Approximation with a fixed number of solutions of some multiobjective maximization problems*

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Abstract

We investigate the problem of approximating the Pareto set of some multiobjective optimization problems with a given number of solutions. Our purpose is to exploit general properties that many well studied problems satisfy. We derive existence and constructive approximation results for the biobjective versions of Max Submodular Symmetric Function (and special cases), Max Bisection, and Max Matching and also for the $k$-objective versions of Max Coverage, Heaviest Subgraph, Max Coloring of interval graphs.

1 Introduction

In multiobjective combinatorial optimization a solution is evaluated considering several objective functions and a major challenge in this context is to generate the set of efficient solutions or the Pareto set (see [12] about multiobjective combinatorial optimization). However, it is usually difficult to identify the efficient set mainly due to the fact that the number of efficient solutions can be exponential in the size of the input and moreover the associated decision problem is NP-complete even if the underlying single-objective problem can be solved in polynomial time (e.g. shortest path [12]). To handle these two difficulties, researchers have been interested in developing approximation algorithms with an a priori provable guarantee such as polynomial time constant approximation algorithms. Considering that all objectives have to be maximized, and for a positive $\rho \leq 1$, a $\rho$-approximation of Pareto set is a set of solutions that includes, for each efficient solution, a solution that is at least at a factor $\rho$ on all objective values. Intuitively, the larger the size of the approximation set, the more accurate it can be.

It has been pointed out by Papadimitriou and Yannakakis [28] that, under certain general assumptions, there always exists a $(1 - \varepsilon)$-approximation, with any given accuracy $\varepsilon > 0$, whose size is polynomial both in the size of the instance and in $1/\varepsilon$ but exponential in the number of criteria. In this result, the accuracy $\varepsilon > 0$ is given explicitly and a general upper

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bound on the size of the approximation set is given. When the number of solutions in the approximation set is limited, not every level of accuracy is possible. So, once the number of solutions is fixed in the approximation set of a multiobjective problem, the following questions are raised: What is the accuracy for which an approximation is guaranteed to exist? Which accuracy can be obtained in polynomial time?

In this paper we are interested in establishing for some multiobjective maximization problems the best approximation ratio of the set of efficient solutions when the size of the approximation set is given explicitly. We give three approaches that deal with biobjective and \( k \)-objective problems that allow us to obtain approximations of the set of efficient solutions with one or several solutions. More precisely, in a first approach, we consider a general maximization problem and establish a sufficient condition that guarantees the construction of a constant approximation of the Pareto set with an explicitly given number of solutions. As a corollary, we can construct a \( (1 - \varepsilon) \)-approximation of the Pareto set with \( O(\frac{1}{\varepsilon}) \) solutions. In a second approach, we establish a necessary and sufficient condition for the construction of a constant approximation of the Pareto set with one solution. In a third approach, we establish a sufficient condition for the construction of one solution with approximation guarantee for \( k \)-objective selection problems. In these three approaches, if the corresponding solutions can be found in polynomial time then the biobjective or \( k \)-objective selection problems admits polynomial time approximation with one solution.

Properties defined in these three approaches apply to several problems previously studied in single-objective approximation. Thus we derive polynomial time constant approximations with one solution for Biobjective Max Bisection, Biobjective Max Partition, Bijective Max Cut, Bijective Max Set Splitting, Bijective Max Matching and \( k \)-objective Heaviest Subgraph, \( k \)-objective Max \( q \) Colorable Subgraph and \( k \)-objective Max Coverage, but also for \( k \)-objective versions of some particular cases of Max Coverage: Max \( q \) Vertex Cover and Max \( q \) Selection. Some instances show that the given biobjective approximation ratios are the best we can expect. In addition Bijective Max Partition, Bijective Max Cut, Bijective Max Set Splitting admit a \( (1 - \varepsilon) \)-approximation of the Pareto set with \( O(\frac{1}{\varepsilon}) \) solutions.

Several results exist in the literature on the approximation of multiobjective combinatorial optimization problems. One can mention the existence of fully polynomial time approximation schemes for biobjective shortest path [19, 34, 33], knapsack [13, 8], minimum spanning tree [28], scheduling problems [6], randomized fully polynomial time approximation scheme for matching [28], and polynomial time constant approximation for max cut [4], a biobjective scheduling problem [32] and the traveling salesman problem [5, 26]. Note that [4], [26] and [32] are approximations with a single solution.

This article is organized as follows. In Section 2, we introduce basic concepts about multiobjective optimization and approximation. Section 3 is devoted to an approach for approximating some biobjective problems with one or several solutions. Section 4 presents a necessary and sufficient condition for approximating some biobjective problems with one solution within a constant factor. In Section 5 we establish an approach for approximating some multiobjective selection problems. Conclusions are provided in a final section.
2 Preliminary on multi-objective optimization and approximation

Consider an instance of a multi-objective optimization problem with \( k \) criteria or objectives where \( X \) denotes the finite set of feasible solutions. Each solution \( x \in X \) is represented in the objective space by its corresponding objective vector \( w(x) = (w_1(x), \ldots, w_k(x)) \). We assume that each objective has to be maximized.

From these \( k \) objectives, the dominance relation defined on \( X \) is defined as follows: a feasible solution \( x \) dominates a feasible solution \( x' \) if and only if \( w_i(x) \geq w_i(x') \) for \( i = 1, \ldots, k \) with at least one strict inequality. A solution \( x \) is efficient if and only if there is no other feasible solution \( x' \in X \) such that \( x' \) dominates \( x \), and its corresponding objective vector is said to be non-dominated. Usually, we are interested in finding a solution corresponding to each non-dominated objective vector. The set of all such solutions is called Pareto set.

For any \( 0 < \rho \leq 1 \), a solution \( x \) is called a \( \rho \)-approximation of a solution \( x' \) if \( w_i(x) \geq \rho \cdot w_i(x') \) for \( i = 1, \ldots, k \). A set of feasible solutions \( X' \) is called a \( \rho \)-approximation of the set of all efficient solutions if, for every feasible solution \( x \in X \), \( X' \) contains a feasible solution \( x' \) that is a \( \rho \)-approximation of \( x \). If such a set exists, we say that the multi-objective problem admits a \( \rho \)-approximate Pareto set with \( |X'| \) solutions.

An algorithm that outputs a \( \rho \)-approximation of a set of efficient solutions in polynomial time in the size of the input is called a \( \rho \)-approximation algorithm. In this case we say that the multi-objective problem admits a polynomial time \( \rho \)-approximate Pareto set.

Consider in the following a single-objective maximization problem \( P \) defined on a ground set \( U \). Every element \( e \in U \) has a non-negative weight \( w(e) \). The goal is to find a feasible solution (subset of \( U \)) with maximum weight. The weight of a solution \( S \) must satisfy the following scaling hypothesis: if \( \text{opt}(I) \) denotes the optimum value of \( I \), then \( \text{opt}(I') = t \cdot \text{opt}(I) \), where \( I' \) is the same instance as \( I \) except that \( w'(e) = t \cdot w(e) \). For example, the hypothesis holds when the weight of \( S \) is defined as the sum of its elements’ weights, or \( \min_{e \in S} w(e) \), etc.

