The minimum reload $s$-$t$ path, trail and walk problems

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Abstract

This paper deals with problems on non-oriented edge-colored graphs. The aim is to find a route between two given vertices $s$ and $t$. This route can be a walk, a trail or a path. Each time a vertex is crossed by a walk there is an associated non-negative reload cost $r_{i,j}$, where $i$ and $j$ denote, respectively, the colors of successive edges in this walk. The goal is to find a route whose total reload cost is minimum. Polynomial algorithms and proofs of \textbf{NP}-hardness are given for particular cases: when the triangle inequality is satisfied or not, when reload costs are symmetric (i.e., $r_{i,j} = r_{j,i}$) or asymmetric. We also investigate bounded degree graphs and planar graphs. We conclude the paper with the traveling salesman problem with reload costs.

Keywords: Edge-colored graphs; \textbf{NP}-hardness; reload optimization; paths, trails and walks; TSP; inapproximation.

1 Introduction

In the last few years a great number of applications have been modelled as problems in edge-colored graphs. More recently, we find interesting applications involving edge-colored graphs and connection (or reload) costs arising at each vertex, according to the pair of colors of the edges used by a walk through that vertex [4, 5, 6, 9]. Despite their apparent importance in the telecommunications and transportation industry, reload costs have not been extensively studied in the literature.

In [4, 9], each color is viewed as a subnetwork and is used to model a cargo transportation network which uses different means of transportation or data transmission costs arising in large communication networks. In all these models, the transportation or communication costs between the subnetworks usually dominate the costs within individual subnetworks. Some applications in satellite networks are also discussed in [5] where the various subnetworks may represent different products offered by the commercial satellite service providers. In [5], terrestrial satellite dishes are required to first capture the radio signals and then special electric-to-fiber converters are required to transform the electric signals from the satellite dishes to optical pulses that can be sent over optical fibers. These

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interface costs are referred as reload costs and depend on the technologies being connected. As another example, imagine a road network with many tolls. A fee (reload cost) must be paid each time we change from one road to another. One may be interested in paying as little as possible to travel from a source to a destination. We call this problem the minimum toll cost $s$-$t$ path problem.

In our case, we will be particularly concerned with problems involving reload paths, trails and walks with both symmetric and asymmetric costs between fixed source $s$ and destination $t$ (with obvious applications in transportation and communication networks as mentioned above). A section is also devoted to Hamiltonian cycles.

1.1 The model

Let $I_c = \{1, 2, \ldots, c\}$ be a given set of colors with $c \geq 1$. In this work, $G^c$ denotes a simple connected non-oriented edge-colored graph containing two particular vertices $s$ and $t$. Each edge has a color of $I_c$.

We recall here some standard graph terminology: the vertex and edge sets of $G^c$ are denoted by $V(G^c)$ and $E(G^c)$, respectively. The order of $G^c$ is the number $n$ of its vertices and the size of $G^c$ is the number $m$ of its edges. For a given color $i$, $E^i(G^c)$ denotes the set of edges of $G^c$ colored by $i$. We denote by $N_{G^c}(x)$ the set of all neighbors of $x$ in $G^c$, and by $N^1_{G^c}(x)$, the set of vertices of $G^c$, linked to $x$ with edges colored by $i$. The degree of $x$ in $G^c$ is $d_{G^c}(x) = |N_{G^c}(x)|$ and the maximum degree of $G^c$, denoted by $\Delta(G^c)$, is $\Delta(G^c) = \max\{d_{G^c}(x) : x \in V(G^c)\}$.

A non-oriented edge between two vertices $x$ and $y$ is denoted by $[x, y]$ while its color is denoted by $c([x, y])$. Given a graph $G = (V, E)$, a walk $\rho$ from $s$ to $t$ in $G$ (called $s$-$t$ walk) is a sequence $\rho = (v_0, e_0, v_1, e_1, \ldots, e_k, v_{k+1})$ where $v_0 = s$, $v_{k+1} = t$ and $e_i = [v_i, v_{i+1}]$ for $i = 0, \ldots, k$. A trail from $s$ to $t$ in $G$ (called $s$-$t$ trail) is a walk $\rho = (v_0, e_0, v_1, e_1, \ldots, e_k, v_{k+1})$ from $s$ to $t$ where $e_i \neq e_j$ for $i \neq j$. A path from $s$ to $t$ in $G$ (called $s$-$t$ path) is a trail $\rho = (v_0, e_0, v_1, e_1, \ldots, e_k, v_{k+1})$ from $s$ to $t$ where $e_i \neq e_j$ for $i \neq j$.

We are also given a $c \times c$ matrix $R = [r_{i,j}]$ (for $i, j \in I_c$) whose entries define the reload cost (or connection cost) when changing from color $i$ to color $j$. It is assumed that each entry $r_{i,j}$ of $R$ is a non-negative integer (i.e. $r_{i,j} \in \mathbb{N}$). Here, we will both consider symmetric and asymmetric matrices. We say that a matrix $R$ satisfies the triangle inequality, if and only if, for all edges $e_i, e_j, e_k \in E(G^c)$ which are adjacent to a common vertex, we have $r_{e_i,e_j} \leq r_{e_i,e_k} + r_{e_k,e_j}$ (see [W4] for similar definitions). Given a path/trail/walk $\rho = (v_0, e_0, v_1, e_1, \ldots, e_k, v_{k+1})$ between vertices $s$ and $t$, we define the reload cost of $\rho$ as:

$$r(\rho) = \sum_{j=0}^{k-1} r_{e_j,e_{j+1}}$$

(1)

Given a (non-colored) digraph $D = (V, \vec{E})$, an arc from $x$ to $y$ is denoted by $(x, y)$. The length of the path, trail or walk $\rho_c$ in $G^c$ (resp. $D$), denoted by $|\rho_c|$, is the number of its edges (resp. arcs).

We will also recall the concept of contraction for non-oriented graphs. Given an induced subgraph $Q$ of a non-colored graph $G$, a contraction of $Q$ in $G$ consists in replacing $Q$ by a new vertex, say $z_Q$, so that each vertex $x$ in $G - Q$ is connected to $z_Q$ by an edge, if and only if, there exists an edge $[x, y]$ in $G$ for some vertex $y$ in $Q$.

An instance of the minimum reload s-t path/trail/walk problem (called minimum reload
s-t path, minimum reload s-t trail and minimum reload s-t walk respectively) consists of a simple connected edge-colored graph $G^c$, a pair $s, t \in V(G^c)$ and a $c \times c$ matrix $R = [r_{i,j}]$ associating a non-negative cost to each pair of colors. The objective is to find a path/trail/walk $\rho$ between $s$ and $t$ with minimum reload cost. For instance, in the minimum toll cost s-t path problem, $r_{i,j} = r_j$ for $i, j \in I_c$ with $i \neq j$ and $r_{i,i} = 0$ where every $r_j$ is a non-negative integer. Finally, notice that if $c = 1$ (i.e., there is only one color in $G^c$), these 3 problems are equivalent to finding an s-t path of minimum length in $G^c$. Thus, we will assume $c \geq 2$.

1.2 Related work

To the best of our knowledge, reload cost optimization has been mainly studied in the context of spanning trees [1,2,5,9], but also very recently for some variants of paths, tours and flow problems [2]. In [9], the authors consider the problem of finding a spanning tree of minimum diameter with respect to the reload costs and they propose inapproximability results for graphs of maximum degree 5 and polynomial results for graphs of maximum degree 3. In [4], the author discusses inapproximability results for the same problem when restricted to graphs with maximum degree 4. In [5,6], the authors give several formulations with computational results to solve the reload cost spanning tree problem.

