Weighted coloring on planar, bipartite and split graphs: complexity and approximation

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Abstract

We study complexity and approximation of \textsc{min weighted node coloring} in planar, bipartite and split graphs. We show that this problem is \textsc{NP}-hard in planar graphs, even if they are triangle-free and their maximum degree is bounded above by 4. Then, we prove that \textsc{min weighted node coloring} is \textsc{NP}-hard in \(P_3\)-free bipartite graphs, but polynomial for \(P_3\)-free bipartite graphs. We next focus on approximability in general bipartite graphs and improve earlier approximation results by giving approximation ratios matching inapproximability bounds. We next deal with \textsc{min weighted edge coloring} in bipartite graphs. We show that this problem remains strongly \textsc{NP}-hard, even in the case where the input graph is both cubic and planar. Furthermore, we provide an inapproximability bound of \(7/6 - \varepsilon\), for any \(\varepsilon > 0\) and we give an approximation algorithm with the same ratio. Finally, we show that \textsc{min weighted node coloring} in split graphs can be solved by a polynomial time approximation scheme.

Keywords: Graph coloring; weighted node coloring; weighted edge coloring; approximability; \textsc{NP}-completeness; planar graphs; bipartite graphs; split graphs.

1 Introduction

We give in this paper some complexity results as well as some improved approximation results for \textsc{min weighted node coloring}, originally studied in Guan and Zhu [15]. A \(k\)-coloring of \(G = (V, E)\) is a partition \(S = (S_1, \ldots, S_k)\) of the node set \(V\) of \(G\) into stable sets \(S_i\). In this case, the objective is to determine a node coloring minimizing \(k\). A natural generalization of this problem is obtained by assigning a strictly positive integer weight \(w(v) > 0\) to any node \(v \in V\), and defining the weight of stable set \(S\) of \(G\) as \(w(S) = \max\{w(v) : v \in S\}\). Then, the objective is to determine a node coloring \(S = (S_1, \ldots, S_k)\) of \(G\) minimizing the quantity \(\text{val}(S) = \sum_{i=1}^{k} w(S_i)\). One of the original motivations for studying this problem is related to batch scheduling. In the typical situation where jobs in a single batch are processed in parallel, the processing time of a batch equals the largest processing time of the jobs inside this batch. In presence of pairwise incompatibilities between jobs (to be in the same batch), then minimizing the overall processing time is exactly an instance of \textsc{min weighted node coloring} (where jobs are nodes, weights are processing times, and edges are incompatibilities).

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This problem is easily shown to be \textbf{NP}-hard; it suffices to consider \( w(v) = 1, \forall v \in V \) and \textbf{MIN WEIGHTED NODE COLORING} becomes the classical node coloring problem. Other applications of \textbf{MIN WEIGHTED NODE COLORING} and other generalizations of batch coloring problems are indicated in Finke et al. \cite{11}. Other versions of weighted colorings have been studied in Hassin and Monnot \cite{17}, Balas and Xue\cite{1}, and Frank \cite{12}.

Consider an instance \( I \) of an \textbf{NP}-hard minimization problem \( \Pi \) and a polynomial time algorithm \( \mathcal{A} \) computing feasible solutions for \( \Pi \). Denote by \( m_k(I, S) \) the value of the \( \Pi \)-solution \( S \) computed by \( \mathcal{A} \) on \( I \) and by \( \text{opt}(I) \), the value of an optimal \( \Pi \)-solution for \( I \). The quality of \( \mathcal{A} \) is expressed by the ratio (called approximation ratio in what follows) \( \rho_k(I) = m_k(I, S)/\text{opt}(I) \), and the quantity \( \rho_k = \inf\{r : \rho_k(I) < r \text{ for all instance } I \text{ of } \Pi\} \). A very favorable situation for polynomial approximation occurs when for any \( \varepsilon > 0 \) there is a polynomial algorithm \( \mathcal{A}_\varepsilon \) achieving a ratio bounded above by \( 1 + \varepsilon \). We call such algorithms \((\mathcal{A}_\varepsilon, \varepsilon > 0)\) a \emph{polynomial time approximation scheme}. and the class of problems admitting such approximation schemes is denoted \textbf{PTAS}. Moreover, when the complexity of \( \mathcal{A}_\varepsilon \) is bounded by a function \( f(\varepsilon)p(|I|) \), for some polynomial \( p \), then the scheme is called \emph{efficient polynomial time approximation scheme}; when the complexity is bounded by a polynomial \( q(1/\varepsilon, |I|) \), then it is called a \emph{fully polynomial time approximation scheme}. The problems admitting such schemes are called respectively \textbf{EP\:TAS} and \textbf{FPTAS}.

\textbf{Related works.}

\textbf{MIN WEIGHTED NODE COLORING} has been introduced in \cite{15}, and further studied from a complexity and approximation viewpoint in \cite{6}. The results in this article, extending and improving previous ones, have been originally presented at the ISAAC 2004 conference, see \cite{9}.

During the last few years, \textbf{MIN WEIGHTED NODE COLORING} has also appeared under the name \textit{max coloring}, in Pemmaraju et al. \cite{25}, and Pemmaraju and R. Raman \cite{26}. In these two papers, several approximation results are given: a \( 4\rho \)-approximation is obtained for \textbf{MIN WEIGHTED NODE COLORING} in any class of graphs for which the coloring problem admits a \( \rho \)-approximation, \cite{26}. For instance, this implies a \( 4 \)-approximation for perfect graphs. Independently from our results, a \( 8/7 \)-approximation for \textbf{MIN WEIGHTED NODE COLORING} in bipartite graphs is also obtained. Very recently, on-line versions of \textbf{MIN WEIGHTED NODE COLORING} were presented in Epstein and Levin \cite{10} and an off-line \( \varepsilon \)-approximation is proposed for perfect graphs, \cite{10, 16}. Moreover, in Halldórssson and Shachnai \cite{16} a polynomial algorithm with time complexity \( O(n \log n) \) in paths and an efficient polynomial time approximation scheme in partial \( k \)-trees are given for \textbf{MIN WEIGHTED NODE COLORING}, improving the time complexities given in \cite{8} for these classes of graphs. In \cite{25, 8}, the problem is shown to be \textbf{NP}-hard in interval graphs, but it is polynomially solvable by a dynamic programming algorithm in co-interval graphs \cite{11}. A \( 2 \)-approximation for \textbf{MIN WEIGHTED NODE COLORING} in interval graphs is also obtained in \cite{25}. The edge coloring problem has been previously studied in \cite{22} as a special case of a non-preemptive scheduling model. In this latter paper, a greedy \( 2 \)-approximation is given and an approximation within a ratio smaller than \( 7/6 \) is proved to be \textbf{NP}-hard.

\textit{Contents of the article.}

We give some complexity and approximation results for \textbf{MIN WEIGHTED NODE COLORING}. We
first deal with planar graphs and we show that, for this family, the problem studied is NP-hard, even if we restrict our attention to triangle-free planar graphs with node-degree not exceeding 4.

We then deal with particular families of bipartite graphs. The NP-hardness of **min weighted node coloring** has been established in [6] for general bipartite graphs. We show here that this remains true even if we restrict ourselves to planar bipartite graphs or to $P_2$-free bipartite graphs, i.e. bipartite graphs that do not contain induced chains on 21 vertices or more (for definitions of graph-theoretical notions used in this paper, the interested reader is referred to Berge [2]).

It is interesting to observe that these results are obtained as corollaries of a kind of generic reduction from the precoloring extension problem shown to be NP-complete in Bodlaender et al. [3], Hujter and Tuza [19, 20], Kratochvil [21]. Then, we slightly improve the last result to $P_3$-free bipartite graphs and show that the problem becomes polynomial in $P_5$-free bipartite graphs. Observe that in [6], we have proved that **min weighted node coloring** is polynomial for $P_4$-free graphs and NP-hard for $P_5$-free graphs.

Then, we focus on approximability of **min weighted node coloring** in (general) bipartite graphs. As proved in [6], this problem is approximable in such graphs within an approximation ratio of $4/3$; in the same paper a lower bound of $8/7 - \varepsilon$, for any $\varepsilon > 0$, was also provided. Here we improve the approximation ratio of [6] by matching the $8/7$-lower bound of [6] with an equal upper bound; in other words, we show here that **min weighted node coloring** in bipartite graphs is approximable within approximation ratio bounded above by $8/7$.

We next deal with **min weighted edge coloring** in bipartite graphs. In this problem we consider an edge-weighted graph $G$ and try to determine a partition of the edges of $G$ into matchings in such a way that the sum of the weights of these matchings is minimum (analogously to the node-weighted model, the weight of a matching is the maximum of the weights of its edges). In [6], it is shown that **min weighted edge coloring** is NP-hard for cubic bipartite graphs. Here, we slightly strengthen this result by showing that this problem remains strongly NP-hard, in cubic and planar bipartite graphs. Furthermore, we strengthen the inapproximability bound provided in [6], by reducing it from $8/7 - \varepsilon$ to $7/6 - \varepsilon$, for any $\varepsilon > 0$. Also, we match it with an upper bound of the same value, improving so the $5/3$-approximation ratio provided in [6].

Finally, we deal with the approximation of **min weighted node coloring** in split graphs. As proved in [6], **min weighted node coloring** is strongly NP-hard in such graphs, even if the nodes of the input graph receive only one of two distinct weights. It followed that this problem could not be solved by fully polynomial time approximation schemes, but no approximation study was addressed there. In this paper we show that **min weighted node coloring** in split graphs can be solved by a polynomial time approximation scheme.