In the \( k \)-objective version, called \( k \)-objective \( P \), \( k \geq 2 \), every element \( e \in U \) has \( k \) non-negative weights \( w_1(e), w_2(e), \ldots, w_k(e) \) and the goal is to find a Pareto set within the set of feasible solutions. Given an instance \( I \) of \( k \)-objective \( P \), we denote by \( \text{opt}_i(I) \) (or simply \( \text{opt}_i \)) the optimum value of \( I \) restricted to objective \( i \), \( i = 1, \ldots, k \). Here, the objective function on objective \( 1 \) is not necessarily of the same kind as on objective \( 2 \), but both satisfy the scaling hypothesis. For example, one objective can be additive (sum of element’s weight) and the other can be bottleneck (min or max of element’s weights).

3 Approximation with a given number of solutions for some boolean problems

Papadimitriou and Yannakakis [28] proved the existence of a \((1-\varepsilon)\)-approximation of size polynomial in the size of the instance and \( \frac{1}{\varepsilon} \). In this general result, the accuracy \( \varepsilon > 0 \) is given explicitly but the size of the approximation set is roughly bounded. In this section we consider a general maximization problem \( \Pi \) and establish a sufficient condition that guarantees the construction of a constant approximation of the Pareto set with an explicitly given number of solutions for \( \Pi \). This result allows to construct a \((1-\varepsilon)\)-approximation of the Pareto set with \( O(\frac{1}{\varepsilon}) \) solutions but not necessarily in polynomial time. Moreover, if the single objective problem is polynomial time constant approximable and the above construction is done in poly-
nominal time then the biobjective version is also polynomial time constant approximable with one solution. Thus we obtain constant approximations and polynomial time constant approximations with one solution for Biobjective Submodular Symmetric Function and also for Biobjective Max Partition, Biobjective Max Cut, Biobjective Max Set Splitting but also for Biobjective Max Matching.

In the following, we are interested in biobjective maximization problems, Biobjective II, which satisfy the following property.

**Property 1** Given any two feasible solutions \( S_1 \) and \( S_2 \), and any real \( \alpha \) satisfying \( 0 < \alpha \leq 1 \), if \( w_2(S_1) < \alpha w_2(S_2) \) and \( w_1(S_2) < \alpha w_1(S_1) \) then there exists a feasible solution \( S_3 \) which satisfies \( w_1(S_3) > (1 - \alpha)w_1(S_1) \) and \( w_2(S_3) > (1 - \alpha)w_2(S_2) \).

We say that Biobjective II satisfies Property 1 polynomially if \( S_3 \) can be constructed in polynomial time.

Property 1 means that if \( S_1 \) is not an \( \alpha \)-approximation of \( S_2 \) and \( S_2 \) is not an \( \alpha \)-approximation of \( S_1 \) for both objective functions \( w_1 \) and \( w_2 \), then there exists a feasible solution \( S_3 \) which simultaneously approximates \( S_1 \) and \( S_2 \) with performance guarantee \( 1 - \alpha \).

Given a positive integer \( \ell \), consider the equations \( x^{2\ell} = 1 - x^\ell \) and \( x^{2\ell-1} = 1 - x^\ell \). Denote by \( \alpha_\ell \) and \( \beta_\ell \) their respective solutions in the interval \([0, 1]\). Note that \( \alpha_\ell = \left(\frac{\sqrt{5} - 1}{2}\right)^{1/\ell} \). Moreover, \( \alpha_\ell < \beta_{\ell+1} < \alpha_{\ell+1}, \ell \geq 1 \). Indeed, since \( \beta_\ell \in (0, 1) \), we have \( 1 - \beta_{\ell+1}^\ell < 1 - \beta_{\ell+1}^{\ell+1} = 2 - \beta_{\ell+1}^{\ell+1} < \beta_{\ell+1}^{2\ell} \). Since the function \( f_\ell(x) = x^{2\ell} + x^\ell - 1 \) is strictly increasing when \( x \in (0, 1) \), for any \( \ell \geq 1 \) and \( f_\ell(\beta_{\ell+1}) > 0 = f_\ell(\alpha_\ell) \), we have \( \beta_{\ell+1} > \alpha_\ell \).

**Theorem 1** If Biobjective II satisfies Property 1, then it admits a \( \beta_\ell \)-approximate Pareto set (resp. an \( \alpha_\ell \)-approximate Pareto set) containing at most \( p \) solutions, where \( p \) is a positive odd integer such that \( p = 2\ell - 1 \) (resp. a positive even integer such that \( p = 2\ell \)).

**Proof:** Let \( S_1 \) (resp. \( S_2 \)) be a solution optimal for the first objective (resp. second one). In the following, \( opt \) denotes the optimal value on the first objective and also on the second objective. This can be assumed without loss of generality because a simple rescaling can make the optimal values coincide (e.g. we can always assume that \( opt_2 \neq 0 \), thus by multiplying each weight \( w_2(e) \) by \( \frac{opt_2}{opt_2} \) we are done; since the result is only existential, the time complexity for the determination of \( opt_1 \) and \( opt_2 \) is not taken into account). Then \( w_1(S_1) = w_2(S_2) = opt \). Consider first the case where \( p \) is odd. Let \( p = \beta_\ell \) with \( p = 2\ell - 1 \). Subdivide the bidimensional value space with coordinates \( \{0\} \cup \{\rho^{\ell}opt : 0 \leq i \leq p\} \). See Figure 1 for an illustration.

Given \( i, 1 \leq i \leq p \), the strip \( s(i, \cdot) \) is the part of the space containing all couples \( (w_1, w_2) \) satisfying \( \rho^{\ell}opt < w_1 \leq \rho^{\ell-1}opt \) and \( 0 \leq w_2 \leq opt \). The strip \( s(p + 1, \cdot) \) is the part of the space containing all couples \( (w_1, w_2) \) satisfying \( 0 \leq w_1 \leq \rho^{\ell}opt \) and \( 0 \leq w_2 \leq opt \). Given \( j, 1 \leq j \leq p \), the strip \( s(\cdot, j) \) is the part of the space containing all couples \( (w_1, w_2) \) satisfying \( \rho^{\ell}opt < w_2 \leq \rho^{\ell-1}opt \) and \( 0 \leq w_1 \leq opt \). The strip \( s(\cdot, p + 1) \) is the part of the space containing all couples \( (w_1, w_2) \) satisfying \( 0 \leq w_2 \leq \rho^{p}opt \) and \( 0 \leq w_1 \leq opt \).

Suppose that \( w_2(S_1) < \rho^{\ell}opt \) and \( w_1(S_2) < \rho^{\ell}opt \). In other words \( S_1 \in s(1, \cdot) \cap s(\cdot, p + 1) \) and \( S_2 \in s(\cdot, 1) \cap s(p + 1, \cdot) \). Using Property 1 there exists a solution \( S_3 \) satisfying \( w_1(S_3) > (1 - \rho^{\ell})opt \) and \( w_2(S_3) > (1 - \rho^{\ell})opt \). Moreover, \( 1 - \rho^{\ell} = 1 - \beta_{\ell+1}^{2\ell-1} = \beta_{\ell+1}^{\ell} = \rho^{\ell} \). Then \( S_3 \) is a \( \rho \)-approximation of any solution \( S \) satisfying \( \max\{w_1(S), w_2(S)\} \leq \rho^{\ell-1}opt \).
One can construct a $\rho$-approximate Pareto set $P$ as follows: $P = \{S_3\}$ at the beginning and for $j = \ell - 1$ down to 1, pick a feasible solution $S$ with maximum weight $w_1$ in $s(\cdot, j)$ (if $s(\cdot, j)$ contains at least one value of a feasible solution) and set $P = P \cup \{S\}$. Afterwards, for $i = \ell - 1$ down to 1, pick a feasible solution $S$ with maximum weight $w_2$ in $s(i, \cdot)$ (if $s(i, \cdot)$ contains at least one value of a feasible solution) and set $P = P \cup \{S\}$. For every strip the algorithm selects a solution which $\rho$-approximates (on both objective functions) any other solution in the strip. Since the solutions of $P$ approximate the whole bidimensional space, $P$ is a $\rho$-approximate Pareto set containing at most $p = 2\ell - 1$ solutions.