In [2], the authors consider several models for reload cost paths, tours and flow problems, which have several applications in telecommunication, transportation, and energy distribution. In particular, they study the following model: given a directed arc-colored graph $D^c = (V, \vec{E})$ where each arc $e \in \vec{E}$ has a non-negative cost $w(e)$ and a color $c(e) \in I_c$, and given a non-negative integer reload cost matrix $R = [r_{i,j}]$ for $i, j \in I_c$, they want to find an oriented s-t trail $\rho = (s, e_1, v_1, e_2, \ldots, e_k, t)$ of $D^c$ minimizing $\sum_{i=1}^k w(e_i) + \sum_{i=1}^{k-1} r_{c(e_i), c(e_{i+1})}$. They prove that this problem, called the minimum reload+weight directed s-t trail problem, is solvable in polynomial time.

A path or trail in $G^c$ is called properly edge-colored if each pair of successive edges differ in color. The properly edge-colored s-t path (or trail) problem and some of its variants, including the determination of a longest properly edge-colored s-t path (or trail) for a particular class of graphs, and the determination of two or more properly edge-colored s-t paths (or trails), have been considered in [1]. The minimum reload s-t path (resp., minimum reload s-t trail) problem is also related to the problem of deciding whether a simple connected edge-colored graph $G^c$ has a properly edge-colored s-t path (resp., s-t trail). For instance, if we set for the reload cost $r_{i,i} = 1$ and $r_{i,j} = 0$ for $i, j \in I_c$ with $i \neq j$, then there exists an s-t path (resp., s-t trail) with reload cost 0 in $G^c$, if and only if, $G^c$ has a properly edge-colored s-t path (resp., trail). Analogously, there exists a monochromatic s-t path in $G^c$, if and only if there exists an s-t path in $G^c$ with $r_{i,i} = 0$, $r_{i,j} = 1$.

The minimum reload s-t path problem can be viewed as a generalization of the classical minimum weight s-t path problem in digraphs.

Lemma 1. There is a polynomial time reduction from the minimum weight directed s-t path problem to the minimum reload s-t path problem.

Proof. Let $(D, w)$ be a weighted digraph where $D = (V, \vec{E})$, $V = \{v_1, \ldots, v_n\}$, $s = v_1$, $t = v_n$, $\vec{E} = \{e_1, \ldots, e_m\}$ and a positive weight $w(e_\ell)$ for each arc $e_\ell \in \vec{E}$. From $(D, w)$ instance of the minimum weight directed s-t path problem, we can build an instance
of the minimum reload $s$-$t$ path problem with asymmetric reload costs as described in the sequel. Initially, without loss of generality, consider $D$ with no incoming arcs at $s$ and with no outgoing arcs at $t$.

- $I_c = \{0, 1, \ldots, m\}$ is the set with $m + 1$ colors.
- The graph $G^c$ is described by $V(G^c) = V \cup \{v'_\ell : \ell = 1, \ldots, m\}$ and $E(G^c) = \{[v_i, v'_\ell], [v'_\ell, v_j] : e_\ell = (v_i, v_j) \in \overline{E}\}$. Edges $[v_i, v'_\ell] \in E(G^c)$ and $[v'_\ell, v_j] \in E(G^c)$ are colored in colors $\ell$ and 0, respectively.
- Finally, the reload costs $r_{i,j}$ are defined as follows: $r_{0,\ell} = 0$ and $r_{\ell,0} = w(e_\ell)$ for $\ell \neq 0$ and $r_{0,0} = (m + 1) \max\{w(e_\ell) : e_\ell \in \overline{E}\} = r_{i,j}$ for colors $i, j \neq 0$.

Figure 1 illustrates this transformation. It is clear that this construction is done within polynomial-time. Moreover, any $s$-$t$ path $\mu$ of $D$ with weight $w(\mu) = \sum_{e_\ell \in E} w(e_\ell)$ corresponds to a path $\mu'$ of $G^c$ with reload cost $r(\mu') = w(\mu)$ and conversely any path $\mu'$ of $G^c$ with reload cost $r(\mu') < (m + 1) \max\{w(e_\ell) : e_\ell \in \overline{E}\}$ corresponds to a path $\mu$ of $D$ with weight $w(\mu) = r(\mu')$.

![Figure 1: Transformation of an weighted digraph $D$ with 4 arcs into a 5-edge-colored graph $G^c$ with reload costs](image)

1.3 Contribution and organization of the paper

In Section 2, we discuss the case of finding a minimum reload $s$-$t$ walk, either with symmetric or with asymmetric reload cost matrix. In Section 3, we deal with paths and trails when reload costs are symmetric. We prove that the minimum reload $s$-$t$ trail problem can be solved in polynomial time. In addition, we show that the minimum reload $s$-$t$ path problem is polynomially solvable either if $c = 2$ and the triangle inequality holds (here $R$ is not necessarily a symmetric matrix), or if $G^c$ has a maximum degree 3. However this latter problem is NP-hard when $c \geq 3$, even for graphs with maximum degree 4 and reload cost matrix satisfying the triangle inequality. We conclude Section 3 by showing that, if $c \geq 4$ and the triangle inequality is satisfied, the minimum symmetric reload $s$-$t$ path problem
remains NP-hard even for planar graphs with maximum degree 4. In Section 4 we deal with asymmetric reload costs. For a reload cost matrix satisfying the triangle inequality, we construct a polynomial time procedure for the minimum reload $s$-$t$ trail problem and we prove that the minimum asymmetric reload $s$-$t$ trail problem is NP-hard even for graphs with 3 colors and maximum degree equal to 3. Table 1 summarizes the main results given in the paper.

Table 1: Summary of main results.

<table>
<thead>
<tr>
<th>Polynomial time problems</th>
<th>NP-hard problems</th>
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<tbody>
<tr>
<td>walk</td>
<td>all cases</td>
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<tr>
<td>trail</td>
<td>(sym. $R$)</td>
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<tr>
<td>(asym. $R$)</td>
<td>(asym. $R$) ∧ ($\Delta(G^c) = 3$) ∧ ($c = 3$)</td>
</tr>
<tr>
<td>(c = 2) ∧ (triangle ineq.)</td>
<td>(sym. $R$) ∧ ($\Delta(G^c) = 4$) ∧ ($c \geq 3$) ∧ (triangle ineq.)</td>
</tr>
<tr>
<td>path</td>
<td>(sym. $R$) ∧ ($\Delta(G^c) \leq 3$)</td>
</tr>
<tr>
<td></td>
<td>($\Delta(G^c) = 4$) ∧ ($c \geq 4$) ∧ (triangle ineq.)</td>
</tr>
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We investigate a version of the traveling salesman problem with reload costs in Section 5. In particular we show that the problem is NP-hard and no approximation with non-trivial performance ratio is likely to exist. Finally some concluding remarks are given in Section 6.