In the remainder of the paper, we shall assume that for any weighted node or edge coloring $S = (S_1, \ldots, S_t)$ considered, we will have $w(S_1) \geq \ldots \geq w(S_t)$.
2 Weighted node coloring in triangle-free planar graphs

The node coloring problem in planar graphs has been shown to be NP-hard by Garey and Johnson [13], even if the maximum degree does not exceed 4. On the other hand, this problem becomes easy in triangle-free planar graphs (see Grötzsch [14]). Here, we show that the weighted node coloring problem is NP-hard in triangle-free planar graphs with maximum degree 4 by using a reduction from 3-sat planar, a problem proved to be NP-complete in Lichtenstein [23]. This problem is defined as follows: Given a collection \( \mathcal{C} = (C_1, \ldots, C_m) \) of clauses over the set \( X = \{x_1, \ldots, x_n\} \) of boolean variables such that each clause \( C_j \) has at most three literals (and at least two), is there a truth assignment \( f \) satisfying \( \mathcal{C} \)? Moreover, the bipartite graph \( BP = (L, R; E) \) is planar where \( |L| = n, |R| = m \) and \( [x_i, c_j] \in E \) if the variable \( x_i \) (or \( \pi_i \)) appears in the clause \( C_j \).

**Theorem 2.1** MIN WEIGHTED NODE COLORING is NP-hard in triangle-free planar graphs with maximum degree 4.

**Proof:** Let \( BP = (L, R; E) \) be the bipartite graph representing an instance \((X, \mathcal{C})\) of 3-sat planar where \( L = \{x_1, \ldots, x_n\}, R = \{c_1, \ldots, c_m\} \). We construct an instance \( I = (G, w) \) of MIN WEIGHTED NODE COLORING by using two gadgets:

- The clause gadgets \( F(C_j) \) are given in Figure 1 for a clause \( C_j \) of size 3 and in Figure 2 for a clause \( C_j \) of size 2. The nodes \( c_j^k \) are those that will be linked to the rest of the graph.

![Figure 1: Graph \( F(C_j) \) representing a clause \( C_j \) of size 3.](image)

- The variable gadgets \( H(x_i) \) are given in Figure 3 for variable \( x_i \). Assume that \( x_i \) appears \( p_1 \) times positively and \( p_2 \) times negatively in \((X, \mathcal{C})\), then in \( H(x_i) \) there are \( 2p = 2(p_1 + p_2) \) special nodes \( x_i^k, \overline{x_i}^k, k = 1, \ldots, p \). These nodes form a path which meets node \( x_i^k, \overline{x_i}^k \) alternatively.

- The weights of nodes which are not given in Figures 1, 2 and 3 are 1.

- These gadgets are linked together by the following process. If variable \( x_i \) appears positively (resp. negatively) in clause \( C_j \), we link one of the variables \( \overline{x_i}^k \) (resp. \( x_i^k \)), with a different \( k \)
for each $C_j$, to one of the three nodes $c_j^i$ of gadget $F(C_j)$. This can be done in a way that preserves the planarity of the graph.

Indeed, consider a planar embedding of the graph $BP$. For each node $v$ of degree $\delta(v)$ in the planar representation of $BP$, let us call $e_{v}^{1}, \ldots, e_{v}^{\delta(v)}$ the endpoints on $v$ of the edges adjacent at $v$ considered in a circular order. Then, for each edge in $BP$ which joins node $x_i$ in endpoint $e_{x_i}^{k}$ to node $C_j$ in endpoint $e_{c_j}^{l}$, we put an edge from $x_i^k$ (if $x_i$ appears negatively in $C_j$, $x_i^k$ otherwise) to $c_j^l$.

Observe that $G$ is triangle-free and planar with maximum degree 4. Moreover, we assume that $G$ is not bipartite (otherwise, we add a disjoint cycle $\Gamma$ with $|\Gamma| = 5$ and $\forall v \in V(\Gamma), w(v) = 1$).

It is then not difficult to check that $(X,C)$ is satisfiable if and only if $opt(I) \leq 6$.

Let $g$ be a truth assignment satisfying $(X,C)$. We set $S'_1 = \{v : w(v) = 3\}$ and $S'_2 = \{v : w(v) = 2\} \cup \{x_i^k : g(x_i) = 1\} \cup \{x_i^k : g(x_i) = 0\}$. Since $g$ satisfies the formula, we can color at least one node $c_j^l$ with color 2 and then easily extend $(S'_1, S'_2)$ to a coloring $S = (S_1, S_2, S_3)$ of $G$ with $S'_i \subseteq S_i$ for $i = 1, 2$. We have $w(S_1) = 3, w(S_2) = 2, w(S_3) = 1$ and then $val(S) \leq 6$.

Conversely, let $S = (S_1, \ldots, S_\ell)$ be a coloring of $G$ with $val(S) \leq 6$. Assume $w(S_1) \geq \ldots \geq w(S_\ell)$. We have $\ell \geq 3$ since $G$ is not bipartite, and $w(S_1) = 3$. We deduce $w(S_2) < 3$ (otherwise $val(S) \geq 3 + 3 + 1$). Moreover, since each node of weight 2 is adjacent to a node of weight 3, we have $w(S_2) = 2$. For the same reasons, we deduce $\ell = 3$ and $w(S_3) = 1$. We claim that for any
j = 1, \ldots, m$, $S_2 \cap \{c_j^1, c_j^2, c_j^3\} \neq \emptyset$ where $c_j^1, c_j^2, c_j^3$ are the nodes of $F(C_j)$ (without $c_j^3$ for a gadget of a clause of size 2). Otherwise, we must have $\{c_j^1, c_j^2, c_j^3\} \subseteq S_3$ but in this case, we cannot color $F(C_j)$ with 3 colors. Thus, setting $g(x_i) = 1$ iff $x_i^k \in S_2, \forall k$, we deduce that $g$ is a truth assignment satisfying $(X, C)$. \hfill \Box

3 Weighted node coloring in bipartite graphs

3.1 Complexity results

NP-hardness of \textsc{min weighted node coloring} in bipartite graphs has been proved in Demange et al. [6, 7]. Here, we show that some more restrictive versions are also \textsc{NP}-hard, namely bipartite planar graphs and $P_{21}$-free bipartite graphs, i.e., bipartite graphs that do not contain induced chains of length 20 or more. We use a generic reduction from the precoloring extension node coloring problem (in short \textsc{PreExt node coloring}). Then, using another reduction we improve this result to $P_8$-free bipartite graphs. This latter problem can be described as follows. Given a positive integer $k$, a graph $G = (V, E)$ and $k$ pairwise disjoint subsets $V_1, \ldots, V_k$ of $V$, we want to decide if there exists a node coloring $S = (S_1, \ldots, S_k)$ of $G$ such that $V_i \subseteq S_i$ for all $i = 1, \ldots, k$. Moreover, we restrict our attention to some specific class of graphs $\mathcal{G}$: we assume that $\mathcal{G}$ is closed under addition of a pendant edge with a new node (i.e., if $G = (V, E) \in \mathcal{G}$ and $x \in V$, $y \notin V$, then $G + [x, y] \in \mathcal{G}$).

**Theorem 3.1** Let $\mathcal{G}$ be a class of graphs which is closed under addition of a pendant edge with a new node. If \textsc{PreExt node coloring} is \textsc{NP}-complete for graphs in $\mathcal{G}$, then \textsc{min weighted node coloring} is \textsc{NP}-hard for graphs in $\mathcal{G}$.

**Proof:** Let $\mathcal{G}$ be such a class of graphs. We shall reduce \textsc{PreExt node coloring} in $\mathcal{G}$ graphs to weighted node coloring in $\mathcal{G}$ graphs. Let $G = (V, E) \in \mathcal{G}$ and consider $k$ pairwise disjoint subsets $V_1, \ldots, V_k$ of $V$. We build an instance $I = (G', w)$ of weighted node coloring using several gadgets $T_i$, for $i = 1, \ldots, k$. The construction of $T_i$ is given by induction as follows:

- $T_1$ is simply a root $v_1$ with weight $w(v_1) = 2^{k-1}$.

- Given $T_1, \ldots, T_{i-1}$, $T_i$ is a tree with a root $v_i$ of weight $w(v_i) = 2^{k-i}$ that we link to tree $T_p$ via edge $[v_i, v_p]$ for each $p = 1, \ldots, i-1$.

Figure 4 illustrates the gadgets $T_1, T_2, T_3$. Now, $I = (G', w)$ where $G' = (V', E')$ is constructed in the following way:

- $G'$ contains $G$.

- For all $i = 1, \ldots, k$, we replace each node $v \in V_i$ by a copy of the gadget $T_i$ where we identify $v$ with root $v_i$.

- For all $v \in V \setminus (\cup_{i=1}^k V_i)$ we set $w(v) = 1$. 

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Note that, by hypothesis, \( G' \in \mathcal{G} \). We prove that the precoloring of \( G' \) (given by \( V_1, \ldots, V_k \)) can be extended to a proper node coloring of \( G \) using at most \( k \) colors if and only if \( \text{opt}(I) \leq 2^k - 1 \).

Let \( S = (S_1, \ldots, S_k) \) with \( V_i \subseteq S_i \) be a node coloring of \( G \). We get \( S' = (S'_1, \ldots, S'_k) \) where each stable \( S'_i \) is given by \( S'_i = (S_i \setminus V_i) \cup \{ v : \exists j \leq k, v \in T_j \text{ and } w(v) = 2^{k-i} \} \). It is easy to check that \( S' \) is a coloring of \( G' \) and \( \text{opt}(I) \leq \text{val}(S') = \sum_{i=1}^{k} 2^{k-i} = 2^k - 1 \).