Now suppose that $w_2(S_1) \geq \rho^2 opt$ (the case $w_1(S_2) \geq \rho^2 opt$ is treated similarly). Solution $S_1$ must be in $s(\cdot, j^*)$ for $1 \leq j^* \leq p$. Since $w_1(S_1) = opt$, $S_1$ is a $\rho$-approximation of any solution $S$ in $s(\cdot, p) \cup s(\cdot, p + 1)$. One can build an $\rho$-approximate Pareto set $P$ as follows: $P = \{S_1\}$ at the beginning and for $j = j^* - 1$ down to 1, pick a feasible solution $S$ with maximum weight $w_1$ in $s(\cdot, j)$ (if $s(\cdot, j)$ contains at least one value of a feasible solution) and set $P = P \cup \{S\}$. Since the strips form a partition of the space, the algorithm returns an $\rho$-approximate Pareto set containing at most $p$ solutions.

The proof is similar in the second case where $p$ is even by considering $\rho = \alpha \ell$ with $p = 2\ell$.

**Corollary 1** If Biobjective II satisfies Property 1, then it admits a $(1 - \varepsilon)$-approximate Pareto set containing $O(\frac{1}{\varepsilon})$ solutions.

**Proof:** According to Theorem 1, we need at most $2\ell$ solutions where $(\frac{\sqrt{5} - 1}{2})^{1/\ell} \geq 1 - \varepsilon$ in order to obtain a $(1 - \varepsilon)$-approximate Pareto set. Thus $\ell = O(\frac{1}{\varepsilon})$. □

Property 1 can be relaxed in the following way:

**Property 2** Given any two feasible solutions $S_1$ and $S_2$, and a real $\alpha$ satisfying $0 < \alpha \leq 1$. If $w_2(S_1) < \alpha w_2(S_2)$ and $w_1(S_1) < \alpha w_1(S_1)$ then there exists a feasible solution $S_3$ which satisfies $w_1(S_3) > (c - \alpha)w_1(S_1)$ and $w_2(S_3) > (c - \alpha)w_2(S_2)$, where $0 < c \leq 1$ is a constant.

We say that Biobjective II satisfies Property 2 polynomially if $S_3$ can be constructed in polynomial time.

Given a positive integer $\ell$, consider the equations $x^{2\ell} = c - x^{\ell}$ and $x^{2\ell - 1} = c - x^\ell$. Denote by $\gamma_\ell$ and $\delta_\ell$ their respective solutions in the interval $[0, 1]$. Note that $\gamma_\ell = \left(\frac{\sqrt{1 + 5} - 1}{2}\right)^{1/\ell}$ and $\gamma_\ell < \delta_\ell < \gamma_{\ell + 1}$, $\ell \geq 1$. 

![Figure 1: Illustration of Theorem 1](image-url)
Theorem 2 If Biobjective II satisfies Property 2, then it admits a $\delta_\ell$-approximate Pareto set (resp. a $\gamma_\ell$-approximate Pareto set) containing at most $p$ solutions, where $p$ is a positive odd integer such that $p = 2\ell - 1$ (resp. a positive even integer such that $p = 2\ell$).

Proof: The proof is similar to the proof of Theorem 1. Suppose that $w_2(S_1) < \rho^p opt$ and $w_1(S_2) < \rho^p opt$. Using Property 2 there exists a solution $S_3$ satisfying $w_1(S_3) > (c-\rho^p)opt$ and $w_2(S_3) > (c-\rho^p)opt$. For the case $\rho = \delta_\ell$ and $p = 2\ell - 1$, we get that $c-\rho^p = c-\delta_\ell^{2\ell-1} = \delta_\ell^\ell = \rho^\ell$. For the case $\rho = \gamma_\ell$ and $p = 2\ell$, we get that $c-\rho^p = c-\gamma_\ell^{2\ell} = \gamma_\ell^\ell = \rho^\ell$. Then $S_3$ is a $\rho$-approximation of any solution $S$ satisfying $\max\{w_1(S), w_2(S)\} \leq \rho^p opt$.

The previous results of this section consider the construction, not necessarily in polynomial time, of an approximate Pareto set with a fixed number of solutions. We give in the following some conditions on the construction in polynomial time of an approximate Pareto set with one solution.

Proposition 1 If II is polynomial time $\rho$-approximable and Biobjective II satisfies Property 1 polynomially (resp. 2), then Biobjective II is polynomial time $\frac{\rho}{2}$-approximable (resp. $\frac{\rho^p}{2}$-approximable) with one solution.

Proof: Consider first the case where Biobjective II satisfies Property 2 polynomially. Let $S_1$ (resp. $S_2$) be a polynomial time $\rho$-approximation solution for the first objective (resp. second one). In the following, $opt_1$ (resp. $opt_2$) denotes the optimal value on the first objective (resp. second one). If $w_2(S_1) \geq \frac{cw_1(S_1)}{2}$ then $w_2(S_1) \geq \frac{\rho^p}{2} opt_1$ and thus $S_1$ is a $\frac{\rho^p}{2}$-approximate Pareto set. If $w_1(S_2) \geq \frac{cw_2(S_2)}{2}$ then $w_1(S_2) \geq \frac{\rho^p}{2} opt_2$ and thus $S_2$ is a $\frac{\rho^p}{2}$-approximate Pareto set. Otherwise, $w_2(S_1) < \frac{cw_1(S_1)}{2}$ and $w_1(S_2) < \frac{cw_2(S_2)}{2}$ and since Biobjective II satisfies Property 2 polynomially, we can construct in polynomial time a feasible solution $S_3$ which satisfies $w_1(S_3) \geq \frac{cw_1(S_1)}{2}$ and $w_2(S_3) \geq \frac{cw_2(S_2)}{2}$, that is a $\frac{\rho^p}{2}$-approximate Pareto set. The proposition also holds in the case where Biobjective II satisfies Property 1 polynomially by replacing $c$ by 1.

In the following, we say that Property 1 is tight if there exist $S_1$, $S_2$ and $\alpha$ such that $w_2(S_1) < \alpha w_2(S_2)$ and $w_1(S_2) < \alpha w_1(S_1)$ and there is no $\alpha' < \alpha$ and a feasible solution $S_4$ satisfying $w_1(S_4) > (1-\alpha')w_1(S_1)$ and $w_2(S_4) > (1-\alpha')w_2(S_2)$.

We consider in Sections 3.1 and 3.2 several examples of problems II that satisfy the scaling hypothesis and such that Biobjective II satisfy Property 1 or Property 2.