2 A walk better than a path

Choosing a walk instead of a path can help in reducing the reload costs. For instance, Figure 2 illustrates two different $s$-$t$ walks, $\rho_1 = (s, e_1, v_1, e_3, t)$ and $\rho_2 = (s, e_1, v_1, e_2, v_2, e_2, v_1, e_3, t)$, with reload costs $r_{i,j} = 1$ for $i,j \in \{1, 2, 3\}$ except for $r_{1,3} = r_{3,1} = 4$. The reload cost of $\rho_1$ is $r(\rho_1) = r_{1,3} = 4$ whereas the reload cost of $\rho_2$ is $r(\rho_2) = r_{1,2} + r_{2,2} + r_{2,3} = 3$. Notice that the minimum reload cost of an $s$-$t$ walk is a lower bound on the minimum reload cost of an $s$-$t$ trail which is a lower bound on the minimum reload cost of an $s$-$t$ path since a path is a trail and a trail is a walk.

We already know that the minimum reload $s$-$t$ walk problem is polynomial since there is a polynomial reduction from the minimum reload $s$-$t$ walk problem to the minimum reload+weight directed $s$-$t$ trail problem (see Subsection 1.2 for a description of this problem). Actually, from $G^c$, $c$, $I_c$ and a reload cost matrix $R = [r_{i,j}]$, an instance of the minimum reload $s$-$t$ walk problem, we build an instance $D^c$, $c'$ $I'_c$, $w$ and a reload cost matrix $R' = [r'_{i,j}]$ of the minimum reload+weight directed $s$-$t$ trail problem as follows: $V(D^c) = V(G^c)$ and we replace each edge $e = [v_i, v_j]$ of $G^c$ by two arcs $e_1 = (v_i, v_j)$ and $e_2 = (v_j, v_i)$ with color $c'(e_1) = c'(e_2) = c(e)$. Thus, $I'_c = I_c$. Finally, $r'_{i,j} = r_{i,j}$ for $i,j \in I_c$ and $w(e) = 0$ for every arc $e \in \overrightarrow{E}(D^c)$. It is not difficult to see that any directed $s$-$t$ trail $\rho$ of $D^c$ with reload+weight cost $r'(\rho) + w(\rho)$ corresponds to an $s$-$t$ walk $\rho_c$ of $G^c$ with reload cost $r(\rho_c) = r'(\rho) + w(\rho)$. On the other side, any optimal $s$-$t$ walk $\rho^*_c$ of $G^c$ using a minimum number of edges can be converted into a directed $s$-$t$ trail $\rho^*$ of $D^c$ with reload+weight cost $r'(\rho^*) + w(\rho^*) = r(\rho^*_c)$. Thus, using the result of [2], our result follows.
We propose another polynomial method to solve the minimum reload \textit{s-t walk} problem. Notice that the construction we propose differs from the one given in [2] for solving the minimum reload+weight directed \textit{s-t trail} problem.

Let \( G^c \) with \( V(G^c) = \{s,t\} \cup \{v_1,\ldots,v_n\} \) be a simple edge-colored connected graph with colors in \( I_c \). We reduce the minimum \textit{s-t walk} problem to the computation of a shortest \( s_0-t_0 \) path in an auxiliary digraph \( H = (V',E') \) with weight \( w \) in the arcs. The digraph \( H \) contains \( |I_c| \) induced subgraphs \( H_\ell \) for \( \ell \in I_c \). The vertex set of each subgraph \( H_\ell \) is \( \{s^\ell,t^\ell\} \cup \{v_1^\ell,\ldots,v_n^\ell\} \). There is an arc from \( v_i^\ell \) to \( v_j^\ell \) if and only if, there is a walk \( (v_j,e_1,v_i,e_2,v_k) \) in \( G^c \) such that \( c(e_1) = \ell \) and \( c(e_2) = \ell'. \) This construction can be done within polynomial time. An example is given in Figure 3.

![Figure 3: Transformation of \( G^c \) into a digraph \( H \).](image)

Formally, the digraph \( H \) is built as follows:

- \( V' = \{s_0,t_0\} \cup \{s^\ell,v_1^\ell,\ldots,v_n^\ell,t^\ell : \ell \in I_c\} \)
- For any pair of edges \( [v_j,v_i] \in E^\ell(G^c) \) and \( [v_i,v_k] \in E^\ell'(G^c) \), with \( \ell,\ell' \in I_c \) and \( v_i \in V(G^c) \setminus \{s,t\} \) (possibly with \( v_j = v_k \)), add arcs \( (v_i^\ell,v_k^\ell) \) and \( (v_i^\ell',v_j^\ell') \) to \( E' \). Next update \( E' \) by deleting all incoming (resp., outgoing) arcs to \( s^\ell \) (resp., from \( t^\ell \)) for every \( \ell \in I_c \). Moreover, add arc \( (s_0,s^\ell) \) to \( E' \) (resp., \( (t^\ell,t_0) \) to \( E' \)), if and only if,
there exists \([s, v_i] \in E^e(G^c)\) (resp., \([v_i, t] \in E^e(G^c)\)).

- \(w(v^e_i, v^e_j) = r^e_{i,j}\) for arc \((v^e_i, v^e_j) \in \overline{E}\) and \(w(s^e_i, v^e_j) = 0\) for arc \((s^e_i, v^e_j) \in \overline{E}\). Finally, \(w(s_0, s^e_i) = 0\) for arc \((s_0, s^e_i) \in \overline{E}\) and \(w(t^e_i, t_0) = 0\) for arc \((t^e_i, t_0) \in \overline{E}\).

**Theorem 2.** For any simple connected edge-colored graph \(G^c\) and any pair \(s, t\) of vertices of \(G^c\), the minimum reload \(s\-t\) walk problem can be solved in polynomial time.

**Proof.** Let \(G^c\) with \(V(G^c) = \{s, t\} \cup \{v_1, \ldots, v_n\}\) be a simple edge-colored connected graph with colors in \(I_c\). We apply the transformation described above. Now, observe that any directed path \(\rho^c\) from \(s_0\) to \(t_0\) in \(H\) with weight \(w(\rho^c) = \sum_{e \in \rho^c} w(e)\) corresponds in \(G^c\) to an \(s\-t\) walk \(\rho_c\) with reload cost \(r(\rho_c) = w(\rho^c)\). Symmetrically any minimum reload \(s\-t\) walk \(\rho^*_c\) of \(G^c\) with reload cost \(r(\rho^*_c)\) and using a minimum number of edges can be converted into a directed path \(\rho'\) from \(s_0\) to \(t_0\) in \(H\) such that \(w(\rho') = r(\rho^*_c)\). Actually, in order to prove this claim we need to show that the directed path \(\rho'\) will not pass twice by vertex \(v^e_i\) for each \(v \in V(G^c)\) and \(\ell \in I_c\). This latter property holds because we have:

**Property 3.** If \(\rho^*_c\) is a minimum reload \(s\-t\) walk of \(G^c\) using a minimum number of edges, then \(\rho^*_c\) does not contain a subsequence \((e_0, v, e_1, \ldots, e_k, v, e_{k+1})\) with \(c(e_0) = c(e_k)\) or \(c(e_1) = c(e_{k+1})\).