Conversely, let \( S' = (S'_1, \ldots, S'_k) \) with \( w(S'_1) \geq \cdots \geq w(S'_k) \) be a weighted node coloring of \( G' \) with weight \( \text{val}(S') \leq 2^k - 1 \). First, we prove by induction that \( V'_i = \{ v : \exists j \leq k, v \in T_j, w(v) = 2^{k-i} \} \) is a subset of \( S'_i \), for all \( i \leq k \). For \( i = 1 \), the result is true since otherwise we have \( w(S'_1) = w(S'_2) = 2^k - 1 \) and then \( \text{val}(S') \geq w(S'_1) + w(S'_j) = 2^k \). Now, assume that \( V'_j \subseteq S'_j \) for \( j < i \) and let us prove that \( V'_i = \{ v : \exists p \leq k, v \in T_p, w(v) = 2^{k-i} \} \subseteq S'_i \). By construction of gadget \( T_j \), \( j \geq i \), each node \( v \) of weight \( 2^{k-i} \) is adjacent to a node of weight \( 2^{k-p} \) for all \( p < i \). Thus, \( v \notin S'_p \). Now, if \( V'_i \not\subseteq S'_i \), then \( w(S'_i) = w(S'_{i+1}) = 2^{k-i} \) and we deduce \( \text{val}(S') \geq w(S'_i) + \ldots + w(S'_{i+1}) = \sum_{j=i}^{k} 2^{k-j} + 2^{k-i} = 2^k \), which is a contradiction. Since \( V'_i \neq \emptyset \) for \( i \leq k \), we deduce \( \ell \geq k \). Consequently, \( \ell = k \), since \( \forall v \in V', \ w(v) \geq 1 \). Now, letting \( S = (S_1, \ldots, S_k) \) where \( S_i = (S'_i \setminus V'_i) \cup V_i \) for each \( i = 1, \ldots, k \), we obtain a node coloring of \( G \).

Using the results of Kratochvíl [21] on the \( \text{NP} \)-completeness of \( \text{PreExt \ node \ coloring} \) in bipartite planar graphs and \( \text{P}_{13} \)-free bipartite graphs, we deduce:

**Corollary 3.2** In bipartite planar graphs, \( \text{MIN \ weighted \ node \ coloring} \) is strongly \( \text{NP} \)-hard and it is not \( \frac{2}{3} - \varepsilon \)-approximable for all \( \varepsilon > 0 \) unless \( \text{P} = \text{NP} \).

**Proof** : \( \text{PreExt \ node \ coloring} \) with \( k = 3 \) has been proved to be \( \text{NP} \)-complete in [21] for bipartite planar graphs. Since these graphs are closed under addition of a pendent edge with a new node, the result follows. Moreover, from the proof of Theorem 3.1 with \( k = 3 \), we deduce that it is \( \text{NP} \)-complete to distinguish between \( \text{opt}(I) \leq 7 \) and \( \text{opt}(I) \geq 8 \).

**Corollary 3.3** In \( \text{P}_{21} \)-free bipartite graphs, \( \text{MIN \ weighted \ node \ coloring} \) is strongly \( \text{NP} \)-hard and it is not \( \frac{32}{31} - \varepsilon \)-approximable for all \( \varepsilon > 0 \) unless \( \text{P} = \text{NP} \).
Proof: \textsc{PreExt Node Coloring} with \( k = 5 \) has been proved \textsc{NP}-complete in [21] for \( P_{13} \)-free bipartite graphs. When we add gadgets \( T_i \) with \( i \leq 5 \) to a \( P_{13} \)-free bipartite graph, we obtain a \( P_{21} \)-free bipartite graph. Moreover, from the proof of Theorem 3.1 with \( k = 5 \), we deduce that it is \textsc{NP}-complete to distinguish between \( \text{opt}(I) \leq 31 \) and \( \text{opt}(I) \geq 32 \). \hfill \( \Box \)

In Hujter and Tuza [20], it is shown that \textsc{PreExt Node Coloring} is \textsc{NP}-complete in \( P_{2k} \)-free bipartite chordal graphs for unbounded \( k \) (a bipartite graph is chordal if the induced cycles of length at least 5 have a chord). Unfortunately, we cannot use this result in Theorem 3.1 since the resulting graph has an induced chain with arbitrarily large length. However, we can adapt their reduction to our problem.

\textbf{Theorem 3.4} \textsc{Min Weighted Node Coloring} is \textsc{NP-hard} in \( P_{2k} \)-free bipartite graphs.

Proof: We shall reduce 3-sat-3, proved to be \textsc{NP}-complete in Papadimitriou [27], to our problem. Given a collection \( C = (C_1, \ldots, C_m) \) of clauses over the set \( X = \{x_1, \ldots, x_n\} \) of boolean variables such that each clause \( C_j \) has at most three literals and each variable appears twice positively and once negatively, we construct an instance \( I = (BP, w) \) in the following way:

- We start from \( BP_1 = (L_1, R_1; E_1) \), a complete bipartite graph \( K_{n,m} \) where \( L_1 = \{x_1, \ldots, x_n\} \) and \( R_1 = \{c_1, \ldots, c_m\} \). Moreover, each node of \( BP_1 \) has weight 1.
- We also introduce another bipartite graph \( BP_2 \) isomorphic to \( K_{2n,2n} \) where a perfect matching has been deleted. More formally, \( BP_2 = (L_2, R_2; E_2) \) where \( L_2 = \{l_1, \ldots, l_{2n}\} \), \( R_2 = \{r_1, \ldots, r_{2n}\} \), and \( [l_i, r_j] \in E_2 \) iff \( i \neq j \). Finally, \( w(l_i) = w(r_i) = 2^{2n-i} \) for \( i = 1, \ldots, 2n \). Indeed, sets \( \{l_{2i-1}, r_{2i-1}\} \) and \( \{l_{2i}, r_{2i}\} \) will correspond to the variable \( x_i \) and \( \overline{x_i} \) respectively.
- Between \( BP_1 \) and \( BP_2 \), there is a set \( E_3 \) of edges defined as follows: \([x_i,r_j] \notin E_3 \) iff \( j = 2i-1 \) or \( j = 2i+1 \) and \([l_i,c_j] \notin E_3 \) iff \( i = 2k-1 \) and \( x_k \) is in \( C_j \) or \( i = 2k \) and \( \overline{x_k} \) is in \( C_j \).

Figure 5 illustrates the construction of the bipartite complement of \( BP \) with the clause \( c_m = \overline{x_1} \lor x_2 \lor \overline{x_n} \).

![Figure 5: Bipartite complement of graph BP with the clause \( c_m = \overline{x_1} \lor x_2 \lor \overline{x_n} \).](image-url)
Let us show that $BP$ is $P_5$-free. We represent in Figure 6 the possible subgraphs in $BP_1$ (configurations $A_1$, $A_2$ and $A_3$) and in $BP_2$ (configurations $B_1$ to $B_9$) induced by a chain in $BP$. In configurations $A_3$ and $B_9$, the number of nodes is arbitrary. Note that the upper line may correspond either to $L_1$ or $R_1$ for the left part (and $L_2$ or $R_2$ for the right part). Now we look at the possible ways to link a configuration $A_i$ to a configuration $B_j$ to obtain a chain on (at least) 8 nodes.

- If we choose $A_1$, we easily see that it is impossible.
- If we choose $A_2$, the only way to have a chain on at least 8 nodes is to choose $B_8$ and link a node of $A_2$ to a node of $B_8$. In this case, if the upper line corresponds to $L_1$ (left part), any node in $R_2$ is adjacent to at least one of these two nodes and we get at least one node of degree 3. If the upper line corresponds to $R_1$ (left part) and $L_2$ (right part), then there is a clause which contains a variable and its negation. Indeed, the two bottom nodes in $B_8$ correspond to a variable and its negation since they are not adjacent to the bottom node of $A_2$.
- If we choose $A_3$, the only possibility to have a chain on at least 8 nodes is to choose $B_9$. But in this case, the chain simply alternates nodes of $R_1$ and nodes of $L_2$. Then, at least one node of $L_2$ is not linked to at least 3 nodes of $R_1$, i.e., a literal appears in at least 3 clauses, which is not possible.

We claim that $(X, C)$ is satisfiable if and only if opt($I$) $\leq 2^{2n} - 1$.

Let $g$ be a truth assignment satisfying $(X, C)$. We build the colors inductively. $S_0 = \emptyset$ and for $i = 1, \ldots, n$, $S_{2i-1} = \{l_{2i-1}, r_{2i-1}\} \cup \{c_j : c_j \notin S_p, p < 2i - 1, g(x_i) = 1 \text{ and } x_i \text{ is in } C_j\}$, $S_{2i} = \{l_{2i}, r_{2i}\} \cup \{c_j : c_j \notin S_p, p < 2i, g(x_i) = 0 \text{ and } \overline{c_j} \text{ is in } C_j\}$. Finally, if $g(x_i) = 1$ then we add $x_i$ to $S_{2i}$; otherwise, we add $x_i$ to $S_{2i-1}$. We can easily see that $S = (S_1, \ldots, S_{2n})$ is a node coloring of $BP$ with $val(S) = 2^{2n} - 1$.