3.1 Max Submodular Symmetric Function

A function $f$ defined on the power set of some ground set $U$ is called

- **submodular** if $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$, for all $X, Y \subseteq U$;
- **symmetric** if $f(X) = f(U \setminus X)$ for all $X \subseteq U$;
- **non negative** if $f(X) \geq 0$ for all $X \subseteq U$.

Max Submodular Symmetric Function is a combinatorial optimization problem which consists of maximizing a non negative, symmetric, and submodular function $f$ computable in polynomial time. The problem is known to generalize many NP-hard problems.
Feige, Mirrokni and Vondrak [14] give a $(\frac{1}{2} - o(1))$-approximation algorithm for Max Submodular Symmetric Function.

**Lemma 1** Biojective Max Submodular Symmetric Function satisfies Property 1 polynomially.

**Proof:** Let $\alpha \in (0, 1]$ and $S_1$, $S_2$ two solutions of an instance of Biojective Max Submodular Symmetric Function satisfying the inequalities: $w_2(S_1) < \alpha w_2(S_2)$ and $w_1(S_2) < \alpha w_1(S_1)$. Consider $S_3 = S_1 \Delta S_2 = (S_1 \setminus S_2) \cup (S_2 \setminus S_1)$, where $\Delta$ is the symmetric difference. We have

\[
w_1(S_3) + w_1(S_2) \geq w_1(S_1 \cup S_2) + w_1(S_2 \setminus S_1) = w_1(U \setminus (S_1 \cup S_2)) + w_1(S_2 \setminus S_1) \geq w_1(U \setminus S_1) + w_1(\emptyset) \geq w_1(S_1) \tag{1}
\]

where submodularity is used on the first and third lines, symmetry is used on the second and third lines and non negativity is used on the third line. Using $w_1(S_2) < \alpha w_1(S_1)$ into Inequality (1), we get that $w_1(S_3) \geq w_1(S_1) - w_1(S_2) \geq (1 - \alpha)w_1(S_1)$. With similar arguments we get that $w_2(S_3) \geq w_2(S_2) - w_2(S_1) \geq (1 - \alpha)w_2(S_2)$.

**Corollary 2** Biojective Max Submodular Symmetric Function admits a

(i) $\beta_\ell$-approximate Pareto set (resp. an $\alpha_\ell$-approximate Pareto set) containing at most $p$ solutions, where $p = 2\ell - 1$ (resp. $p = 2\ell$).

(ii) $(1 - \varepsilon)$-approximate Pareto set containing $O(\frac{1}{\varepsilon})$ solutions.

As indicated above, Corollary 2 deals with the possibility to reach some approximation bounds when the number of solutions in the Pareto set is fixed. We give in the following an approximation bound that we can obtain in polynomial time with one solution.

**Corollary 3** Biojective Max Submodular Symmetric Function admits a polynomial time $(\frac{1}{4} - o(1))$-approximate Pareto set with one solution.

**Proof:** The results follows from Lemma 1 and Proposition 1 with $\rho = \frac{1}{2} - o(1)$.

Now we review some special cases of Max Submodular Symmetric Function and use Proposition 1 to derive the existence of an approximate Pareto set with one solution for the bijective version.

**Special cases of Max Submodular Symmetric Function**

Max Pos NAE consists of a set of clauses $C$ defined on a set $U$ of boolean variables $x_1, \ldots, x_n$. The clauses are composed of two or more positive variables and they are endowed with a non negative weight. The Max Pos NAE problem consists of finding an assignment of the variables such that the total weight of the clauses that are satisfied is maximum, where a positive clause is satisfied by an assignment if it contains at least a true variable and at least a false variable. It is NP-hard and 0.7499-approximable [36]. Max Pos NAE is also known under the name Max SET Splittting or Max HYPERGRAPH CUT [36]. The special case in
which every clause contains exactly $k$ variables is denoted Max Pos $k$NAE. Max Pos $3$NAE is 0.908-approximable [37]. For $k \geq 4$, Max Pos $k$NAE is $(1 - 2^{1-k})$-approximable [3, 22] and this is the best possible since it is hard to approximate within a factor of $1 - 2^{1-k} + \varepsilon$, for any constant $\varepsilon > 0$ [20]. Another special case of Max Pos NAE in which every clause contains exactly 2 variables corresponds to Max Cut (given a graph $G = (V, E)$ with non negative weights on its edges, find $V' \subset V$ such that the total weight of the edges having exactly one extremity in $V'$ is maximum) which is 0.878-approximable [16].

The Max Partition problem consists of a set $J$ of $n$ items $1, \ldots, n$ where each item $j$ has a non negative weight $w(j)$. A solution $S$ is a bipartition $J_1 \cup J_2$ of the items. The goal is to find a solution $S$ such that $w(S) = \min\{\sum_{j \in J_1} w(j), \sum_{j \in J_2} w(j)\}$ is maximized. This NP-hard problem was also studied in the context of scheduling, where the number of partitions is not fixed, and consists of maximizing the earliest machine completion time [35]. Max Partition is a special case of the Max Subset Sum problem. An input of Max Subset Sum is formed by a set $J$ of $n$ items $1, \ldots, n$, each item $j$ has a non negative weight $w(j)$, and an integer $t$. The problem consists of finding a subset $S$ of $J$ whose sum $w(S)$ is bounded by $t$ and maximum. Max Subset Sum has an FPTAS [10]. We can obtain a FPTAS for Max Partition using the previous FPTAS for $t = \sum_{i=1}^n w(i)/2$.

Observe that Biobjective Max Partition is not $(1/2+\varepsilon)$-approximable with one solution. In order to see this, consider 3 items of weights $w_1(1) = 2, w_2(1) = 1, w_3(2) = 1, w_2(2) = 2, w_1(1) = 1, w_2(3) = 1$. The two efficient solutions $S_i$, $i = 1, 2$ consists of placing $i$ in a part and the other items in the other part and have weights $w_1(S_1) = 2, w_2(S_1) = 1, w_1(S_2) = 1, w_2(S_2) = 2$. Any other solution is either dominated by one of these two or has weights equal to 1 on both criteria.

Using Proposition 1, we obtain the following approximation ratios for the the biobjective versions with one solution of special cases of Max Submodular Symmetric Function:

<table>
<thead>
<tr>
<th>Problem</th>
<th>approx. ratio</th>
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<tbody>
<tr>
<td>Max Pos NAE</td>
<td>0.374</td>
</tr>
<tr>
<td>Max Pos $3$NAE</td>
<td>0.454</td>
</tr>
<tr>
<td>Max Pos $k$NAE, $k \geq 4$</td>
<td>$1/2 - 1/2^k$</td>
</tr>
<tr>
<td>Max Cut</td>
<td>0.439</td>
</tr>
<tr>
<td>Max Partition</td>
<td>$1/2 - \varepsilon$</td>
</tr>
</tbody>
</table>

Note that the result about Max Cut is the same as the one given in [4]. Moreover, Biobjective Max Cut is not $(1/2 + \varepsilon)$-approximable with one solution [4], meaning that we are close to the best possible approximation result.

### 3.2 Max Matching

Given a complete graph $G = (V, E)$ with non negative weights on the edges, the Max Matching problem is to find a matching of the graph of total weight maximum. Max Matching is solvable in polynomial time [11]. We study in this part the biobjective Max Matching problem and consider instances where the graph is a collection of complete graphs inside which the weights satisfy the triangle inequality, since otherwise the biobjective Max Matching problem is not at all approximable with one solution. In order to see this, consider a complete graph on 3 vertices with weights $(1,0),(0,1),(0,0)$. The optimum value on each
objective is 1. Nevertheless, any solution has value 0 with respect to at least one objective. Clearly Property 1 is not satisfied in this case.