**Proof.** Let us show Property 3 by contradiction. Let \(\rho^*_c\) be a minimum reload \(s\-t\) walk of \(G^c\) using a minimum number of edges and assume that \(\rho^*_c\) contains a subsequence \((e_0, v, e_1, \ldots, e_k, v, e_{k+1})\) with \(c(e_0) = c(e_k)\) or \(c(e_1) = c(e_{k+1})\). Let \(\rho'_c\) be the walk in which the subsequence \((e_0, v, e_1, \ldots, e_k, v, e_{k+1})\) is replaced by \((e_0, v, e_{k+1})\). In this case, the sequence \(\rho'_c\) is an \(s\-t\) walk in \(G^c\) with reload cost \(r(\rho'_c) \leq r(\rho^*_c)\), contradiction with the definition of \(\rho^*_c\). Thus, we deduce that \(\rho^*_c\) can be converted into an oriented path from \(s_0\) to \(t_0\) in \(H\) since this path will pass through vertices \(v^{c(e_0)}\) and \(v^{c(e_k)}\) which are different. Notice that Property 3 also implies that \(\rho^*_c\) contains at most twice the same edge and if an edge \(e\) appears twice in \(\rho^*_c\) then it is used in both directions (see for instance the walk \(\rho_1\) in Figure 2) This figure illustrates two different \(s\-t\) walks, \(\rho_1 = (s, e_1, v_1, e_3, t)\) and \(\rho_2 = (s, e_1, v_1, e_2, v_2, e_2, v_1, e_3, t)\), with reload costs \(r_{i,j} = 1\) for \(i, j \in \{1, 2, 3\}\) except for \(r_{1,3} = r_{3,1} = 4\). The reload cost of \(\rho_1\) is \(r(\rho_1) = r_{1,3} = 4\) whereas the reload cost of \(\rho_2\) is \(r(\rho_2) = r_{1,2} + r_{2,2} + r_{2,3} = 3\).

In conclusion, a shortest directed path from \(s_0\) to \(t_0\) in \(H\) corresponds to a minimum reload \(s\-t\) walk in \(G^c\) and thus it can be computed within polynomial time.

**3 Trails and paths with symmetric reload costs**

Let \(R\) be a symmetric matrix with non-negative integer reload costs. Here, we prove that the minimum reload \(s\-t\) trail problem can be solved in polynomial time for every \(c \geq 2\). In addition, we show that the minimum reload \(s\-t\) path problem can be solved in polynomial time either if \(c = 2\) and the triangle inequality holds (here \(R\) is not necessarily a symmetric matrix) or if \(G^c\) has a maximum degree 3. However the problem is NP-hard when \(c \geq 3\) for graphs satisfying the triangle inequality and with maximum degree equal to 4. We conclude the section by showing that, if \(c \geq 4\) and the triangle inequality is satisfied, then the minimum reload \(s\-t\) path problem remains NP-hard even for planar graphs with maximum degree 4.
In the sequel, we show how to turn the minimum reload s-t trail problem into a minimum perfect matching problem in a weighted non-colored graph $G$ defined as follows.

Given two vertices $s$ and $t$ in $V(G^c) = \{v_1, \ldots, v_n\}$, set $W = V(G^c) \setminus \{s, t\}$. Now, for each $v_i \in W$, we define a subgraph $G_i$ with vertex and edge sets as illustrated in Figure 4. Formally:

- $V(G_i) = \{v_{i,j}, v'_{i,j} : v_j \in N_{G^c}(v_i)\} \cup \{p_{j,k}, q_{j,k} : j < k \text{ and } v_j, v_k \in N_{G^c}(v_i)\}$
- $E(G_i) = \{[v_{i,j}, v'_{i,j}] : v_j \in N_{G^c}(v_i)\} \cup \{[v'_{i,j}, p_{j,k}], [p_{j,k}, q_{j,k}], [q_{j,k}, v'_{i,j}] : j < k \text{ and } v_j, v_k \in N_{G^c}(v_i)\}$

The non-colored graph $G = (V', E')$ edge weighted by $w$ is constructed as follows:

- $V' = \{s', t'\} \cup (\bigcup_{v_i \in W} V(G_i))$, and
- $E' = \{[v_{i,j}, v_{i,x,y}] : j = x \text{ and } i = y\} \cup \{[s', v_{i,j}] : v_j = s \text{ and } [v_i, v_j] \in E(G^c)\} \cup \{[v_{i,j}, t'] : v_j = t \text{ and } [v_i, v_j] \in E(G^c)\} \cup \{[v'_{i,j}, p_{j,k}], [p_{j,k}, q_{j,k}], [q_{j,k}, v'_{i,j}] : j < k \text{ and } v_j, v_k \in N_{G^c}(v_i)\}$

- $w([v'_{i,j}, p_{j,k}]) = \frac{1}{2}r_{G^c}(v_{i,j}, v_k) \cdot c(v_{i,j}, v_k)$, $w([v'_{i,j}, q_{j,k}]) = \frac{1}{2}r_{G^c}(v_{i,j}, v_k) \cdot c(v_{i,j}, v_k)$ and all remaining edges have a weight 0.

After $G$ is constructed, we have to find a minimum weighted perfect matching $M^*$ in $G$. A matching is a set of pairwise non adjacent edges. A matching $M$ is called perfect if $M$ touches all vertices of $G$. The weight of matching $M$ is $w(M) = \sum_{e \in M} w(e)$ (computing a minimum weighted perfect matching is polynomial, see [8] for a good reference on general matchings). Notice that $G$ has a perfect matching since there exists an s-t path in $G^c$, which is assumed to be connected. We can prove that perfect matchings in $G$ will be associated to reload s-t trails in $G^c$ and vice-versa. Formally:

**Theorem 4.** For any simple connected edge-colored graph $G^c$ and any pair $s, t$ of vertices of $G^c$, the minimum symmetric reload s-t trail problem can be solved in polynomial time.
Proof. From $G^c$ instance of the the minimum symmetric reload $s$-$t$ trail problem, we polynomially build a weighted undirected graph $G = (V', E')$ as indicated above (see Figure 4). Let $M$ be a weighted perfect matching in $G$ with weight $w(M) = \sum_{e \in M} w(e)$. The associated reload $s$-$t$ trail $\rho_c$ in $G^c$ can be obtained after the contraction of all subgraphs $G_i$ in $G$ and by associating the remaining non-colored edges with colored edges in $G^c$. Since the reload cost matrix is symmetric and $w([v_{i,j}', p_{i,k}']) + w([v_{i,k}', q_{j,k}']) = r_{c([v_i,v_j]),c([v_i,v_k])}$ we can easily see that $w(M) = r(\rho_c)$.

Conversely, given an $s$-$t$ trail $\rho_c$ of $G^c$ with reload cost $r(\rho_c)$, we construct the associated perfect matching $M$ in the following manner: (a) for every vertex $v_i$ of $G^c$, out of $\rho_c$, we choose all edges with weight 0 in $G_i$; and (b), for every vertex $v_i$ of $G^c$, belonging to $\rho_c$, if $\rho_c$ contains the subsequence $(v_a, e, v_i, e', v_b)$ with $e \neq e'$, we choose edges $[v_{i,a}', p_{i,a}]$ and $[q_{a,b}', v_{i,b}']$ (we assume $a < b$) of $G_i$; and finally, (c) we choose all the remaining edges of $G$ (with cost 0), in order to obtain a perfect matching of $G$. In this way, it is easy to see that $w(M) = r(\rho_c)$. Therefore, a minimum reload $s$-$t$ trail corresponds in $G$ to a minimum weighted perfect matching. Note that the complexity of the minimum reload $s$-$t$ trail is dominated by the complexity of the minimum perfect matching problem in $G$. Since the construction of each $G_i$ depends on the number of neighbors of $v_i$, we can say that a minimum reload $s$-$t$ can be obtained in polynomial time in the size of $G^c$. \hfill \Box

Corollary 5. For any simple connected edge-colored graph $G^c$ of maximum degree 3 and any pair $s, t$ of vertices of $G^c$, the minimum symmetric reload $s$-$t$ path problem can be solved in polynomial time.