Conversely, let $S = (S_1, \ldots, S_{\ell})$ be a node coloring of $BP$ with $val(S) = 2^{2n} - 1$. An inductive proof on $i$ shows that $\{l_i, r_i\} \subseteq S_i$ (otherwise, we have $val(S) > 2^{2n}$); consequently, $\ell = 2n$. Thus, setting $g(x_i) = 1$ if $x_i \in S_{2i}$ and $g(x_i) = 0$ if $x_i \in S_{2i-1}$, we obtain a truth assignment satisfying $(X, C)$. □

3.2 A polynomial result

We now prove that MIN WEIGHTED NODE COLORING is polynomial for $P_5$-free bipartite graphs, i.e., bipartite graphs without induced chains on 5 nodes. Notice that in general $P_5$-free graphs, the weighted node coloring problem is NP-hard since on the one hand, the split graphs are $P_5$-free and
on the other hand, we have proved in Demange et al. [6] that the weighted node coloring problem is NP-hard for split graphs. There are several characterizations of $P_5$-free bipartite graphs, see for example, Hammer et al. [18], Chung et al. [4] and Hujter and Tuza [19]. In particular, $BP$ is a $P_5$-free bipartite graph if and only if $BP$ is bipartite and each connected component of $BP$ is $2K_2$-free, i.e., its complement is $C_4$-free.

**Lemma 3.5** In a $P_5$-free bipartite graph, any optimal weighted node coloring uses at most 3 colors.

**Proof:** Let $BP = (L, R; E)$ be a $P_5$-free bipartite graph with connected components $BP_1, \ldots, BP_p$. Assume the converse and let us consider an optimal solution $S^* = (S^*_1, \ldots, S^*_p)$ with $\ell \geq 4$ and $w(S^*_i) \geq \ldots \geq w(S^*_\ell)$. Observe that, without loss of generality, we can assume that there exists a connected component $BP_{k_0}$ colored with $\ell$ colors and any connected component $BP_{k}$ using $j$ colors is colored with colors $1, \ldots, j$. Moreover, we also suppose that in any connected component $BP_{j_0}$, each node colored with color $i \geq 2$ is adjacent to nodes with colors $1, \ldots, i-1$ (if $v \in S^*_j$ is not adjacent to any node of $S^*_j$ with $j < i$, we recolor $v$ with color $j$. This node coloring remains optimal since $w(S^*_j) \geq w(S^*_i) \geq w(v)$).

We claim that there exist $1 \leq i < j \leq \ell$ such that $S^*_k \cap L \neq \emptyset$ and $S^*_k \cap R \neq \emptyset$ for $k = i, j$. Otherwise, since $\ell \geq 4$, we must have $S^*_i \subseteq L$ (resp., $S^*_j \subseteq R$) and $S^*_j \subseteq L$ (resp., $S^*_i \subseteq R$) for some $i_0 < j_0$. In this case, by merging $S^*_i$ with $S^*_j$, we obtain a better node coloring than $S^*$, which is a contradiction.

So, consider a connected component $BP_{k_0}$ and let $l_j \in S^*_j \cap L$ and $r_j \in S^*_j \cap R$ be two nodes of $BP_{k_0}$. From the previous claim, we deduce that there exist other nodes $l_i, r_i$ of $BP_{k_0}$ such that $l_i \in S^*_i \cap L$, $r_i \in S^*_i \cap R$ and $[l_i, r_j] \in E$, $[l_j, r_i] \in E$. Since $BP$ is bipartite, these 2 edges form an induced $2K_2$ which is a contradiction with the characterization of $P_5$-free bipartite graphs. □

Let $BP_1, \ldots, BP_p$ be the connected components of $BP$ where $BP_i = (L_i, R_i; E_i)$. Let $S^* = (S^*_1, S^*_2, S^*_3)$ (with possibly some $S^*_i = \emptyset$) be an optimal solution with $w(S^*_1) \geq w(S^*_2) \geq w(S^*_3)$ and denote by $S^*_i = (S^*_1, S^*_2, S^*_3)$ the restriction of $S^*$ to the subgraph $BP_i$. Remark that we may assume $w(S^*_i) \geq w(S^*_2) \geq w(S^*_3)$ (otherwise, we can flip the color without increasing the weight).

Moreover, we have:

**Lemma 3.6** One of the following cases occurs necessarily, for any $i = 1, \ldots, p$:

(i) $S^*_1 = L_i$ (resp., $S^*_1 = R_i$), $S^*_2 = R_i$ (resp., $S^*_2 = L_i$) and $S^*_3 = \emptyset$.

(ii) $S^*_1 \cap L_i \neq \emptyset$ and $S^*_1 \cap R_i \neq \emptyset$, $S^*_2 \subseteq R_i$ (resp., $S^*_2 \subseteq L_i$) and $S^*_3 \subseteq L_i$ (resp., $S^*_3 \subseteq R_i$).

**Proof:** Let $BP = (L, R; E)$ be a $P_5$-free bipartite graph with connected components $BP_1, \ldots, BP_p$. Assume that $S^*_1 \cap L_i = \emptyset$ or $S^*_1 \cap R_i = \emptyset$. In this case, it is clear that we are in the first case (i) (since we have assumed $w(S^*_1) \geq w(S^*_2) \geq w(S^*_3)$). Now, suppose $S^*_1 \cap L_i \neq \emptyset$ and $S^*_1 \cap R_i \neq \emptyset$; from the proof of Lemma 3.5, the result follows. □

The algorithm computing an optimal solution is described as follows:
\(P_3\)-FREEBIPARTITECOLOR

1 For all \(k_1, k_2 \in \{w(v) : v \in V\}, k_1 \geq k_2\), do

1.1 For all connected components \(BP_i = (L_i, R_i; E_i), i = 1, \ldots, p\), do

1.1.1 If \(L_i \cup R_i \setminus (L'_i \cup R'_i)\) is an independent set where \(L'_i = \{v \in L_i : w(v) \leq k_1\}\) and \(R'_i = \{v \in R_i : w(v) \leq k_2\}\), then set \(S_{2,i}^{k_1,k_2} = L'_i, S_{3,i}^{k_1,k_2} = R'_i\) and \(S_{1,i}^{k_1,k_2} = L_i \cup R_i \setminus (L'_i \cup R'_i)\);

1.1.2 Otherwise, if \(L_i \cup R_i \setminus (L'_i \cup R'_i)\) is an independent set where \(L'_i = \{v \in L_i : w(v) \leq k_2\}\) and \(R'_i = \{v \in R_i : w(v) \leq k_1\}\), then set \(S_{2,i}^{k_1,k_2} = R'_i, S_{3,i}^{k_1,k_2} = L'_i\) and \(S_{1,i}^{k_1,k_2} = L_i \cup R_i \setminus (L'_i \cup R'_i)\);

1.1.3 Otherwise go to step 1;

1.1.4 Set \(S_{j,i}^{k_1,k_2} = \bigcup_{i=1}^p S_{j,i}^{k_1,k_2}\) for \(j = 1, 2, 3\) and \(S_i^{k_1,k_2} = (S_i^{k_1,k_2,1}, S_i^{k_1,k_2,2}, S_i^{k_1,k_2,3})\) (with maybe \(S_i^{k_1,k_2} = \emptyset\));

2 Output \(S = \arg\min\{val(S_i^{k_1,k_2}) : k_2 \leq k_1\}\);

This algorithm has a time complexity of \(O(n|w|^2)\) where \(|w| = |\{w(v) : v \in V\}|\). By applying a sorting procedure by dichotomy on \(k_2\), we can improve it to \(O(n|w|\log|w|)\). Note that this algorithm also computes the best node 2-coloring among the colorings using at most 2 colors (when \(k_1 = w_{\max}\)).

Theorem 3.7 MIN WEIGHTED NODE COLORING is polynomial in \(P_3\)-free bipartite graphs.

Proof: Let \(S^* = (S_1^*, S_2^*, S_3^*)\) (with maybe \(S_1^* = \emptyset\)) be an optimal solution satisfying Lemmas 3.5 and 3.6. We assume \(w(S_2^*) \geq w(S_3^*)\) and if \(S^*\) is a node 3-coloring, then we have \(w(S_1^*) = w_{\max}\); otherwise \(w(S_i^*) = 0\). Let \(k_1 = w(S_2^*)\) and \(k_2 = w(S_3^*)\); consider the step of algorithm corresponding to \(k_1, k_2\). If \(S^*\) is a node 2-coloring, then the result is true. So, assume \(S_1^* \neq \emptyset\); by construction, \(P_3\)-FREEBIPARTITECOLOR finds a feasible solution \(S_i^{k_1,k_2}\) with \(w(S_i^{k_1,k_2}) \leq w_{\max}, w(S_2^{k_1,k_2}) \leq k_1\) and \(w(S_3^{k_1,k_2}) \leq k_2\). Thus, we deduce the expected result. \(\square\)

Let us conclude this subsection by the following observation: the clique-width of a \(P_3\)-free bipartite graphs is at most 5, but to our knowledge the complexity of MIN WEIGHTED NODE COLORING in graphs of bounded clique-width is not known.