Biobjective Max Matching problem is NP-hard [30]. It remains NP-hard even on instances where the graph is a collection of complete graphs inside which the weights satisfy the triangle inequality. In order to see this, we reduce in polynomial time Partition (proved NP-hard in [23]) to our problem. Given an instance \(I\) of Partition with \(2n\) non negative integers \(a_1, \ldots, a_{2n}\) such that \(\sum_{i=1}^{n} a_i = 2B\), we construct a graph \(G\), instance of our problem as follows: \(G\) contains \(6n\) vertices, at each integer \(a_i\) we associate a triangle with vertices \(v_i^1, v_i^2, v_i^3\) and weights \(w_1(v_i^1, v_i^2) = a_i, w_2(v_i^1, v_i^2) = a_i, w_1(v_i^2, v_i^3) = a_i, w_2(v_i^2, v_i^3) = 2a_i, w_1(v_i^3, v_i^3) = 2a_i\). For every edge between 2 different triangles we associate an weight 0 on both objectives. Clearly, there is a partition of the \(2n\) integers into two sets of sum \(B\) if and only if there is a matching in \(G\) of weight at least \(3B\) on each objective. It remains an open problem to decide if Biobjective Max Matching remains NP-hard on complete graphs where the weights satisfy the triangle inequality.

**Lemma 2** Biobjective Max Matching satisfies Property 2 polynomially with \(c = 1/3\).

**Proof:** Let \(\alpha \in (0, 1]\) and \(S_1, S_2\) two solutions of an instance of biobjective Max Matching satisfying the inequalities: \(w_2(S_1) < \alpha w_2(S_2)\) and \(w_1(S_2) < \alpha w_1(S_1)\). The set of edges of \(S_1 \cup S_2\) constitutes several connected components: cycles of size \(4\ell\), cycles of size \(4\ell + 2\), \(\ell \geq 1\), and paths of length \(\ell \geq 1\). In order to construct \(S_3\) we proceed in two steps. Firstly, starting from \(S_1 \cup S_2\), we construct in polynomial time a collection of vertex disjoint paths \(\mathcal{P}\) of length at most 2 such that \(w_i(S_i \cap \mathcal{P}) \geq \frac{w_i(S_i)}{3}\). Secondly, we construct \(S_3\) from \(\mathcal{P}\).

Given such a collection \(\mathcal{P}\) of paths of length at most 2, we construct \(S_3\) as follows: for each path \(P_j = v_1, v_2\) of length 1 from \(\mathcal{P}\), we add \((v_1, v_2)\) to \(S_3\); for each path \(P_j = v_1, v_2, v_3\) of length 2 from \(\mathcal{P}\), we add \((v_1, v_3)\) in \(S_3\). It is easy to see, since \(\mathcal{P}\) is vertex disjoint, that \(S_3\) is a matching. In this last case, using triangle inequality, we have

\[
 w_i(S_{3-i} \cap P_j) + w_i(S_3 \cap P_j) \geq w_i(S_i \cap P_j) \quad \text{for} \quad i = 1, 2
\]

and thus the sum over all paths \(P_j \in \mathcal{P}\), we obtain

\[
 w_i(S_{3-i} \cap \mathcal{P}) + w_i(S_3) \geq w_i(S_i \cap \mathcal{P}) \quad \text{for} \quad i = 1, 2.
\]

Moreover, \(w_i(S_{3-i} \cap \mathcal{P}) < w_i(S_{3-i}) < \alpha w_i(S_i)\). Thus

\[
 w_i(S_3) \geq w_i(S_i \cap P_j) - \alpha w_i(S_i) \geq \frac{w_i(S_i)}{3} - \alpha w_i(S_i) = \left(\frac{1}{3} - \alpha\right) w_i(S_i) \quad \text{for} \quad i = 1, 2.
\]

We show in the following how to construct the collection \(\mathcal{P}\).

Consider a cycle of size \(4\ell\), \(C_j = v_1, v_2, \ldots, v_{4\ell}\) from \(S_1 \cup S_2\) and suppose \(S_1 \cap C_j = \{v_{2i+1}, v_{2i+2}\}, i = 0, \ldots, 2\ell - 1\). Let \(S'_1\) be the matching of maximum weight \(w_1\) between \(M = \{(v_{4i+1}, v_{4i+2}), i = 0, \ldots, \ell - 1\}\) and \((S_1 \cap C_j) \setminus M\). Thus \(w_1(S'_1) \geq \frac{1}{2} w_1(S_1 \cap C_j)\). Similarly we construct \(S'_2\) obtaining \(w_2(S'_2) \geq \frac{1}{2} w_2(S_2 \cap C_j)\). \(S'_1 \cup S'_2\) consists of paths of length two and we add \(S'_1 \cup S'_2\) to \(\mathcal{P}\). Thus, \(w_1(S'_1 \cup S'_2) \geq \frac{1}{2} w_1(S_1 \cap C_j)\) and \(w_2(S'_1 \cup S'_2) \geq \frac{1}{2} w_2(S_2 \cap C_j)\).

On a path of length \(\ell \geq 1\) from \(S_1 \cup S_2\), we proceed as in the previous case, obtaining the same inequalities as before.
Consider a cycle $C_j$ of size $4\ell + 2$ from $S_1 \cup S_2$. We remove from $C_j$ the edge with the smallest $w_1$ from $S_1 \cap C_j$ and the one with the smallest $w_2$ from $S_2 \cap C_j$. On the path or the two remaining paths we proceed as in the previous case for the construction of $S'_1$ and $S'_2$. $S'_1 \cup S'_2$ consists of paths of length at most two and we add $S'_1 \cup S'_2$ to $P$. In this case, $w_1(S'_1) \geq \frac{1}{2}(1 - \frac{1}{2\ell + 1})w_1(S_1 \cap C_j) \geq \frac{1}{3}w_1(S_1 \cap C_j)$ and $w_2(S'_2) \geq \frac{1}{2}(1 - \frac{1}{2\ell + 1})w_2(S_2 \cap C_j) \geq \frac{1}{3}w_2(S_2 \cap C_j)$.

Thus, summing all these inequalities we obtain $w_i(S_i \cap P) \geq \frac{w_i(S_i)}{3}$ for $i = 1, 2$. \hfill \Box

**Corollary 4** Biobjective Max Matching admits a $\delta$-approximate Pareto set (resp. an $\gamma$-approximate Pareto set) containing at most $p$ solutions, where $p = 2\ell - 1$ (resp. $p = 2\ell$).

**Corollary 5** Biobjective Max Matching admits a polynomial time $\frac{1}{c}$-approximate Pareto set with one solution.

**Proof:** It follows from Lemma 2 and Proposition 1 considering $p = 1$. \hfill \Box

## 4 Approx mat on w th one solut on for b object ve problems

In this section, we establish a necessary and sufficient condition for constructing, not necessarily in polynomial time, a constant approximation with one solution of the Pareto set for biobjective maximization problems. Moreover, if the construction can be done in polynomial time and the single-objective problem is polynomial time constant approximable, then the biobjective version is polynomial time constant approximable with one solution. Using this condition, we establish a polynomial time 0.174-approximation with one solution for Biobjective Max Bisection.

