mathcal{S}$ is the juxtaposition of these hypocolorings $\mathcal{S}_i^*$.

Theorem 6. Min Hypocoloring *is* $\lceil \frac{\Delta(G)+1}{3} \rceil$*-approximable.*

Proof. We have $val(\mathcal{S}) = \sum_{i=1}^{s} opt(G_i)$ and $opt \geq opt(G_i)$ for any $i = 1, \ldots, s$. Then, $val(\mathcal{S}) \leq s \times opt$.

5 Polynomial cases

In this section, we consider two polynomial cases of Min Hypocoloring: when the input is a tree with maximum degree at most Δ and when the input is a 2-regular graph. For sake of convenience, we assume that $w(v) \geq 0, \forall v$ (so, it may exist some nodes v with $w(v) = 0$); in this case, as we will show, the first case is equivalent to Min Hypocoloring in forests with degree at most Δ whereas the second case is equivalent to Min Hypocoloring in graphs with $\Delta(G) = 2$. Thus, since a tree is a particular bipartite graph, we have a boundary for the hardness of Min Hypocoloring between trees with maximum degree at most 39 and bipartite graphs with maximum degree at most 39. Finally, there is also another hardness gap for general graphs between graphs with maximum degree at least 3 and graphs with maximum degree at most 2.

Before establishing these results, we shall give some results on Min Hypocoloring in $(t+1)$-clique free graphs. For a hypostable set S, the *characteristic value* will be the integer number q such that $q = w(S)$. More generally, for a hypocoloring $\mathcal{S} = (S_1, \ldots, S_k)$ with $w(S_1) \geq \ldots \geq w(S_k)$ we call *vector of characteristic values*, the vector $(q_1, \ldots, q_k)$ such that for any $i \leq k, q_i = w(S_i)$. The Min Hypocoloring problem is close to the List-hypocoloring$_t$ problem.
List-hypocoloring$_t$:
Instance: a graph $G = (V, E)$, a set $\mathcal{C}$ of colors and, for every clique K with size at most t, $C_K \subseteq \mathcal{C}$ is a set of colors such that each one of them may occur on some nodes of the clique K but not on all nodes at a time.
Question: does G admit a hypocoloring such that for any clique K, not all the nodes of K have the same color i with $i \in C_K$?

Clearly, we must have $C_K \subseteq C_{K'}$ when $K \subseteq K'$ and List-hypocoloring$_t$ polynomially reduces to List-hypocoloring$_{t'}$ when $t \leq t'$. Moreover, we have:

Proposition 4. *In the graphs with maximum degree Δ,* Min Hypocoloring *polynomially reduces to* List-hypocoloring$_{\Delta+1}$.

Proof. A minimum hypocoloring can be computed by the following algorithm:

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1 For every vector $(q_1, \ldots, q_\Delta)$ with $q_1 \geq \ldots \geq q_\Delta$ and such that $q_i = \sum_{v \in V(K_i)} w(v)$ for some clique K_i of G do
  1.1 Solve the related List-hypocoloring$_{\Delta+1}$ instance;
  1.2 If the answer is yes, construct such a hypocoloring;
2 Select a minimum weight hypocoloring among feasible hypocolorings computed during an execution of step 1.2;

The complexity-time of this algorithm is $O(n^{\Delta^2})$ times the complexity-time to solve List-hypocoloring$_{\Delta+1}$.

Corollary 2. *If* List-hypocoloring$_{\Delta+1}$ *is polynomial on Ψ-graphs, then* Min Hypocoloring *is also polynomial on Ψ-graphs.*

Using Proposition 2 and a slight modification of Proposition 4, we deduce:

Corollary 3. *Let us consider a class Ψ of $(t+1)$-clique free graphs satisfying $\chi(G) \leq k$ and such that* List-hypocoloring$_t$ *is polynomial on Ψ. Then,* Min Hypocoloring *is also polynomial on Ψ when $|w|$ is bounded by a constant.*

5.1 Trees with maximum degree Δ

In trees, there are at most $2n-1$ characteristic values for the different hypostable sets. Thus, the complexity of the algorithm of Proposition 4 is in this case in $O(n^{\Delta})$ times the complexity-time of List-hypocoloring$_{\Delta+1}$. We now show how we can solve List-hypocoloring$_{\Delta+1}$ in trees by using dynamic programming. Let $\mathcal{C} = \{1, \ldots, \Delta\}$ be the set of colors. Let us then consider $(T = (V, E); (C_K)_{K \in V \cup E})$ an instance of List-hypocoloring$_{\Delta+1}$ where T is a tree. Given a node v, we respectively denote by $H_v(T)$ and $H'_v(T)$ the sets of colors defined by:

$h \in H_v(T)$ $(resp. H'_v(T))$ *if and only if there is a feasible hypocoloring for which v is colored by h and no (resp. exactly one) neighbor of v is colored by h.*

We denote by $v_1, \ldots, v_d$ the neighbors of v. The deletion of v induces a forest with d connected components $T_1, \ldots, T_d$ where T_i is the subtree containing v_i.