Proof. Obvious since in graphs of maximum degree 3, any $s$-$t$ path is an $s$-$t$ trail. The reload cost matrix being symmetric, one can apply Theorem 4. \hfill \Box

A complete example with $G^c$ and its associated non-colored graph $G$ is illustrated at the end of the paper (see Figure 10).

Now, we deal with graphs $G^c$ colored with two colors. We show that the minimum reload $s$-$t$ path problem is polynomial if the reload cost matrix $R$ satisfies the triangle inequality ($R$ is not necessarily symmetric).

Theorem 6. For any simple connected edge-colored graph $G^c$ with $c = 2$ colors and such that the associated matrix $R$ of reload costs satisfies the triangle inequality and for any pair $s, t$ of vertices of $G^c$, the minimum reload $s$-$t$ path problem can be solved in polynomial time.

Proof. Let $G^c = (V, E)$ with $I_c = \{1, 2\}$ be an instance of the minimum reload $s$-$t$ path problem. We also assume that the reload cost matrix $R = [r_{i,j}]$ satisfies the triangle inequality. Here, $R$ is not necessarily symmetric. We first show that any minimum reload $s$-$t$ walk of $G^c$ using a minimum number of edges is an $s$-$t$ path of $G^c$. Let $\rho^*_c$ be a minimum reload $s$-$t$ walk of $G^c$ using a minimum number of edges and assume that $\rho^*_c$ passes twice through some vertices. Consider the first vertex $v$ visited twice by $\rho^*_c$. This means that $\rho^*_c$ contains the subsequence $C = (v_0, e_0, v, e_1, \ldots, e_k, v, e_{k+1}, v_k)$ (see Figure 5 for an illustration). Let $\rho'_c$ be the $s$-$t$ walk in which the subsequence $C$ is replaced by $(v_0, e_0, v, e_{k+1}, v_k)$. We show that $r(\rho'_c) \leq r(\rho^*_c)$ which leads to a contradiction since on the one hand $|\rho'_c| < |\rho^*_c|$ and on the other hand $\rho^*_c$ is supposed to get a minimum number of edges. We consider two cases:
Case (1) $c(e_1) \neq c(e_k)$. If $c(e_0) = c(e_{k+1})$ then $r_{c(e_0),c(e_{k+1})} \leq r_{c(e_0),c(e_1)} + r_{c(e_k),c(e_{k+1})}$ (recall that $|I_c| = 2$); thus $r(\rho'_c) \leq r(\rho^*_c)$ and we get a contradiction. So, $c(e_0) \neq c(e_{k+1})$ and moreover $c(e_0) = c(e_1)$ for the same reasons (otherwise, $c(e_0) \neq c(e_1)$ and then $c(e_1) = c(e_{k+1})$ since $|I_c| = 2$. Hence $rc(e_0),c(e_1)$ and we deduce $r(\rho'_c) \leq r(\rho^*_c)$. Now, since $|I_c| = 2$ there exists $i \in \{2, \ldots, k\}$ such that $c(e_1) = c(e_{i-1}) \neq c(e_i)$. We deduce that $r(\rho'_c) \leq r(\rho^*_c)$. See case (1) of Figure 5.

Case (2) Now, assume $c(e_1) = c(e_k)$. Since edges $e_0, e_k, e_{k+1}$ are adjacent to a common vertex $v$, by applying the triangle inequality we obtain $r_{c(e_0),c(e_{k+1})} \leq r_{c(e_0),c(e_1)} + r_{c(e_k),c(e_{k+1})} = r_{c(e_0),c(e_1)} + r_{c(e_k),c(e_{k+1})}$. Thus, $r(\rho'_c) \leq r(\rho^*_c)$. See case (2) of Figure 5.

In conclusion, any optimal reload $s$-$t$ walk of $G^c$ using a minimal number of edges is an $s$-$t$ path.

Finally, we apply the transformation made in Theorem 2 from instance $G^o$ with $|I_c| = 2$ except that we replace $w(e)$ by $w'(e) = (2m+1)w(e) + 1$ for each arc $e$ of $H$. Let $\rho'$ be a shortest directed $s_0$-$t_0$ path in $H$ with weight $w'(\rho')$. The path $\rho'$ corresponds in $G^c$ to an optimal reload $s$-$t$ walk $\rho'_c$ of $G^c$ using a minimum number of edges which conclude the proof. Otherwise, let $\rho^*_c$ be an optimal reload $s$-$t$ walk of $G^c$ using a minimum number of edges $|\rho^*_c| < |\rho'_c|$. We have $|\rho^*_c| \leq 2m$ since any edge of $G^c$ is used at most twice (see Property 3 of Theorem 2). The sequence $\rho^*_c$ corresponds to a directed path $\rho^*$ in $H$ with weight $w'(\rho^*) = (2m+1)w(\rho^*) + |\rho^*_c| + 2 = (2m+1)r(\rho^*_c) + |\rho^*_c| + 2$. We deduce $r(\rho^*_c) = r(\rho'_c)$ since otherwise $w(\rho^*) = r(\rho^*_c) - 1 = w(\rho'_c) - 1$ (recall that $r_{i,j} \in \mathbb{N}$) and $w'(\rho^*) = (2m+1)w(\rho'_0) - 1 + |\rho^*_c| + 2 = (2m+1)w(\rho'_0) + |\rho^*_c| + 2 = w'(\rho'_c)$ (recall that $|\rho^*_c| \leq 2m$). Thus, $w'(\rho^*) < w'(\rho'_c)$, contradiction since $\rho'$ is assumed to be a shortest directed $s_0$-$t_0$ path of $H$.

A possible application of Theorem 6 is the following. Consider a $(2, 2)$-matrix $R$ satisfying $r_{1,1} = r_{2,2} = 0$. It is easy to see that $R$ satisfies the triangle inequality, and then one can apply Theorem 6 (on the other hand, this restriction becomes $\text{NP}$-hard for a $(3, 3)$-matrix with $r_{i,i} = 0$, see the proof of item (i) of Corollary 3). We also deduce that the minimum toll cost $s$-$t$ path problem (see Section 1) is polynomial for two colors since it is a subproblem of the case considered above. Notice that, the minimum toll cost $s$-$t$ path problem for $r_j = 1 \forall j \in I_c$, is equivalent to minimizing the number of color flips in an $s$-$t$ path. Actually, the minimum toll cost $s$-$t$ path problem is polynomially solvable (without constraints on the number of colors).

Theorem 7. For any simple connected edge-colored graph $G^c$ and any pair $s, t$ of vertices of $G^c$, the minimum toll cost $s$-$t$ path problem can be solved in polynomial time.
Proof. The proof is quite identical to Theorem 6. Let \( R = [r_{i,j}] \) be a reload cost matrix satisfying \( r_{i,j} = 0 \) and \( r_{i,j} = r_j \) if \( i \neq j \). We only show that any minimum reload \( s-t \) walk of \( G_c \) using a minimum number of edges is an \( s-t \) path of \( G_c^e \). Let \( \rho^*_c \) be a minimum reload \( s-t \) walk of \( G_c^e \) using a minimum number of edges and assume that \( \rho^*_c \) contains the subsequence \( C = (v_0, e_0, v, e_1, \ldots, e_k, v, e_{k+1}, v_k) \) (possibly with \( e_1 = e_k \)). Let \( \rho^*_c \) be the \( s-t \) walk in which the subsequence \( C \) is replaced by \( C' = (v_0, e_0, v, e_{k+1}, v_k) \). If \( c(e_0) = c(e_{k+1}) \), then \( r(C') = 0 \leq r(C) \). If \( c(e_0) \neq c(e_{k+1}) \), then \( r(C') = r(c(e_0), c(e_{k+1})) \leq r(C) \). The rest of the proof is similar to proof of Theorem 6.