3.3 Approximation

In Demange et al. [6], a \(\frac{4}{3}\)-approximation is given for MIN WEIGHTED NODE COLORING and it is proved that a \((\frac{5}{4} - \varepsilon)\)-approximation is not possible, for any \(\varepsilon > 0\), unless \(P = NP\), even if we consider arbitrarily large values of \(\text{opt}(I)\). Using Corollary 3.2, we deduce that this lower bound also holds if we consider bipartite planar graphs. Here, we shall give a \(\frac{8}{7}\)-approximation in bipartite graphs.
**BIPARTITECOLOR**

1. Sort the nodes of $BP$ in non-increasing weight order (i.e., $w(v_1) \geq \ldots \geq w(v_n)$);

2. For $i = 1$ to $n$ do
   
   2.1 Set $V_i = \{v_1, \ldots, v_i\}$;
   
   2.2 Compute an optimal weighted node coloring $S^*_i = (S^*_1, S^*_2)$ ($S^*_2$ may be empty) in $BP[V_i]$ among the colorings using at most two colors;

   2.3 Define node coloring $S^{i} = (S^*_1, S^*_2, L \setminus V_i, R \setminus V_i)$ ($L \setminus V_i$ and $R \setminus V_i$ may be empty);

3. Output $S = \text{argmin}\{\text{val}(S^i) : i = 1, \ldots, n\}$;

Step 2.2 consists of computing the (unique) 2-coloring $(S^*_1, S^*_2)$ (with $w(S^*_1) \geq w(S^*_2)$) of each connected component $BP_j$, $j = 1 \ldots p$ of $BP[V_i]$ (with $S^*_2 = \emptyset$ if $BP_j$ is an isolated node).

Then it merges the most expensive sets, i.e., it computes $S^*_i = \bigcup_{j=1}^{p} S^*_i[j]$ for $i = 1, 2$. It is easy to observe that $S^*_i = (S^*_1, S^*_2)$ is the best weighted node coloring of $BP[V_i]$ among the colorings using at most 2 colors; such a coloring can be found in $O(m)$ time where $m = |E|$.

**Theorem 3.8** Algorithm BIPARTITECOLOR (polynomially) solves in $O(nm)$ time weighted node coloring in bipartite graphs within approximation ratio bounded above by $\frac{8}{7}$.

**Proof:** Let $I = (BP, w)$ be a weighted bipartite-graph where $BP = (L, R; E)$ and $S^* = (S^*_1, \ldots, S^*_i)$ be an optimal node coloring of $I$ with $w(S^*_1) \geq \ldots \geq w(S^*_i)$. If $l < 3$, then BIPARTITECOLOR finds an optimal weighted node coloring which is $S^n$ (corresponding to the step $i = n$). Now, assume $l \geq 3$ and let $i_j = \min\{k : v_k \in S^*_j\}$. Then $w(v_{i_j}) = w(S^*_j)$. We have $i_1 = 1$ and

$$\text{opt}(I) \geq w(v_{i_1}) + w(v_{i_2}) + w(v_{i_3})$$  \hspace{1cm} (3.1)

Let us examine the various steps of this algorithm:

- when $i = i_2 - 1$, the algorithm produces a node 3-coloring $S_2^{i_2 - 1} = (S_1^{i_2 - 1}, L \setminus S_1^{i_2 - 1}, R \setminus S_1^{i_2 - 1})$. Indeed, by construction $V_{i_2 - 1} \subseteq S_1^{i_2 - 1}$ is an independent set, and then, $S_1^{i_2 - 1}$ is defined by $S_1^{i_2 - 1} = V_{i_2 - 1}$ and $S_2^{i_2 - 1} = \emptyset$. Moreover, $\forall v \notin V_{i_2 - 1}$, $w(v) \leq w(v_{i_2})$ and then

$$\text{val}(S_2^{i_2 - 1}) \leq w(v_{i_1}) + 2w(v_{i_2})$$  \hspace{1cm} (3.2)

- when $i = i_3 - 1$, the algorithm produces in $BP[V_{i_3 - 1}]$ a node 2-coloring $S_3^{i_3 - 1}$ with a weight $\text{val}(S_3^{i_3 - 1}) \leq w(v_{i_1}) + w(v_{i_2})$ since the coloring $(S_1^* \cap V_{i_3 - 1}, S_2^* \cap V_{i_3 - 1})$ is a feasible node 2-coloring of $BP[V_{i_3 - 1}]$ with weight $w(v_{i_1}) + w(v_{i_2})$. Finally, since the weights are sorted in non-increasing order, we obtain:

$$\text{val}(S_3^{i_3 - 1}) \leq w(v_{i_1}) + w(v_{i_2}) + 2w(v_{i_3})$$  \hspace{1cm} (3.3)
• when \( i = n \) (the last step), the algorithm just produces a node 2-coloring satisfying:

\[
\text{val}(S^n) \leq 2w(v_i)
\]  

(3.4)

Using (3.2), (3.3) and (3.4), we deduce:

\[
\text{val}(S) \leq \min\{2w(v_1); w(v_1) + w(v_2) + 2w(v_3); w(v_1) + 2w(v_2)\}
\]  

(3.5)

The convex combination of these 3 values with coefficients \( \frac{1}{7}, \frac{4}{7} \) and \( \frac{2}{7} \) respectively and the inequality (3.1) give the expected result, i.e.:

\[
\text{val}(S) \leq \frac{1}{7} \times 2w(v_1) + \frac{4}{7} \times (w(v_1) + w(v_2) + 2w(v_3)) + \frac{2}{7} \times (w(v_1) + 2w(v_2)) \leq \frac{8}{7} \text{opt}(I)
\]

\[\square\]

In [8], it is more generally proved that there is a polynomial algorithm within approximation ratio bounded above by \( \frac{\ell^3}{3\ell^2 - 3\ell + 1} \) for weighted node coloring in \( k \)-partite graphs.

### 4 Weighted edge coloring in bipartite graphs

The weighted edge coloring problem on a graph \( G \) can be viewed as the weighted node coloring problem on \( L(G) \) where \( L(G) \) is the line graph of \( G \). Here, for simplicity, we refer to the edge model.

### 4.1 Complexity results

Demange et al. [6] have proved that \textsc{min weighted edge coloring} in bipartite cubic graphs is strongly \textsc{NP}-hard and a lower bound of \( \frac{8}{7} \) is given for the approximation. Here, we slightly improve these complexity results. Indeed, we show that weighted edge coloring in bipartite cubic planar graphs is strongly \textsc{NP}-hard and we deduce that it is \textsc{NP}-complete to obtain an approximation within a ratio \( \frac{7}{6} - \varepsilon \), for any \( \varepsilon > 0 \).

**Theorem 4.1** \textsc{min weighted edge coloring} is strongly \textsc{NP}-hard in bipartite cubic planar graphs.

**Proof:** We shall reduce \textsc{prext edge coloring} in bipartite cubic planar graphs to our problem. Given a bipartite cubic planar graph \( BP = (V, E) \) and 3 pairwise disjoint matchings \( E_1, \ldots, E_3 \) of \( E \), the question of \textsc{prext edge coloring} is to determine if it is possible to extend the edge precoloring \( E_1, \ldots, E_3 \) to a proper edge 3-coloring of \( G \). Recently, this problem has been shown to be \textsc{NP}-complete in Marx [24].

Let \( BP = (V, E) \) and \( E_1, \ldots, E_3 \) be an instance of \textsc{prext edge coloring}; we construct an instance \( I = (BP', w) \) of weighted edge coloring such that the answer of \textsc{prext edge coloring} instance is yes if and only if there exists an edge coloring \( S \) of \( I \) with weight \( \text{val}(S) \leq 6 \).

The construction of instance \( I \) is the following:
• Each edge in $E_1$ receives weight 3.

• Each edge $[x, y] \in E_2$ is replaced by a gadget $F_2$ described in Figure 7, where we identify $x$ and $y$ with $v_0$ and $v_9$ respectively.

• Each edge in $E_3$ is replaced by a gadget $F_3$ which is the same as gadget $F_2$ except that we have exchanged weights 1 and 2.

• The other edges of $G$ receive weight 1.

![Figure 7: Gadget $F_2$ for $e \in E_2$.](image)

Remark that $BP'$ is still a bipartite cubic planar graph.

First of all, assume that $BP$ admits an edge 3-coloring $S = (M_1, M_2, M_3)$ where $E_i \subseteq M_i$ for any $i = 1, 2, 3$. We get a coloring $S' = (M'_1, M'_2, M'_3)$ of $BP'$ where $M'_1 = M_1 \cup \{e \in F_2 \cup F_3 : w(e) = 3\}$ and, for $i = 2, 3$, $M'_i = (M_i \setminus E_i) \cup \{e \in F_2 \cup F_3 : w(e) = 4 - i\}$. We can easily check that $\text{opt}(I) \leq \text{val}(S') = 3 + 2 + 1 = 6$.

Conversely, consider an edge coloring $S' = (M'_1, \ldots, M'_p)$ of $G'$ with $\text{val}(S') \leq 6$ and assume $w(M'_1) \geq \ldots \geq w(M'_p)$. We have $\ell \geq 3$ since $\Delta(BP') = 3$. Then, all the edges of weight 3 must be in the matching $M'_1$, and no edge of weight 2 is in $M'_p$ with $p \geq 3$, since otherwise we have $\text{val}(S') \geq 7$ ($3 + 3 + 1$ in the first case and $3 + 2 + 2$ in the second case). Moreover, each edge of weight 2 is adjacent to an edge of weight 3, and then, the edges of weight 2 are necessarily in $M'_2$. Finally, remark that the edges of weight 1 of the gadgets are adjacent to an edge of weight 2 and an edge of weight 3 and must be in $M'_p$ with $p \geq 3$. Moreover, $p = 3$ and more generally $\ell = 3$ since $\text{val}(S') \leq 6$. Now, consider the edge coloring $(M_1, M_2, M_3)$ of $BP$ where for any $i = 1, 2, 3$ we have $M_i = (M'_1 \setminus \{e \in F_2 \cup F_3 : w(e) = 4 - i\}) \cup E_i$. We can easily see that $(M_1, M_2, M_3)$ is a solution for the edge precoloring extension problem. \hfill {}\square

From the proof of Theorem 4.1, we deduce that computing an optimal weighted edge 3-coloring of a cubic bipartite graphs among all edge 3-colorings is $\text{NP}$-hard. Using the same technique, we can prove that more generally, finding an optimal weighted edge $k$-coloring of a cubic bipartite graph among the edge colorings using at most $k$ colors is $\text{NP}$-hard for any $k = 3, 4, 5$.

