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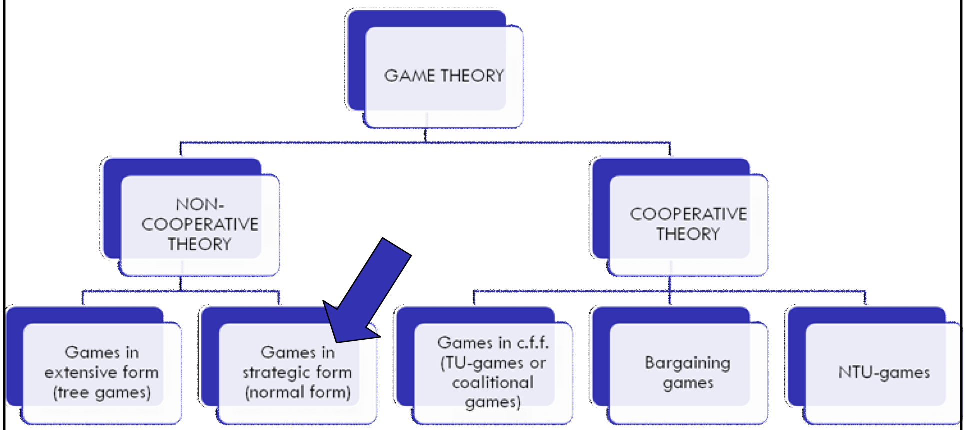
# Key concepts

- **Dominant strategies**
- **Nash equilibrium**
- **Mixed extension**
- **Strategic form vs. extensive form**
- **Information**




**No binding agreements**
**No side payments**
**Goal: Optimal behaviour in conflict situations**

**binding agreements**
**side payments are possible (sometimes)**
**Goal: Reasonable sharing**

# Looking for a solution

- What a player will/should do?
- άwillέ: the descriptive point of view. Aiming at predicting what players will do in the model and hence in the real game
- άshouldέ: the normative point of view. Rationality is based on a teleological description of the players. Players have an άendέ (not like apples, stones, molecules). So, they could do the άwrongέ thing. We could give them suggestions.
- Can we say something on the basis of our assumptions?

# Domination among strategies

- From decision theory we borrow the idea of domination among strategies:
- $x_1$ is (obviously) better than $x_2$ if:

  $h(x_1, y)$ ˉ $h(x_{2,}\ y)$ for every $y \in Y$
- We shall say that $x_1$ (strongly) dominates $x_2$.
- So, if $x_1$ dominates any other $x \in X$, then $x_1$ is the solution

# Prisoner's dilemma

- The game is:

| I \ II | L | R |
|---|---|---|
| T | (3,3) | (1,4) |
| B | (4,1) | (2,2) |

- Obviously B and R are dominant strategies (for I and II respectively). So, we have the solution (B,R). Nice and easy.
- But... **the outcome is inefficient!**
- Both players prefer the outcome deriving from (T, L). And so? The problem is that players are (assumed to be) rational and intelligent.

## Strategies to avoid

- A strategy which is (strongly) dominated by another one will not be played.
- So we can delete it. But then could appear new (strongly) dominated strategies for the other player. We can delete them.
- And so on...
- Maybe players are left with just one strategy each.
- Well, a new way to get a solution for the game.
- Technically: solution via iterated elimination of dominated strategies.

## Strategies to avoid: example

| I \ II | L | R |
|---|---|---|
| T | (2,1) | (1,0) |
| M | (1,1) | (2,0) |
| B | (0,0) | (0,1) |

| I \ II | L | R |
|---|---|---|
| T | (2,1) | (1,0) |
| M | (1,1) | (2,0) |
| ~~B~~ | ~~(0,0)~~ | ~~(0,1)~~ |

| I \ II | L | ~~R~~ |
|---|---|---|
| T | (2,1) | ~~(1,0)~~ |
| M | (1,1) | ~~(2,0)~~ |

| I \ II | L |
|---|---|
| T | (2,1) |
| ~~M~~ | ~~(1,1)~~ |

Solution: (T, L)