Now, we show that the previous restrictions on \( G_c^e \) are almost the best ones to obtain polynomial cases for the minimum reload \( s-t \) path problem.

**Theorem 8.** The minimum symmetric reload \( s-t \) path problem is \( \text{NP}-\text{hard} \) if \( c \geq 3 \), the triangle inequality holds and the maximum degree of \( G_c^e \) is equal to 4.

Proof. We are given a set \( C \) of CNF clauses defined over a set \( X \) of boolean variables. An instance of the \((3, B2)\)-sat problem, called 2-balanced 3-SAT, is such that each clause has exactly 3 literals, each of them appearing exactly 4 times in the clauses, twice negated and twice unnegated. Deciding whether an instance of \((3, B2)\)-sat is satisfiable is \( \text{NP}\)-complete [9]. We are going to reduce \((3, B2)\)-sat to the existence of an \( s-t \) path with reload cost at most \( L \). Let \( I \) be an instance of \((3, B2)\)-sat. For \( h \in \{1, 2, 3, 4\} \), we say that \( C_j \) is the \( h \)-th clause of \( x_i \) if \( x_i \) appears in \( C_j \) and \( x_i \) appears in exactly \( h-1 \) other clauses \( C_j \) with \( j' < j \). For \( \ell \in \{1, 2, 3\} \), we say that \( x_i \) is the \( \ell \)-th variable of \( C_j \), if and only if, \( x_i \) and exactly \( \ell-1 \) other variables \( x_{i'} \) with \( i' < i \) appear in \( C_j \). We build \( G_c^e \) instance of the \( s-t \) path with reload cost at most \( L \) — as follows. We have \( I_c = \{1, 2, 3\} \) and \( L = 11|X| + 3|C| + 1 \). The matrix \( R \) is defined as \( r_{1,2} = r_{2,1} = M \) where \( M > L \). The other entries of \( R \) are set to 1.

The graph \( G_c^e \) has a source vertex \( s \) and a sink vertex \( t \). In addition, for each \( x_i \in X \) (resp. \( C_j \in C \)) we build a gadget as depicted on the left (resp. right) of Figure 6. The gadget of a variable \( x_i \) consists of a left part (vertices \( f_i^0 \) to \( f_i^3 \) and \( d_i^0 \) to \( d_i^3 \)), a right part (vertices \( t_i^0 \) to \( t_i^4 \) and \( k_i^0 \) to \( k_i^3 \)), an entrance \( a_i \) and an exit \( b_i \). The left (resp. right) part corresponds to the case where \( x_i \) is set to \text{false} (resp. \text{true}). The gadget of a clause \( C_j \) consists of an entrance \( q_j \), an exit \( w_j \) and three vertices \( u_j^1 \), \( u_j^2 \), and \( u_j^3 \) which correspond to the first, second and third variable of \( C_j \) respectively. We link the gadgets by adding the following edges (see Figure 7):

- \([s, a_1], [b_1, a_2], [b_2, a_3], \ldots, [b_{|X|-1}, a_{|X|}]\) with color 2 (bold);
- \([b_{|X|}, q_1]\) with color 3 (dashed);
- \([w_1, q_2], [w_2, q_3], \ldots, [w_{|C|-1}, q_{|C|}], [w_{|C|}, t]\) with color 1 (thin).

For each pair \( x_i, C_j \) such that \( x_i \) is the \( \ell \)-th variable of \( C_j \) and \( C_j \) is the \( h \)-th clause of \( x_i \) we proceed as follows. If \( x_i \) appears negated in \( C_j \) then add \([t_i^{h-1}, u_j^1], [t_i^h, u_j^2], [f_i^{h-1}, d_i^0], [f_i^h, d_i^3]\) and \([f_i^{h-1}, d_i^0], [f_i^h, d_i^3]\) with color 2 (bold). If \( x_i \) appears unnegated in \( C_j \) then add \([f_i^{h-1}, u_j^1], [f_i^h, u_j^2], [t_i^{h-1}, k_i^0], [t_i^h, k_i^3]\) with color 2 (bold). It is not difficult to see that each vertex’s degree is at most 4. Moreover the triangle inequality holds.

Since \( r_{1,2} > L \) and \( r_{2,1} > L \), any \( s-t \) path \( \rho_c \) with reload cost at most \( L \) starts at \( s \), enters the gadget of \( x_1 \) and visits the variable-gadgets in turn. When \( b_{|X|} \) is reached, \( \rho_c \) uses \([b_{|X|}, q_1]\) and visits the clause-gadgets in turn. Finally \( t \) is reached from \( w_{|C|} \). Then
Figure 6: Gadgets for a variable $x_i$ (left) and a clause $C_j$ (right).

Figure 7: Left: How the gadgets are linked. Right: How to link the gadget of $x_7$ if it appears in $C_3 = (x_1 \lor x_7 \lor x_8)$, $C_4 = (\overline{x}_3 \lor x_5 \lor \overline{x}_7)$, $C_5 = (\overline{x}_7 \lor \overline{x}_8 \lor x_9)$ and $C_6 = (\overline{x}_1 \lor \overline{x}_6 \lor x_7)$. 
the number of vertices (resp., intersections between two edges) in the graph of \( G^c \).

If a truth assignment \( \tau \) satisfies \( I \) then \( G^c \) admits an \( s \)-\( t \) path \( \rho_c \) with reload cost \( 11|X| + 3|C| \). Indeed, if \( x_i \) is false (resp. true) in \( \tau \) then \( \rho_c \) goes across the left (resp. right) part of \( x_i \)'s gadget. Since \( \tau \) satisfies \( I \) we know that at least one literal per clause is true. If the \( \ell \)-th literal of \( C_j \) is true (choose \( \ell \) arbitrarily if it is not unique) then \( \rho_c \) passes through \( u_i^\ell \). Conversely an \( s \)-\( t \) path \( \rho_c \) with reload cost \( 11|X| + 3|C| + 1 \) induces a truth assignment that satisfies \( I \): set \( x_i \) to false (resp. true) if \( \rho_c \) passes through the left (resp. right) part of \( x_i \)'s gadget.

\[ \square \]

**Corollary 9.** The two following statements hold:

(i) In the general case, the minimum symmetric reload \( s \)-\( t \) path problem is not approximable at all if \( c \geq 3 \), the triangle inequality holds and the maximum degree of \( G^c \) is equal to 4.

(ii) If \( r_{i,j} \geq 1 \) for every \( i, j \in I_c \), the minimum symmetric reload \( s \)-\( t \) path problem is not \( O(2^{P(n)}) \)-approximable for every polynomial \( P \) if \( c \geq 3 \), the triangle inequality holds and the maximum degree of \( G^c \) is equal to 4.

**Proof.** We show that the reduction built in Theorem 8 is a gap reduction, which means that it is \( \text{NP} \)-complete to distinguish between \( \text{OPT}(G^c) \leq B \) and \( \text{OPT}(G^c) > B \) for some \( B > 0 \). Let us denote by \( \text{OPT}(G^c) \) the reload cost of an optimal solution of \( G^c \), the instance built in Theorem 8.