**Corollary 4.2** For all $\varepsilon > 0$, \textsc{min weighted edge coloring} is not $7/6 - \varepsilon$ approximable in bipartite cubic planar graphs unless $P = \text{NP}$. 

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4.2 Approximation

In Demange et al. [6], a $\frac{2}{3}$-approximation is given for \textsc{min weighted edge coloring} in bipartite graphs with maximum degree 3. Here, we give a $\frac{7}{9}$-approximation.

We need some notations: If $BP = (V, E)$ is a bipartite graph with node set $V = \{v_1, \ldots, v_n\}$, we always assume that its edges $E = \{e_1, \ldots, e_m\}$ are sorted in non-increasing weight order (i.e., $w(e_1) \geq \ldots \geq w(e_m)$). If $V'$ is a subset of nodes and $E'$ a subset of edges, $BP[V']$ and $BP[E']$ denote the subgraph of $BP$ induced by $V'$ and the partial graph of $BP$ induced by $E'$ respectively.

For any $i \leq m$, we set $E_i = \{e_1, \ldots, e_i\}$ and $E_i = E \setminus E_i$. Finally, $V_i$ denotes the set of nodes of $BP$ incident to an edge in $E_i$ (so it is the subset of non-isolated nodes of $BP[E_i]$).

Consider the following algorithm \textsc{bipartiteedgecolor}, which uses as subroutines algorithms \textsc{sol1}, \textsc{sol2} and \textsc{sol3} presented later.

\begin{algorithm}
\caption{\textsc{bipartiteedgecolor}}
1 For $i = m$ downto 1 do
   \begin{enumerate}
   \item Apply algorithm \textsc{sol1} on $BP[E_i]$;
   \item If $\textsc{sol1}(BP[E_i]) \neq \emptyset$, complete in a greedy way all the colorings produced by \textsc{sol1} on the edges of $E_i$. Let $S_{1,i}$ be a best coloring among these edge colorings of $BP$;
   \end{enumerate}
2 For $j = i$ downto 1 do
   \begin{enumerate}
   \item Apply algorithm \textsc{sol2} on $BP[E_j]$;
   \item If $\textsc{sol2}(BP[E_j]) \neq \emptyset$, complete in a greedy way all the colorings produced by \textsc{sol2} on the edges of $E_j$. Let $S_{2,j,i}$ be a best coloring among these edge colorings of $BP$;
   \end{enumerate}
3 Output $S = \arg\min\{\text{val}(S_{1,i}), \text{val}(S_{k,j,i}) : k = 2, 3, \ j = 1, \ldots, i, \ i = 1, \ldots, m\}$.
\end{algorithm}

The greedy steps 1.2, 1.3.2 and 1.3.4 can be described as follows: for each edge not yet colored, try to color it with an existing color, and otherwise take a new color. A simple argument shows that these edge colorings do not use more than 5 colors. Indeed, assume the converse and let us consider an edge with color 6. Since the maximum degree of $BP$ is 3, this edge is adjacent to at most 4 edges and then to at most 4 colors. Thus, we can recolor this edge with a missing color in $1, \ldots, 5$. Obviously, this result also holds for an optimal solution. More generally, in [6], we have proved that, in any graph $G$, there is an optimal weighted node coloring using at most $\Delta(G) + 1$ colors, where $\Delta(G)$ denotes the maximum degree of $G$. In our case, we have $G = L(H)$, the line graph of $H$, and we deduce
that an optimal weighted node coloring uses at most \( \Delta(L(H)) + 1 \leq 2(\Delta(H) - 1) + 1 = 2\Delta(H) - 1 \) colors.

The three algorithms SOL1, SOL2 and SOL3 are used on several partial graphs \( BP' \) of \( BP \). In the following, \( V', E' \) and \( m' \) denote respectively the node set, the edge set and the number of edges of the current graph \( BP' \). Moreover, we set \( \overline{V}_i = V' \setminus V_i \) and \( \overline{E}_i = E' \setminus E_i \). If \( M = (M_1, \ldots, M_l) \) is an edge coloring of \( BP' \), we note \( i_j = \min\{k : e_k \in M_j\} \) for \( j = 1, \ldots, l \). We assume, for readability, that some color classes \( M_j \) may be empty (in this case \( i_j = m' + 1 \)). The principle of these algorithms is to find a decomposition of \( BP' \) (a subgraph of \( BP \)) into two subgraphs \( BP'_1 \) and \( BP'_2 \) such that each one is of maximum degree 2. When there exists such a decomposition, we can color \( BP'_1 \) and \( BP'_2 \) with at most 2 colors respectively since \( BP \) is bipartite.

For Algorithm SOL1, in order to find such a decomposition, we will solve an instance of the following problem \( P1 \) known to be polynomial (see for instance exercise 5.57 page 191 of the book [5]): given a graph \( G = (V, E) \) and \( 2|V| \) integers \( a_v, b_v \) for \( v \in V \), the problem consists of computing (if any) an edge subset \( \tilde{E} \subseteq E \) of maximum size such that in \( \tilde{G} = (V, \tilde{E}) \) we have \( \forall v \in V \ a_v \leq d_{\tilde{G}}(v) \leq b_v \).

Now, we are ready to describe Algorithm SOL1.

---

**SOL1**

1. For \( j = m' \) downto 1 do
   
   1.1 If the maximum degree of \( BP'|E'_j| \) is at most 2 then
      
      1.1.1 Find an optimal solution (if any) \( \tilde{E}_j \subseteq E'_j \) of problem \( P1 \) with inputs \( BP'|E'_j| \) and \( a_v, b_v \) for \( v \in V' \) where the integers \( a_v \) and \( b_v \) are given by:
         
         - Set \( W_i = \{v \in V' : d_{BP'|E'_j|}(v) = i\} \) for \( i = 0, 1, 2 \).
         - \( a_v = 1 \) if \( v \in W_0 \) and \( d_{BP'}(v) = 3 \), \( a_v = 0 \) otherwise.
         - \( b_v = 0 \) if \( v \in W_2 \), \( b_v = 1 \) if \( v \in W_1 \) and \( b_v = 2 \) otherwise.
      
      1.1.2 If a subset \( \tilde{E}_j \) has been found, do
         
         1.1.2.1 Set \( M^j = \tilde{E}_j \) and for all paths \( \mu \subseteq M^j \) between two nodes of \( W_1 \) delete one edge \( e \in \mu \) of \( M^j \) (i.e., set \( M^j := M^j \setminus \{e\} \));
         
         1.1.2.2 Consider the decompositions \( BP'_{1,j} \) and \( BP'_{2,j} \) of \( BP' \) induced by \( E'_j \cup M^j \) and \( E'_j \setminus (E'_j \cup M^j) \) respectively;
         
         1.1.2.3 Find an optimal edge coloring \( (M^j_1, M^j_2) \) among the edge 2-colorings of \( BP'_{1,j} \);
         
         1.1.2.4 Color greedily the edges of \( BP'_{2,j} \) with two colors and let \( (M^j_3, M^j_4) \) be the 2-coloring obtained;
         
         1.1.2.5 Let \( S'_j = (M^j_1, M^j_2, M^j_3, M^j_4) \) be the resulting edge 4-coloring of \( BP' \);
      
   2 Output \( \{S'_j : j = 1, \ldots, m'\} \);
Lemma 4.3 If $S = (M_1, M_2, M_3, M_4)$ with $w(M_1) \geq \ldots \geq w(M_4)$ is an edge coloring of $BP'$, then algorithm SOL1 produces a solution $S'_1$ satisfying: $\text{val}(S'_1) \leq w(M_1) + w(M_2) + 2w(M_3)$

Proof: Let $S = (M_1, M_2, M_3, M_4)$ with $w(M_1) \geq \ldots \geq w(M_4)$ be an edge coloring of $BP'$ (where we assume that $BP'$ is of maximum degree 3). Let us examine the step of SOL1 corresponding to $j = i_3 - 1$. By construction, $BP'[E'_{i_3-1}]$ is 2 edge colorable since we have $E'_{i_3-1} \subseteq M_1 \cup M_2$. Moreover, in the subgraph induced by $E'_{i_3-1}$, each node $v$ of degree 3 is adjacent to at least one edge $e_v \in M_1 \cup M_2$. Then $\tilde{E}' = \{e_v : d_{BP'[E'_{i_3-1}]}(v) = 3\} \neq \emptyset$ is a feasible solution of problem P1 defined as in step 1.1. Hence, solution $\tilde{E}_{i_3-1}$ exists in $BP'[E'_{i_3-1}]$. Now, let us show that $M'_{i_3-1}$ remains a feasible solution of problem P1. In $(V, E_{i_3-1})$ all the paths linking two nodes of $W_1$ are node disjoint since by construction the graph $(V, E_{i_3-1})$ is of maximum degree 2. When we delete the edge $e = [x, y] \in \mu$ from $E_{i_3-1}$, the degree of $x$ and $y$ remains at least one.