Lemma 2. *For $h \in \mathcal{C}$, we have:*

- $h \in H_v(T) \Leftrightarrow h \notin C_v$ *and* $\forall j, (H_{v_j}(T_j) \cup H'_{v_j}(T_j)) \setminus [(H_{v_j}(T_j) \cup H'_{v_j}(T_j)) \cap \{h\}] \neq \emptyset$.
- $h \in H'_v(T) \Leftrightarrow h \notin C_v$ *and* $\exists j \leq d, h \in H_{v_j}(T_j) \setminus C_{[v,v_j]}$ *and* $\forall j' \neq j, (H_{v_{j'}}(T_{j'}) \cup H_{v_{j'}}(T_{j'})) \setminus [(H_{v_{j'}}(T_{j'}) \cup H'_{v_{j'}}(T_{j'})) \cap \{h\}] \neq \emptyset$.

Proposition 5. *For any $t \geq 2$,* List-hypocoloring$_t$ *in trees is polynomial.*

Proof. Let us consider the following polynomial-time algorithm:

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1. Choose a root $r \in V$ and orient the tree from r to leaves (T_v denotes the subtree induced by v and its successors);
2. Compute, for every node v and from leaves to the root, sets $H_v(T_v)$ and $H'_v(T_v)$ (by using Lemma 2);
3. For every color in $H_r(T) \cup H'_r(T)$ compute a feasible hypocoloring by using Lemma 2 (from the root to leaves);

5.2 Graphs with maximum degree two

We shall examine here the special situation where the graph G has maximum degree $\Delta(G) = 2$ (the case $\Delta(G) = 1$ being trivial). From Proposition 1, there exists an optimal hypocoloring $\mathcal{S} = (S_1, S_2)$ of G with $w(S_1) \geq w(S_2)$. The case $S_2 = \emptyset$ is trivial and can be solved in linear-time. Thus, we will suppose $S_i \neq \emptyset$ for

$i = 1, 2$. We prove by a technique similar of the one described earlier that the case of maximum degree two is also polynomial. However, the method presented here is slightly more involved than the previous one. First, observe that solving MIN HYPOCOLORING in graphs with $\Delta(G) = 2$ or in 2-regular graphs are equivalent. As a consequence, we may restrict our attention to graphs $G = (V, E)$ whose connected components are cycles; let $n = |V| = |E|$. We will define the weight $w(e)$ of an edge $e = [x, y]$ as the sum $w(x) + w(y)$. From (ii) of Proposition 1, we know that S_2 does not contain any K_3; then, we notice that there are at most $n + t + 1$ possible values for $w(S_1)$ where t is the number of triangles of G and $2n$ possible values for $w(S_2)$. It is important to notice that we cannot solve separately the problem in each connected component.

The algorithm is the following: starting with the smallest possible value of p and the smallest possible value of $q \leq p$, we apply Properties 1 to 4 (given below) to get the smallest q for which a solution (S_1, S_2) exists such that $w(S_1) = p$ and $w(S_2) = q$. If such a hypocoloring can be found, we store the current solution $\mathcal{S} = (S_1, S_2)$ with $val(\mathcal{S}) = p + q$ if it is better than the best solution found so far. Whenever such a solution has been found, we increase p to the next possible value and we start again with the minimum q. An optimal hypocoloring (S_1, S_2) will be given by the solution stored.

Property 1. If $w(v) > q$, then $v \in S_1$; if x, y, z are three consecutive nodes on an induced P_3 with $x, y \in S_i$ then $z \in S_{3-i}$ for $i = 1, 2$.

Property 2. If for some edge $e = [x, y]$, we have $w(e) > p$, then x, y are neither both in S_1 nor both in S_2; if $w(e) > q$, then x, y are not both in S_2. In such situations, we shall simply say that the color i is not feasible for edge $e = [x, y]$.

Starting from G with given values p, q we will apply the above properties as long as possible to derive consequences on the colors to be assigned to the nodes and to the edges of G. If we arrive to a situation where no solution exists then, we increase the value of p. Now, each cycle C_i has at least one node with a fixed color. We can describe C_i by the sequence $(F_1, D_1, F_2\ , \ldots, F_k, D_k)$ where F_i and D_i are chains. Moreover, for any i, all nodes of F_i have a fixed color and each D_i has two endpoints with a fixed color and all intermediate nodes are uncolored. Let $a_1, \ldots, a_s$ be the nodes of the chain D_i.

Property 3. If $a_1, a_s \in S_j$ with s odd or $a_1 \in S_j,\ a_s \in S_{3-j}$ with s even for some $j = 1, 2$, then we can alternate the colors 1 and 2 in D_i.

Property 4. If $a_1, a_s \in S_j$ with s even or $a_1 \in S_i,\ a_s \in S_{3-j}$ with s odd for some $j = 1, 2$, then $[a_1, a_2]$ gets one of its feasible colors.

By applying Properties 3 and 4 for each chain D_i, we color properly the remaining cycles. Now, when a value of p is fixed, we observe that the consequence of Properties 1 and 2 can be obtained in $O(n^2)$ steps and this gives a feasible value of q (if there exists). Then again in $O(n)$ steps, we can apply Properties 3 and 4 to determine a 2-hypocoloring. It should be observed that cases where no solution can be found occur only when consequence of Properties 1 and 2 are drawn.

Proposition 6. *The previous algorithm solves* MIN HYPOCOLORING *in graphs with $\Delta(G) \leq 2$ in $O(n^3)$ time.*

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