For (i) we modify the reload costs as follows: \( r_{2,3} = r_{3,2} = r_{3,1} = r_{1,3} = 0 \), \( r_{i,j} = 0 \) for \( i \in I_c = \{1, 2, 3\} \) and \( r_{1,2} = r_{2,1} = M \). Notice that the reload cost matrix \( R \) is symmetric and satisfies the triangle inequality. We have \( \text{OPT}(G^c) = 0 \) iff \( I \) is satisfiable. Thus, it is \( \text{NP} \)-complete to distinguish between \( \text{OPT}(G^c) = 0 \) and \( \text{OPT}(G^c) \geq 1 \).

For (ii), let \( P \) be a polynomial. Set \( M = O(2^{P(n)}) \) where \( n \) is the number of vertices of \( G^c \) in the proof of Theorem 8. We deduce that it is \( \text{NP} \)-complete to distinguish between \( \text{OPT}(G^c) \leq L \) and \( \text{OPT}(G^c) \geq O(2^{P(n)}) \).

\[ \square \]

**Corollary 10.** The minimum symmetric reload \( s \)-\( t \) path problem is \( \text{NP} \)-hard if \( c \geq 4 \), the graph is planar, the triangle inequality holds and the maximum degree is equal to 4.

**Proof.** We use the instance \( G^c \) in the proof of Theorem 8 and make it planar. To do so we use an additional color 4 such that \( r_{3,4} = r_{4,3} = M \) and \( r_{1,4} = r_{4,1} = r_{2,4} = r_{4,2} = 1 \). Let \( G^c \) be an embedding of the graph built in Theorem 8. Here \( M > n + 3p \) where \( n \) (resp. \( p \)) is the number of vertices (resp. intersections between two edges) in the graph of \( G^c \). Note that \( p \) is polynomially bounded in the order of \( G^c \).

One can suppose w.l.o.g. that \([b, |X|, q_1]\) (with color 3) is not intersected by another edge of \( G^c \) (see Figure 7). If some edge \([a, b]\) with color 1 intersects \([c, d]\) with color 2 in \( G^c \) we add a new vertex \( f \) and replace \([a, b]\) by \([a, f], [f, b]\) with color 1, and edge \([c, d]\) by \([c, f], [f, d]\) with color 2. If \([a, b]\) with color 1 (resp. 2) intersects \([c, d]\) with color 1 (resp. 2), we add five new vertices \([f, a', b', c', d']\) and replace \([a, b]\) by \([a, a'], [b', b]\) with color 1 (resp. 2), replace \([c, d]\) by \([c, c'], [d', d]\) with color 1 (resp. 2), add \([a', f], [f, b']\) with color 3 and add \([c', f], [f, d']\) with color 4. In this way, the graph of the resulting instance—denoted by \( G^c \)—is planar.

It is not difficult to see that \( I \) (the instance of \((3, B2) - \text{sat}\) from which \( G^c \) is built) is satisfiable iff there is an \( s \)-\( t \) path \( \rho_c \) in \( G^c \) such that \( r(\rho_c) < M \).
4 Trails and paths with asymmetric reload costs

We now deal with asymmetric reload costs. We mainly prove that the minimum reload $s$-$t$ trail problem is NP-hard in this case.

**Theorem 11.** The minimum asymmetric reload $s$-$t$ trail problem is NP-hard if $c \geq 3$ and the maximum degree of $G^c$ is equal to 3.

**Proof.** This proof is similar to the one of Theorem 8, i.e. we reduce $(3, B2)$-SAT to the existence of an $s$-$t$ path with reload cost at most $L$. Hence we use the same notations and only describe how $G^c$ is built upon $I$. A trail must be a path in the graph of $G^c$ since a vertex’s maximum degree is 3. Hence we only deal with paths in this proof.

We have $I_c = \{1, 2, 3\}$ and $L = 15|X| + 6|C| + 1$. The matrix $R$ is defined as $r_{1,2} = r_{2,3} = r_{3,1} = M$ where $M > L$. The other entries of $R$ are set to 1. The graph $G^c$ has a source $s$ and a sink $t$. In addition, for each $x_i \in X$ (resp. $C_j \in C$) we build a gadget as depicted on the left (resp. middle) of Figure 8. The gadget of a variable $x_i$ consists of a left part (vertices $f_i$, $d_i$ and $e_i$), a right part (vertices $t_i$, $k_i$ and $o_i$), an entrance $a_i$ and an exit $b_i$. The left (resp. right) part corresponds to the case where $x_i$ is set to false (resp. true). The gadget of a clause $C_j$ consists of a left part (vertices $u_j^1$ and $v_j^1$), a middle part (vertices $u_j^2$ and $v_j^2$), a right part (vertices $u_j^3$ and $v_j^3$), an entrance $q_j$, an exit $w_j$ and four intermediate vertices $z_j^1$, $z_j^2$, $y_j^1$ and $y_j^2$. The left, middle and right parts correspond to the first, second and third variable of $C_j$ respectively.

We link the gadgets by adding the following edges with color 3 (dashed): $[s, a_1], [b_1, a_2], [b_2, a_3], \ldots, [b_{|X|}, a_{|X|}]; [b_{|X|}, q_1]; [w_1, q_2], [w_2, q_3], \ldots, [w_{|C| - 1}, q_{|C|}], [w_{|C|}, t]$ (this construction is similar to the one described in the left part of Figure 7 except for the colors of the edges). For each pair $x_i$, $C_j$ such that $x_i$ is the $\ell$-th variable of $C_j$ and $C_j$ is the $h$-th clause
of $x_i$ we proceed as follows. If $x_i$ appears negated in $C_j$ then add $[t_i^{h-1}, v_j^c]$ with color 1 (thin), $[t_i^h, u_j^c]$ with color 2 (bold), $[f_i^{h-1}, d_i^{h-1}]$ with color 1 and $[f_i^h, e_i^c]$ with color 2. If $x_i$ appears unnegated in $C_j$ then add $[t_i^{h-1}, v_j^c]$ with color 1, $[f_i^h, u_j^c]$ with color 2, $[t_i^h, d_i^{h-1}]$ with color 1 and $[t_i^h, e_i^c]$ with color 2. Now $G^c$ is fully described. An example is given on the right of Figure 8. It is not difficult to see that each vertex’s degree of $G^c$ is at most 3.

As in the proof of Theorem 5 it is not difficult to see that a truth assignment that satisfies $I$ corresponds to an $s$-$t$ path with reload cost $15|X| + 6|C| + 1$ in $G^c$ and vice-versa. A complete example of our reduction is illustrated in the Figures 11 and 12.

For graphs of maximum degree 3, trails and paths are identical. Thus, using Theorem 11 we deduce:

**Corollary 12.** The minimum asymmetric reload $s$-$t$ path problem is NP-hard if $c \geq 3$ and the maximum degree of $G^c$ is equal to 3.

**Corollary 13.** The two following statements hold:

(i) In the general case, the minimum asymmetric reload $s$-$t$ trail and the minimum asymmetric reload $s$-$t$ path problems are not approximable at all if $c \geq 3$ and the maximum degree of $G^c$ is equal to 3.