The subgraph $BP'_{1,i_3-1}$ has a maximum degree 2 and contains by construction the subgraph $BP'[E'_{i_3-1}]$. Moreover, no pair of connected components of $BP'[E'_{i_3-1}]$ have been merged in $BP'_{1,i_3-1}$ (since by step 1.1.2.1 we have destroyed all paths between two nodes of $W_1$ in $(V', M'_{i_3-1})$). Thus, any edge 2-coloring of $BP'[E'_{i_3-1}]$ can be extended to an edge 2-coloring of $BP'_{1,i_3-1}$. So, since $\forall e \in M'_{i_3-1}$, $\forall e' \in E'_{i_3-1}$ $w(e) \leq w(e')$, and $(M'_{i_3-1}, M'_{i_3-1})$ is an optimal weighted 2-edge coloring of $BP'_{1,i_3-1}$, we deduce:

$$w(M'_{1,i_3-1}) + w(M'_{2,i_3-1}) \leq w(M_1) + w(M_2) \tag{4.1}$$

By construction, $BP'_{2,i_3-1}$ has no node with degree 3, and then $BP'_{2,i_3-1}$ has maximum degree 2. Moreover, $\forall e \notin (M'_{i_3-1} \cup E'_{i_3-1})$ we have $w(e) \leq w(e_{i_3}) = w(M_3)$. Thus, any edge coloring of $BP'_{2,i_3-1}$ using at most 2 colors and in particular $(M'_{i_3-1}, M'_{i_3-1})$ satisfies:

$$w(M'_{3,i_3-1}) + w(M'_{4,i_3-1}) \leq 2w(M_3) \tag{4.2}$$

Combining (4.1) and (4.2), we obtain:

$$\text{val}(S'_{i_3-1}) \leq w(M_1) + w(M_2) + 2w(M_3)$$

\[\square\]
1.1.2.2 Color $BP'_{1,k}$ with one color and let $M_k^1$ be the set of colored edges;

1.1.2.3 Color greedily $BP'_{2,k}$ with two colors to get color classes $M_k^2$ and $M_k^3$;

1.1.2.4 Let $S_k^2 = (M_1^k, M_2^k, M_3^k)$ be the edge coloring of $BP'$ obtained in this way;

2 Output $\{S_k^2 : k = 1, \ldots, m'\}$;

---

Step 1.1.1 of $SOL2$ is well known to be polynomial. It is also a special case of problem $P1$ with inputs $BP'([V]$ and $a_v, b_v$ for $v \in V$ given by: $a_v = 1$ if $d_{BP'}(v) = 3$, $a_v = 0$ otherwise and $b_v = 1$ for all $v \in V$.

**Lemma 4.4** If $S = (M_1, M_2, M_3)$ with $w(M_1) \geq w(M_2) \geq w(M_3)$ is an edge coloring of $BP'$, then algorithm $SOL2$ produces a solution $S_k^2$ satisfying: $\text{val}(S_k^2) \leq w(M_1) + 2w(M_2)$.

**Proof:** Let $S = (M_1, M_2, M_3)$ with $w(M_1) \geq w(M_2) \geq w(M_3)$ be an edge coloring of $BP'$. Let us examine the step of $SOL2$ corresponding to $k = i_2 - 1$. By construction, $E'_{i_3-1} \subseteq M_1$ and in $M_1 \setminus E'_{i_3-1}$ there is a matching of $BP'([V_{i_3-1}])$ where each node of degree 3 is saturated (otherwise, $BP'$ would not be 3-colored). Thus, $BP'_{1,i_2-1}$ can be considered and colored with one color to get $M_1^2$, and we have:

$$w(M_1^2) = w(M_1)$$  \hspace{1cm} (4.3)

We also deduce that $BP'_{2,i_2-1}$ has maximum degree 2. Then, it can be edge colored with 2 color classes $M_2^2$ and $M_3^2$. Moreover, since $\forall e \in E'_{i_3-1}$, $w(e) \leq w(e_{i_3}) = w(M_2)$, we obtain:

$$w(M_2^2) + w(M_3^2) \leq 2w(M_2)$$  \hspace{1cm} (4.4)

Using (4.3) and (4.4), we obtain:

$$\text{val}(S_k^2) \leq w(M_1) + 2w(M_2) \quad \square$$

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**SOL3**

1 For $k = m'$ down to 1 do

1.1 Determine if there is a matching $M_k$ in $BP'([E_{k}])$ such that each node of degree 3 in $BP'$ is saturated.

1.2 If such a matching is found:

1.2.1 Consider the decomposition $BP'_{1,k}$ and $BP'_{2,k}$ of $BP'$ induced by $M_k$ and $E' \setminus M_k$ respectively;

1.2.2 Color $BP'_{1,k}$ with one color to get $M_k^1$;

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1.2.3 Color greedily $BP'_{2,k}$ with two colors and let $M^1_k$ and $M^2_k$ be the color classes;

1.2.4 Let $S^k_3 = (M^1_k, M^2_k, M^3_k)$ be the edge coloring of $BP'$ obtained;

2 Output $\{S^k_3 : k = 1, \ldots, m'-1\}$;

---

**Lemma 4.5** If $S = (M_1, M_2, M_3)$ with $w(M_1) \geq w(M_2) \geq w(M_3)$ is an edge coloring of $BP'$, then algorithm SOL3 produces a solution $S^k_3$ satisfying: $\text{val}(S^k_3) \leq 2w(M_1) + w(M_3)$

**Proof:** Let $S = (M_1, M_2, M_3)$ with $w(M_1) \geq w(M_2) \geq w(M_3)$ be an edge coloring of $BP'$. As previously, let us consider one particular iteration of SOL3. We examine now the case where $k = i_3 - 1$. By construction, we have $M_3 \subseteq E_{i_3-1}$ and $M_3$ contains a matching where each node of $BP'[E_{i_3-1}]$ having a degree 3 in $BP'$ is saturated. Thus, $BP'_{2,i_3-1}$ exists. Moreover, since $\forall e \in E_{i_3-1}$, $w(e) \leq w(e_{i_3}) = w(M_3)$, we obtain:

$$w(M_{i_3}^{i_3-1}) \leq w(M_3) \quad (4.5)$$

As previously, we deduce that $BP'_{1,i_3-1}$ can be 2-edge colored with color classes $M_{i_3}^{i_3-1}$ and $M_{2}^{i_3-1}$ and we have:

$$w(M_{i_3}^{i_3-1}) + w(M_{2}^{i_3-1}) \leq 2w(M_1) \quad (4.6)$$

Combining (4.5) and (4.6), we obtain:

$$\text{val}(S^{i_3-1}_3) \leq 2w(M_1) + w(M_3)$$

\[\square\]

**Remark 4.6** Observe that if a color class $M_j^{i_3-1}$ is empty, then we can improve the bound; in this case, $\text{val}(S^{i_3-1}_3) \leq 2w(M_1)$. This remark is also valid for algorithms SOL1 and SOL2, and if several color classes are empty. For SOL1 for instance, if $M_{2}^{i_3-1}$ and $M_{3}^{i_3-1}$ are empty, then $\text{val}(S^{i_3-1}_3) \leq w(M_1) + w(M_3)$.

**Theorem 4.7** BIPARTITEEDGECOLOR produces a $\frac{7}{5}$ approximation for MIN WEIGHTED EDGE COLORING in bipartite graphs with maximum degree 3.

**Proof:** Let $S^* = (M_1^*, \ldots, M_k^*)$ with $w(M_1^*) \geq \ldots \geq w(M_k^*)$ be an optimal weighted edge coloring of $BP$. Denote by $i_k^*$ the smallest index of an edge in $M_k^*$ ($i_k^* = m+1$ if the color class is empty).

Consider the iteration of BIPARTITEEDGECOLOR corresponding to the cases $i = i_k^*-1$ and $j = i_4^*-1$. Then:

- Applying Lemma 4.3, we produce in $BP' = BP[E_i]$ an edge coloring of weight at most $w(M_1^*) + w(M_2^*) + 2w(M_3^*)$. Then a greedy coloring of the edges of $E_i$ produces a coloring of weight at most

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\[w(M_1^*) + w(M_2^*) + 2w(M_3^*) + w(M_4^*)\]  \hspace{1cm} (4.7)

- Applying Lemma 4.4, we produce in \(BP' = BP[E_j]\) an edge coloring of weight at most \(w(M_1^*) + 2w(M_2^*)\). Then a greedy coloring of the edges of \(\overline{E}_j\) produces a coloring of weight at most
\[w(M_1^*) + 2w(M_2^*) + 2w(M_4^*)\]  \hspace{1cm} (4.8)

- Applying Lemma 4.5, we produce in \(BP' = BP[E_j]\) an edge coloring of weight at most \(2w(M_1^*) + w(M_3^*)\). Then a greedy coloring of the edges of \(\overline{E}_j\) produces a coloring of weight at most
\[2w(M_1^*) + w(M_3^*) + 2w(M_4^*)\]  \hspace{1cm} (4.9)

Note that if some color class(es) is (are) empty in the colorings produced by one of the algorithms SOLL, then the bounds are still valid. Indeed, for SOLL3 for instance, according to Remark 4.6, the value of the coloring computed at step \(j = i_3 - 1\) has a weight of at most \(2w(M_1^*)\), and the greedy step produces a coloring of value at most \(2w(M_1^*) + 3w(M_1^*) \leq 2w(M_1^*) + w(M_3^*) + 2w(M_4^*)\).