In the following, we are interested in biobjective maximization problems, Biobjective II, which satisfy the following property.

**Property 3** We can construct three solutions $S_1$, $S_2$, $S_3$ such that $S_i$ is a $\rho_i$-approximation for problem II on objective $i$, $i = 1, 2$, and $S_3$ is such that $w_1(S_2) + w_1(S_3) \geq \alpha \cdot w_1(S_1)$ and $w_2(S_1) + w_2(S_3) \geq \alpha \cdot w_2(S_2)$ for some fixed constant $\alpha \leq 1$.

We say that Biobjective II satisfies Property 3 polynomially if $S_1$, $S_2$, $S_3$ can be constructed in polynomial time.

The aim of solution $S_3$ in Property 3 is to compensate the potential inefficiency of $S_i$ on criterion $3 - i$, $i = 1, 2$.

**Theorem 3** Biobjective II is constant approximable with one solution if and only if it satisfies Property 3. Moreover, if Biobjective II is polynomial time constant approximable with one solution then it satisfies Property 3 polynomially. More precisely, if Biobjective II satisfies Property 3 polynomially such that $S_i$ is a polynomial time $\rho_i$-approximation for problem II on objective $i$, $i = 1, 2$, then Biobjective II admits a polynomial time $\alpha \cdot \min\{\rho_1, \rho_2\}$-approximation algorithm with one solution.
Proof: Suppose that Biobjective II is \( \rho \)-approximable with one solution. Let \( S_3 \) be this solution and \( S_1 \) and \( S_2 \) any two solutions. Then \( w_1(S_3) \geq \rho \cdot opt_1 \geq \rho \cdot w_1(S_1) \) and thus by setting \( \alpha = \rho \) we have \( w_1(S_2) + w_1(S_3) \geq \alpha \cdot w_1(S_1) \). The second inequality holds also.

Suppose now that Biobjective II satisfies Property 3. Since \( S_i \) is a \( \rho_i \)-approximation for problem \( \Pi \) on objective \( i, i = 1, 2 \), we have \( w_1(S_1) \geq \rho_1 \cdot opt_1 \) and \( w_2(S_2) \geq \rho_2 \cdot opt_2 \).

Since Property 3 is satisfied, we can construct \( S_3 \) such that
\[ w_1(S_2) + w_1(S_3) \geq \alpha \cdot w_1(S_1) \] (2)

and
\[ w_2(S_1) + w_2(S_3) \geq \alpha \cdot w_2(S_2) \] (3)

Now, we study different cases:

- If \( w_1(S_2) \geq \frac{\alpha}{2} w_1(S_1) \), then we deduce that \( S_2 \) is a good approximation of the Pareto set. From the hypothesis, we have \( w_1(S_2) \geq \frac{\alpha}{2} w_1(S_1) \geq \alpha \cdot \frac{\min(\rho_1, \rho_2)}{2} \cdot opt_1 \). On the other hand, we also have \( w_2(S_2) \geq \rho_2 \cdot opt_2 \geq \alpha \cdot \frac{\min(\rho_1, \rho_2)}{2} \cdot opt_2 \).

- If \( w_2(S_1) \geq \frac{\alpha}{2} w_2(S_2) \), then we deduce that \( S_1 \) is a good approximation of the Pareto set. From the hypothesis, we have \( w_2(S_1) \geq \frac{\alpha}{2} w_2(S_2) \geq \alpha \cdot \frac{\min(\rho_1, \rho_2)}{2} \cdot opt_2 \). On the other hand, by the construction of \( S_1 \) we also have \( w_1(S_1) \geq \rho_1 \cdot opt_1 \geq \alpha \cdot \frac{\min(\rho_1, \rho_2)}{2} \cdot opt_1 \).

- If \( w_1(S_2) \leq \frac{\alpha}{2} w_1(S_1) \) and \( w_2(S_1) \leq \frac{\alpha}{2} w_2(S_2) \), then it is \( S_3 \) which is a good approximation of the Pareto set. Indeed, from inequality (2), we deduce \( w_1(S_3) \geq \frac{\alpha}{2} w_1(S_1) \geq \alpha \cdot \frac{\min(\rho_1, \rho_2)}{2} \cdot opt_1 \) and on the other hand, from inequality (3), we also get \( w_2(S_3) \geq \frac{\alpha}{2} w_2(S_2) \geq \alpha \cdot \frac{\min(\rho_1, \rho_2)}{2} \cdot opt_2 \).

In any of these three cases, we obtain a \( \alpha \cdot \frac{\min(\rho_1, \rho_2)}{2} \)-approximation with one solution.

Clearly, if \( S_1, S_2, S_3 \) are computable in polynomial time, then Biobjective II is approximable in polynomial time. \( \square \)

Note that we can extend Theorem 3 to the case where \( \rho_i \) are not constant.

The interest of Property 3 is to find a simple method in order to construct a polynomial time constant approximation for Biobjective II. This method does not allow us to obtain the best polynomial time constant approximation for Biobjective II with one solution, but only to prove the fact that the problem is polynomial time constant approximable with one solution.

In Proposition 1 we prove that if a problem II is (resp. polynomial time) constant approximable and if Biobjective II satisfies (resp. polynomially) Property 1, then Biobjective II is (resp. polynomial time) constant approximable with one solution, and thus Biobjective II satisfies (resp. polynomially) Property 3 by Theorem 3. Thus all problems studied in Section 3 satisfy Property 3.

There exist problems which are polynomial time constant approximable and thus satisfy Property 3 and do not satisfy Property 1. One example is Biobjective Max TSP, which is polynomial time \( \frac{1}{27} \)-approximable with one solution [26] and does not satisfy Property 1.

**Proposition 2** Biobjective Max TSP does not satisfy Property 1.

**Proof:** Consider the complete graph \( K_5 \) where a fixed \( K_4 \) is decomposable into 2 edge-disjoint Hamiltonian paths \( P_1 \) and \( P_2 \). For every edge \( e \in E(K_5) \), set \( w_1(e) = 1 \) and \( w_2(e) = 0 \) if
\( e \in P_1, w_1(e) = 0 \) and \( w_2(e) = 1 \) if \( e \in P_2 \) and \( w_1(e) = 0 \) and \( w_2(e) = 0 \) if \( e \notin P_1 \cup P_2 \). We can check that there are four non-dominated tours \( T_i, i = 1, \ldots, 4 \) with \( w_1(T_1) = 3, w_2(T_1) = 0, w_1(T_2) = 0, w_2(T_2) = 3, w_1(T_3) = 2, w_2(T_3) = 1 \) and \( w_1(T_4) = 1, w_2(T_4) = 2 \). Consider \( S_i = T_i, i = 1, 2 \) and \( \alpha = 1/2 \). Clearly \( w_2(S_1) < \alpha w_2(S_2) \) and \( w_1(S_2) < \alpha w_1(S_1) \). Moreover there is no solution \( S_3 \) such that \( w_1(S_3) > (1 - \alpha) w_1(S_1) \) and \( w_2(S_3) > (1 - \alpha) w_2(S_2) \). \( \square \)

We consider in the following a problem that satisfies Property 3 and for which we are not able to prove that it satisfies Property 1.