(ii) If $r_{i,j} \geq 1$ for every $i, j \in I_c$, the minimum asymmetric reload $s$-$t$ trail and the minimum asymmetric reload $s$-$t$ path problems are not $O(2^{P(n)})$-approximable for every polynomial $P$ if $c \geq 3$ and the maximum degree of $G^c$ is equal to 3.

**Proof.** The proofs are quite identical to the proof of Corollary 9. For (i) replace the entries of $R$ equal to 1 by 0 and for (ii) replace $M$ by $M = O(2^{P(n)})L$. □

We know that the minimum symmetric reload $s$-$t$ trail problem is polynomially solvable (see Theorem 4). We now prove that this result also holds with asymmetric reload costs if the triangle inequality is satisfied.

**Theorem 14.** For any simple connected edge-colored graph $G^c$ and any pair $s, t$ of vertices of $G^c$, the minimum asymmetric reload $s$-$t$ trail problem can be solved in polynomial time, if the triangle inequality holds.

**Proof.** The proof is similar to the one of Theorem 6 except that we deal with trails instead of paths. In other words, we can prove that any minimum reload $s$-$t$ walk $\rho^*_c$ of $G^c$ using a minimal number of edges is indeed an $s$-$t$ trail of $G^c$. We recall that $\rho^*_c$ contains at most twice the same edge (see Property 3 of Theorem 2). Actually, assume that there exists a minimum reload $s$-$t$ walk $\rho^*_c$ of $G^c$ using a minimal number of edges which is not an $s$-$t$ trail of $G^c$. This means that when we walk around $\rho^*_c$, we meet an edge $e_1 = [v_1, v_2]$ at least twice. It is clear that we can suppose that $e_1$ is not met twice in the same way (i.e., both from $v_1$ to $v_2$ or both from $v_2$ to $v_1$). Thus, w.l.o.g., assume that $\rho^*_c$ contains the subsequence $\rho' = (v_0, e_0, v_1, e_1, v_2, \ldots, v_2, e_1, v_1, e_k, v_k)$. Let $\rho'_c$ be the $s$-$t$ walk where from $\rho^*_c$, the subsequence $\rho'$ is replaced by $(v_0, e_0, v_1, e_k, v_k)$. We obtain $r(\rho'_c) \leq r(\rho^*_c)$ since $r(\rho') \geq r(e_0, e_1, e_k) \geq r(e_0, e_k)$ ($r$ satisfies the triangular inequality) and we have $|\rho'_c| \leq |\rho^*_c|$, which leads to a contradiction. □
Traveling salesman problem with reload costs

The reload traveling salesman problem is defined upon a complete graph $K_n$ on vertices $\{1, \ldots, n\}$ where edges are colored in $I_c$. The goal is to find a vertex permutation $\pi$ (i.e., a Hamiltonian cycle) of $K_n$ minimizing its reload cost $r(\pi) = \sum_{i=1}^{n} r_{c(e_i),c(e_{(i+1)\mod n})}$ with $e_i = (\pi(i), \pi((i + 1) \mod n))$ for $i = 1, \ldots, n$. In [2] it is already proved that the reload traveling salesman problem is NP-hard, even if the reload cost is symmetric. Here, we prove that the result holds even if $c = 2$ and the reload cost satisfies the triangular inequality.

**Theorem 15.** The reload traveling salesman problem is NP-hard even if $c = 2$, the reload cost is symmetric and satisfies the triangular inequality.

**Proof.** The reduction is very simple and is done from the Hamiltonian cycle problem ($HC$ in short). $HC$ consists in deciding whether a simple graph $G$ contains an Hamiltonian cycle. $HC$ is known to be NP-complete [7]. Starting from a graph $G = (V, E)$ on $n$ vertices, instance of $HC$, we complete it into $K_n$ where the initial edges (i.e., edges of $E$) are colored in color 1 and added edges are colored in color 2. We set $r_{1,1} = 1$ and $r_{1,2} = r_{2,1} = r_{2,2} = M$ where $M > n$. Clearly, $K_n$ is colored with two colors and the reload cost $r_{i,j}$ for $i, j \in I_c$ is symmetric and satisfies the triangular inequality.

It is clear that $G$ has an Hamiltonian cycle iff there is an acyclic permutation $\pi$ of $V(K_n)$ with reload cost $r(\pi) \leq n$. See the example of Figure 9.

From this theorem, we deduce the following results.

**Corollary 16.** The two following statements hold:

(i) In the general case, the reload traveling salesman problem is not approximable at all even if $c = 2$, the reload cost matrix is symmetric and satisfies the triangular inequality.

(ii) If $r_{i,j} \geq 1$ for every $i, j \in I_c$, the reload traveling salesman problem is not $O(2^{P(n)})$-approximable for every polynomial $P(n)$ even if $c = 2$, the reload cost matrix is symmetric and satisfies the triangular inequality.

**Proof.** The proofs are quite identical to the proof of Corollary 9. For (i), replace the entries of $R$ equal to 1 (i.e., $r_{1,1} = 1$) by 0, and for (ii) replace $M$ by $M = O(2^{P(n)}) n$. 

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**Figure 9:** Instance of the Hamiltonian Cycle, where all edges are colored 1 (left). Complete graph, instance of the The reload traveling salesman problem (right).
6 Conclusion

In this paper we give a rather complete description of the complexity of the minimum reload $s$-$t$ walk, minimum reload $s$-$t$ trail and minimum reload $s$-$t$ path problems. When $c = 2$, we do not know the complexity of the minimum symmetric reload $s$-$t$ path and the minimum asymmetric reload $s$-$t$ trail problems if the matrix of reload costs does not satisfy the triangle inequality. These open problems seem important to better understand the complexity of the properly edge-colored $s$-$t$ trail / $s$-$t$ path problems when $G^c$ does not have a properly edge-colored $s$-$t$ trail / $s$-$t$ path. In this case, one could be interested in seeking an $s$-$t$ trail / $s$-$t$ path with a minimum number of vertices for which the adjacent edges have the same color. As a future direction, one could be interested in finding heuristic or exact solutions for the minimum reload $s$-$t$ path problem. In this case, the polynomial problems regarding $s$-$t$ trails/walks could be used in the determination of good lower bounds for the value of the minimum reload $s$-$t$ path problem.

Finally, notice that if we study the min-max reload $s$-$t$ walk/trail/path problems, all the results presented here also hold. In this case, we replace the reload cost of a path/trail/walk $\rho = (v_1, e_1, v_2, e_2, \ldots, e_k, v_{k+1})$ between vertices $s$ and $t$ defined as in (1) by the following reload cost $r(\rho) = \max\{r_{c(e_j),c(e_{j+1})} : j = 1, \ldots, k - 1\}$.

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References


Figure 10: A 2-edge-colored graph $G^c$ (top). Associated Weighted non-colored graph $G$ (bottom).
Figure 11: An example of the 3-edge-colored graph $G^c$ associated with the instance $\mathcal{I} = \{(x_1 \lor x_2 \lor x_3) \land (x_1 \lor \overline{x}_2 \lor x_3) \land (\overline{x}_1 \lor \overline{x}_2 \lor \overline{x}_3) \land (\overline{x}_1 \lor x_2 \lor \overline{x}_3)\}$ of the (3, B2)-sat problem.
Figure 12: An example of a solution of the Figure 11, where the variable $x_1$ is set to false and the variables $x_2$ and $x_3$ are set to true. The reload costs are $r_{1,2} = r_{2,3} = r_{3,1} = M > L$, the others entries are set to 1 and $L = 70$. 