Using (4.7), (4.8) and (4.9), we deduce that the coloring \(S\) computed by BIPARTITEEDGECOLOR satisfies:
\[\text{val}(S) \leq \min\{ w(M_1^*) + w(M_2^*) + 2w(M_3^*) + w(M_4^*); w(M_1^*) + 2w(M_2^*) + 2w(M_4^*); 2w(M_1^*) + w(M_3^*) + 2w(M_4^*)\}\]  \hspace{1cm} (4.10)

The convex combination of these 3 values with coefficients \(\frac{3}{6}, \frac{2}{6}\) and \(\frac{1}{6}\) respectively and the inequality (4.10) give the expected result, that is:
\[w(S) \leq \frac{7}{6}w(M_1^*) + \frac{7}{6}w(M_2^*) + \frac{7}{6}w(M_3^*) + w(M_4^*) + \frac{1}{2}w(M_5^*) \leq \frac{7}{6}\text{opt}(I)\]

\[\square\]

5 Weighted node coloring in split graphs

The split graphs are a class of graphs related to bipartite graphs in the sense that its vertex set can be partitioned into two subsets \(K_1, V_2\) with some properties. More precisely, \(G = (K_1, V_2; E)\) is a split graph if \(K_1\) is a clique of \(G\) (with size \(|K_1| = n_1\)), \(V_2\) is an independent set (with size \(|V_2| = n_2\)). Since split graphs form a subclass of perfect graphs, the node coloring problem on split graphs is polynomial. On the other hand, in [6], it is proved that the weighted node coloring problem is strongly NP-hard in split graphs, even if the weights take only two values. Thus, we deduce that there is no fully polynomial time approximation scheme in such a class of graphs, unless \(\text{P=NP}\). Here, we propose a polynomial time approximation scheme using structural properties of
optimal solutions. An immediate observation about split graphs is that any optimal node coloring $S^* = (S_1^*, \ldots, S_\ell^*)$ satisfies $|K_1| \leq \ell \leq |K_1| + 1$ and any color class $S_i^*$ is a subset of $V_2$ with possibly one node of $K_1$. In particular, for any optimal node coloring $S^* = (S_1^*, \ldots, S_\ell^*)$, there exists at most one index $i(S^*)$ such that $S_i^* \cap K_1 = \emptyset$.

**Lemma 5.1** Let $G = (K_1, V_2; E)$ be a split graph. There is an optimal weighted node coloring $S^* = (S_1^*, \ldots, S_\ell^*)$ with $w(S_1^*) \geq \ldots \geq w(S_\ell^*)$ and an index $i_0 \leq \ell + 1$ such that:

- $i \leq i_0$, $S_i^* = \{v_i\} \cup \{v \in V_2 : v \notin \bigcup_{k=1}^{i-1} S_k^* \text{ and } [v, v_i] \notin E\}$ for some $v_i \in K_1$.
- $S^*_{i_0} = V_2 \setminus (S_1^* \cup \ldots \cup S_{i_0-1}^*)$.
- $\forall j > i_0$ $S_j^* = \{v_j\}$ for some $v_j \in K_1$.

**Proof:** Let $G = (K_1, V_2; E)$ be a split graph and let $S^* = (S_1^*, \ldots, S_\ell^*)$ with $w(S_1^*) \geq \ldots \geq w(S_\ell^*)$ be an optimal weighted node coloring of $G$. If $\ell = n_1$ (we recall that $n_1 = |K_1|$), then we set $i_0 = \ell + 1$ otherwise let $i_0$ be the unique $i$ such that $S_i^* \cap K_1 = \emptyset$. We build the set $S_i^*$ in the following way:

- For $i = 1, \ldots, i_0 - 1$, $S_i^* = \{v_i\} \cup \{v \in V_2 : v \notin \bigcup_{k=1}^{i-1} S_k^* \text{ and } [v, v_i] \notin E\}$ where we assume that $S_i^* \cap K_1 = \emptyset$.
- $S_{i_0}^* = V_2 \setminus (S_1^* \cup \ldots \cup S_{i_0-1}^*)$.
- For $i = i_0 + 1, \ldots, \ell$, $S_i^* = S_i^* \cap K_1$.

Thus, when $i_0 = \ell + 1$, the sets resulting from the second and the third items above are respectively empty and not defined. Let us prove that:

$$\forall i = 1, \ldots, \ell, \ w(S_i^*) \leq w(S_i^*) \quad (5.1)$$

Since $w(S_1^*) \geq \ldots \geq w(S_\ell^*)$, we have $w(S_i^*) = \max\{w(v) : v \in K_1 \cup V_2 \setminus (S_1^* \cup \ldots \cup S_{i_0-1}^*)\}$. Moreover, by construction $\bigcup_{j=1}^{i_0-1} S_j^* \subseteq \bigcup_{j=1}^{i_0-1} S_j^*$. Thus, the result follows.

Using inequality (5.1), we deduce that node coloring $S^* = (S_1^*, \ldots, S_\ell^*)$ has a weight $\text{val}(S^*) \leq \sum_{i=1}^\ell w(S_i^*) = opt(I)$ and then, $S^*$ is an optimal weighted node coloring satisfying Lemma 5.1. $\square$

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**SPLITNODECOLORk**

1. For all subsets $K_1^k \subseteq K_1$ with $|K_1^k| \leq k$ do

1.1. For all bijections $f$: $\{1, \ldots, p\} \rightarrow K_1^k$ (where we assume that $p = |K_1^k|$) do

1.1.1. For $i = 1$ to $p$ do

1.1.1.1. Set $S_i^{K_1^k, f} = \{f(i)\} \cup \{v \in V_2 : v \notin \bigcup_{k=1}^{i-1} S_k^{K_1^k, f} \text{ and } [v, f(i)] \notin E\}$;
1.1.2 Set \( S_{p+1}^{K_i,f} = V_2 \setminus (S_1^{K_i,f} \cup \ldots \cup S_p^{K_i,f}) \);

1.1.3 For \( i = p + 2 \) to \( n_1 + 1 \) (assume \( K_1 \setminus K'_1 = \{v_{p+2}, \ldots, v_{n_1+1}\} \)) do

1.1.3.1 Set \( S_i^{K_i,f} = \{v_i\} \);

1.1.4 Set \( S_{K_i,f} = (S_1^{K_i,f}, \ldots, S_{n_1+1}^{K_i,f}) \);

2 Output \( S = \text{argmin}\{\text{val}(S_{K_i,f})\}; \)

This algorithm has a time complexity of \( O(k!n^{k+1}) \).

**Theorem 5.2** For all \( \varepsilon > 0 \), SPLITNODECOLOR \([\varepsilon]\) produces a \( 1 + \varepsilon \) approximation for MIN WEIGHTED NODE COLORING in split graphs.

**Proof:** Let \( G = (K_1, V_2; E) \) be a split graph and let \( S^* = (S_1^*, \ldots, S_\ell^*) \) with \( w(S_1^*) \geq \ldots \geq w(S_\ell^*) \) be an optimal weighted node coloring of \( G \) satisfying Lemma 5.1. Let \( k = \lceil \frac{1}{\varepsilon} \rceil \). If \( i_0 \leq k \), then by construction the solution \( S \) returned by SPLITNODECOLOR \( k \) is optimal. So, assume \( i_0 > k \) and let \( K_1' = (\cup_{j=1}^k S_j^*) \setminus V_2 \). Obviously, \( |K_1'| = k \) and let \( f^*(i) = S_i^* \cap K_1 \) for \( i = 1, \ldots, k \).

Let us examine the solution \( S_{K'_i,f^*} \) corresponding to the step \( K'_i = K'_1 \) and \( f = f^* \) of SPLITNODECOLOR \( k \).

By construction, we have

\[
\forall i = 1, \ldots, k, \quad S_i^{K'_i,f^*} = S_i^* \tag{5.2}
\]

Moreover, since \( K_1 \setminus K'_1 \subseteq S_{k+1}^* \cup \ldots \cup S_\ell^* \) and \( K_1 \setminus K'_1 \) is a clique, we obtain:

\[
\sum_{j=k+2}^{n_1+1} w(S_j^{K'_i,f^*}) \leq \sum_{j=k+1}^{\ell} w(S_j^*) \tag{5.3}
\]

Thus, combining (5.2) and (5.3), we deduce:

\[
\text{val}(S_{K'_i,f^*}) - w(S_{k+1}^{K'_i,f^*}) \leq \text{opt}(I) \tag{5.4}
\]

Moreover, by construction \( w(S_{k+1}^{K'_i,f^*}) \leq w(S_k^*) \leq \ldots \leq w(S_1^*) \) and then

\[
w(S_{k+1}^{K'_i,f^*}) \leq \frac{1}{k} \times \text{opt}(I) \tag{5.5}
\]

Finally, using these two last inequalities with \( \frac{1}{k} \leq \varepsilon \), we obtain the expected result. \( \square \)

### 6 Conclusion

As mentioned in the introduction, the MIN WEIGHTED NODE COLORING problem has received a significant attention in the last few years. Indeed, it is both a relevant model for scheduling problems in particular, and a natural generalization of the famous coloring problem, leading to challenging theoretical questions, some of them being still open.
For instance, the complexity of the problem in trees has been raised by Guan and Zhu [15] in the first article on this problem. Partial answers have been given, such as the polynomiality in chains, the existence of approximation schemes ([8, 16]), and a tight bound of $\theta(\log n)$ on the number of colors in an optimum solution ([26]), but the complexity is still unknown.

Another example is the approximability in perfect graphs (or more generally in graphs where the usual coloring problem is polynomial). Indeed, the problem is NP-hard (since it is hard for bipartite graphs [6] or interval graphs [8]) and even not $8/7 - \varepsilon$ approximable. On the other hand, a 4-approximation algorithm has been proposed in [26], recently improved to an $\varepsilon$-approximation algorithm in [10]. Reducing this gap will be another interesting challenge.

References