### 4.1 Max Bisection

Given a graph \( G = (V, E) \) with non negative weights on the edges, the Max Bisection problem consists of finding a bipartition of the vertex set \( V \) into two sets of equal size such that the total weight of the cut is maximum. We establish in this part a polynomial time \( \frac{6}{4} \)-approximation algorithm for Biobjective Max Bisection where \( \rho \) is any polynomial time approximation ratio given for Max Bisection. Max Bisection is NP-hard [23] and the best approximation ratio known for Max Bisection is \( \rho = 0.701 \) [17].

**Lemma 3** Biobjective Max Bisection on graphs with \( 4n \) vertices satisfies Property 3 polynomially with \( \alpha = 1 \) and \( \rho_1 = \rho \) and \( \rho_2 = \frac{\rho}{2} \), where \( \rho \) is any polynomial time approximation ratio given for Max Bisection.

**Proof:** Formally, let \( S_1 = (V_1, V_2) \) be a bisection of \( I = (G, w_1) \) given by a polynomial time \( \rho \)-approximation algorithm for Max Bisection. Let \( G_i \) be the subgraph of \( G \) induced by \( V_i, i = 1, 2 \) and let \( (A, B) \) (resp., \( (C, D) \)) be a bisection of \( I_1 = (G_1, w_2) \) (resp., \( I_2 = (G_2, w_2) \)) given by a polynomial time \( \rho \)-approximation algorithm. We produce two other bisections \( S_2 \) and \( S_3 \) of \( G \) described by \( (A \cup C, B \cup D) \) and \( (A \cup D, B \cup C) \). W.l.o.g., assume \( w_2(S_2) \geq w_2(S_3) \). We show in the following that \( S_1, S_2, S_3 \) satisfy inequalities of Property 3 with \( \alpha = 1 \).

Let \( S^* = (V^*_1, V^*_2) \) be an optimal bisection on \( (G, w_2) \) using edge set \( E^* \). Thus, \( w_2(S^*) = w_2(E^*) = opt_2 \). Let \( V'_1 = V^*_1 \cap (A \cup B) \) (resp., \( V''_1 = V^*_1 \cap (C \cup D) \)) and \( V'_2 = V^*_2 \cap (A \cup B) \) (resp., \( V''_2 = V^*_2 \cap (C \cup D) \)). Let \( E^*_1 \) (resp., \( E^*_2 \)) be the edge set given the cut \( S^*_1 = (V'_1, V'_2) \) (resp., \( S^*_2 = (V''_1, V''_2) \)). Hence, for \( i = 1, 2 \), \( E^*_i \) is the restriction of \( S^* \) on \( G_i = (V_i, E_i) \).

Finally, let \( E^*_3 \) be the remaining edges, \( E^*_3 = E^* \setminus (E^*_1 \cup E^*_2) \). Note that \( E^*_3 \) belongs to the crossing edges between \( G_1 \) and \( G_2 \), i.e., are in the cut \( S_1 \). Thus, we get:

\[
\text{opt}_2 = w_2(E^*_1) + w_2(E^*_2) + w_2(E^*_3)
\]

(4)

Since \( F_1 = (A, B) \) (resp., \( F_2 = (C, D) \)) is the solution given by a polynomial time \( \rho \)-approximation algorithm on \( I_1 = (G_1, w_2) \) (resp., \( I_2 = (G_2, w_2) \)), we obtain for \( i = 1, 2 \):

\[
w_2(F_i) \geq \rho \cdot \text{opt}_2(I_i) \geq \frac{\rho}{2} w_2(E^*_i)
\]

(5)

Actually, consider the cut \( S^*_1 = (V'_1, V'_2) \) and assume \( |V'_2| \geq |V'_1| \). Let \( U = \arg\min \{ w_2(U, V'_2) : U \subseteq V'_1 \text{ and } |U| = n - |V'_2| \} \). Since \( |V'_1| + |V'_2| = 2n \), we get \( 2|U| \leq |V'_1| \). Thus, we deduce \( 2w_2(U, V'_2) \leq w_2(V'_1, V'_2) = w_2(E^*_1) \). Now, observe that \( S_1 = (V'_1 \setminus U, V'_2 \cup U) \) is a bisection of \( G_1 \). Hence \( \text{opt}_2(I_1) \geq w_2(S_1) \geq w_2(E^*_1) - w_2(U, V'_2) \geq \frac{\rho}{2} w_2(E^*_1) \). Obviously, the same arguments for the cut \( S^*_2 = (V''_1, V''_2) \) on \( G_2 \) lead to a similar conclusion.

On the other hand, since \( w_2(S_2) \geq w_2(S_3) \), we have:
\[ 2(w_2(A, D) + w_2(B, C)) \geq w_2(S_1) \geq w_2(E_3^*) \]  

Finally, using \( w_2(S_2) = w_2(F_1) + w_2(F_2) + w_2(A, D) + w_2(B, C) \), equality (4), (5), (6) and \( \rho \leq 1 \), we obtain that \( S_2 \) is a \( \rho/2 \)-approximation.

Due to the construction of \( S_1, S_2 \) and \( S_3 \), we get \( w_1(S_1) + w_1(S_2) + w_1(S_3) = 2(w_1(S_1) + w_1(A, B) + w_1(C, D)) \) and \( w_2(S_1) + w_2(S_2) + w_2(S_3) = 2(w_2(S_2) + w_2(A, C) + w_2(B, D)) \). Thus, we deduce:

\[ w_1(S_2) + w_1(S_3) \geq w_1(S_1) \]  

and

\[ w_2(S_1) + w_2(S_3) \geq w_2(S_2) \]

\[ \square \]

**Corollary 6** *Biobjective Max Bisection on graphs with 4n vertices admits a polynomial time 0.174-approximate Pareto set with one solution.*

**Proof:** The results follows from Theorem 3 and Lemma 3 and using the polynomial time 0.701-approximation algorithm for Max Bisection [17]. \( \square \)

## 5 Approx mat on w th one solut on for some mult object ve se-lect on problems

In this section we establish a sufficient condition for the construction, not necessarily in polynomial time, of a solution with performance guarantee for \( k \)-objective selection problems. Moreover, if all steps of the construction are feasible in polynomial time, then the \( k \)-objective version is polynomial time approximable with one solution. The general result is applied to the \( k \)-objective versions of three problems: Max Coverage (and its special cases Max q Vertex Cover and Max q Selection), Heaviest Subgraph and Max Coloring of interval graphs.

Consider a general single-objective maximization problem II defined as follows: a set \( \mathcal{U} \) of elements, a non negative weight \( w(e) \) for every element \( e \in \mathcal{U} \), a family \( \mathcal{F} \) of subsets of \( \mathcal{U} \) (either \( \mathcal{F} \) is given explicitly or there is a function that can decide if a subset \( S \in \mathcal{F} \)), a positive integer \( q \), a covering function \( c \) such that \( c(S) \subseteq \mathcal{U} \) for every subset \( S \subseteq \mathcal{F} \). It is assumed that \( c(\mathcal{F}) = \mathcal{U} \) and \( S \subseteq S' \Rightarrow c(S) \subseteq c(S') \). A feasible solution is a set \( S \) of \( q \) subsets of \( \mathcal{F} \). Its weight, to be maximized, is denoted by \( w(S) \) and defined as \( \sum_{e \in c(S)} w(e) \).

We are interested in problems II which satisfy the following property.

**Property 4** For any integer \( t \), \( 1 \leq t \leq q \), there exists a function \( \rho(t, q) \in (0, 1] \) such that for any feasible solution \( S \), one can always select \( t \) subsets among the \( q \) subsets of \( S \) such that the weight of these \( t \) subsets is at least \( \rho(t, q)w(S) \).

We say that II satisfies Property 4 polynomially if the \( t \) subsets can be found in polynomial time. In the \( k \)-objective version of II, every element \( e \) has \( k \) non negative weights \( w_1(e), \ldots, w_k(e) \).
**Theorem 4** If $\Pi$ satisfies Property 4, then $k$-objective $\Pi$ admits a $\rho([q/k], q)$-approximate Pareto set with one solution. Moreover, if $\Pi$ satisfies Property 4 polynomially and if $\Pi$ has a polynomial time $r$-approximation algorithm then $k$-objective $\Pi$ has a $r\rho([q/k], q)$-approximate Pareto set with one solution computable in polynomial time.

**Proof:** Using $S \subseteq S' \Rightarrow c(S) \subseteq c(S')$ and the fact that every element $e$ has a non-negative weight $w_i(e)$ for every $i$, $1 \leq i \leq k$, we deduce that $S \subseteq S' \Rightarrow w_i(S) \leq w_i(S')$. Hence when a partial solution containing less than $q$ subsets is completed with additional subsets, its weight cannot decrease.

Let $S_i$ (resp. $\hat{S}_i$) be an optimal (resp. $r$-approximate) solution with respect to $w_i$. Using Property 4 one can choose $\lfloor \frac{k}{q} \rfloor$ subsets in each $S_i$ (resp. $\hat{S}_i$) to build a new solution $S_0$ (resp. $\hat{S}_0$) containing at most $q$ subsets. We complete, if necessary, $S_0$ (resp. $\hat{S}_0$) in order to obtain a solution with exactly $q$ subsets. Thus, $w_i(S_0) \geq \rho([q/k], q)w_i(S_i)$ (resp. $w_i(\hat{S}_0) \geq \rho([q/k], q)w_i(\hat{S}_i)$) for every $i$, $1 \leq i \leq k$.

Clearly, if $S_i$ is computable in polynomial time and $\Pi$ satisfies Property 4 polynomially, then $\hat{S}_0$ is a polynomial time approximation for $k$-objective $\Pi$.

Several examples of $\Pi$ which satisfy Property 4 are given in Sections 5.1, 5.2 and 5.3.

### 5.1 Max Coverage and special cases

The input of **Max Coverage** is a set $U$ of elements, a non-negative weight $w(e)$ for every element $e \in U$, a family $F$ of subsets of $U$ and a positive integer $q$. The goal is to select $q$ elements of $F$ so that the total weight of the elements of $F$ covered by the union of these $q$ elements of $F$ is maximum. **Max Coverage** is NP-hard and $(1 - 1/e)$-approximable in polynomial time [1]. Clearly, **Max Coverage** is a special case of $\Pi$ and it is not difficult to prove that **Max Coverage** satisfies Property 4 polynomially for $\rho(t, q) = \frac{1}{q}$.

**Max $q$ Vertex Cover** is a special case of **max coverage**. It consists of finding $q$ vertices from an undirected and edge-weighted graph $G = (V, E)$, where $q \leq |V|$, such that the total edge weight covered by the $q$ vertices is maximized. **Max $q$ Vertex Cover** is NP-hard and $\frac{q}{4}$-approximable [18].

**Max $q$ Selection** is another special case of **Max Coverage**. It consists of finding $q$ items from a set of $n$ weighted items of maximum weight. **Max $q$ Selection** is trivial but $k$-objective **Max $q$ Selection** is NP-hard even for $k = 2$ [7].

Using Theorem 4, we get that $k$-objective **Max Coverage**, **Max $q$ Vertex Cover** and **Max $q$ Selection** admit a $\frac{|q/k|}{q}$-approximate Pareto set with one solution. In addition, a $\rho q$-approximate Pareto set with one solution can be computed in polynomial time where $\rho = (1 - 1/e)$, $\rho = 3/4$ and $\rho = 1$ for **Max Coverage**, **Max $q$ Vertex Cover** and **Max $q$ Selection** respectively.

### 5.2 Heaviest Subgraph

The input of **Heaviest Subgraph** is a complete graph $G = (V, E)$ with a non-negative weight $w(e)$ for each edge $e \in E$ and a positive integer $q \leq |V|$. The goal is to select $q$ nodes of $V$ such that the total weight of the subgraph induced by these $q$ nodes is maximum. The weight of a subgraph induced by $V' \subset V$ is denoted by $w(V')$ and defined as $w(V') = \sum_{\{i,j\} \in E(V')} w((i,j))$. **Heaviest Subgraph** is an NP-hard problem and it remains NP-hard even when the weights satisfy the triangle inequality [29]. It is also known under
the name max edge-weighted clique problem and densest subgraph problem (when all the edge weights are equal to 1) [31, 15, 25, 21, 2] and its approximation was studied in [31, 15]. Heaviest Subgraph is also a special case of II. The problem satisfies Property 4 for \( \rho(t, p) = \frac{t(t-1)}{q(q-1)} \) [31]. Hence \( k \)-objective Heaviest Subgraph admits a \( \frac{|q/k|(|q/k|-1)}{q(q-1)} \)-approximate Pareto set with one solution.

5.3 Max Coloring of interval graphs

The input of the maximum coloring problem in interval graphs (Max Coloring for short) is a set of intervals \( \mathcal{U} \) from the real line, each interval \( u \in \mathcal{U} \) has a non-negative weight \( w(u) \) and \( q \) colors. The goal is to find a coloring of the intervals of maximum total weight such that two intersecting intervals must receive distinct colors. The problem is solvable in polynomial time [9]. Clearly, Max Coloring is a special case of II and it is not difficult to prove that it satisfies Property 4 polynomially for \( \rho(t, q) = \frac{t}{q} \). Then, \( k \)-objective Max Coloring admits a \( \frac{|q/k|}{q} \)-approximate Pareto set with one solution which can be computed in polynomial time.

6 Conclus on

In this paper, we have established some sufficient conditions that allow to conclude on the existence of constant approximations of the Pareto set with an explicitly given number of solutions for several biobjective maximization problems. The results we obtained establish a polynomial time approximation when we ask for a single solution in the approximation set except for Heaviest Subgraph. A possible future work would be to give a polynomial time approximation for any explicitly given number of solutions. A necessary and sufficient condition is given for the construction of (polynomial time) constant approximation with one solution for biobjective maximization problems. It would be interesting to generalize this result to maximization problems with more than two objectives. Another interesting future work would be to establish lower bounds for any explicitly given number of solutions for multiobjective maximization problems. We also established in this paper a sufficient condition that allows to conclude on the existence of approximations of the Pareto set with one solution for multiobjective selection problems. A possible future work would be to establish polynomial time approximation for any explicitly given number of solutions.

Our approaches deal with maximization problems and they do not seem to apply to minimization problems. A possible explanation is that, in the maximization framework, adding elements to a partial solution rarely deteriorates it. Minimization problems rarely satisfy this property. Establishing constant approximation of the Pareto set with a given number of solutions or show that this is not possible for minimization problems is an interesting open question.

References


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