# Nash equilibrium

- Basic solution concept, for games in strategic form.
- (2 players only) Given $G = (X, Y, f, g)$, $(x^*,y^*) \in X \times Y$ is a Nash equilibrium for G if:

$f(x^*,y^*) \geq f(x,y^*)$ for all $x \in X$

$g(x^*,y^*) \geq g(x^*,y)$ for all $y \in Y$

Interpretation: $x^*$ is a best reply (max utility f) for player I when player II plays strategy $y^*$, and $y^*$ is a best reply (max utility g) for player II when player I plays strategy $x^*$.

- Existence: Nash's theorem: mixed strategies... See later
- Difficulties:

  not uniqueness

  inefficiency (Adam Smith was wrong?)

## Nash equilibrium: examples (3)

- Example (battle of the sexes, BoS): (T, L) and (B, R) are Nash Equilibria (N.E.). Not unique!

| I \ II | L | R |
|---|---|---|
| T | (2,1) | (0,0) |
| B | (0,0) | (1,2) |

## Nash equilibrium and dominance

Theorem: If a game has a unique couple that survives iterated elimination of dominated strategies, that this couple is a Nash equilibrium.

- In particular, a couple of dominating strategies is a Nash equilibrium.
- So, in the prisoner's dilemma, (B, R) is the (unique) Nash equilibrium.
- So, Nash equilibrium can give an outcome which is inefficient.

## Nash equilibrium: examples (4)

- Consider the following game (coordination game):

| I \ II | L | C | R |
|---|---|---|---|
| T | (0,0) | (1,1) | (0,0) |
| M | (0,0) | (0,0) | (1,1) |
| B | (1,1) | (0,0) | (0,0) |

- (B, L), (T, C) and (M,R) are N.E.

## Nash equilibrium: not unique

- The battle of the sexes and the coordination game (and many others) have more than one NE.
- BIG ISSUE.
- players may have different (opposite) preferences on the equilibrium outcomes (see BoS)
- it is not possible to speak of equilibrium strategies. In the BoS, T is an equilibrium strategy? Or B?

# One more problem

- Example: matching pennies (MP)

| I \ II | L | R |
|---|---|---|
| T | (-1,1) | (1,-1) |
| B | (1,-1) | (-1,1) |

- There is no equilibrium?
- But Nash is famous (also) because of his existence thm (1950).
- But MP is a zero-sum game. So, even vN (1928) guarantees that it has an equilibrium.
- Where do we find it? Usual math trick: extend ( Ïto Ж, sum to integral, solution to weak solution).

# Mixed strategies

- The basic idea is that the player does not choose a strategy, but a probability distribution on strategies.
- Example: I have an indivisible object and I must assign it in a fair way to one of my children. It is quite possible that the best solution is to decide to assign it randomly (with a uniform probability distribution).

# Mixed extension of a game

- Let's apply it to games in strategic form.
- Given a game G = (X, Y, f, g), assume X, Y are finite, and let

$$X = \{x_1, X, x_m\},\ Y = \{y_1, X, y_n\}.$$

- The mixed extension of G is ũ=(æ(X), æ(Y), f̃, g̃).
- Where:
- æ(X) (æ(Y))is the set of probability distributions on X (Y). An element of æ(X) is p = ($p_1$, X, $p_m$)∈ Ɥ$^m$, where $p_i$ is the probability to play strategy $x_i$ andX

$$\tilde{f}(p, q) = \Sigma_{i\in\{1,\dots m\}} \Sigma_{j\in\{1,\dots n\}}\ p_i\, q_j\, f(x_i,y_j)$$

$$\tilde{g}(p, q) = \Sigma_{i\in\{1,\dots m\}} \Sigma_{j\in\{1,\dots n\}}\ p_i\, q_j\, g(x_i,y_j)$$

- Of course, (p,q)∈æ(X)×æ(Y)
- Notice that ũ is itself a game in strategic form. So, no need to redefine concepts (in particular, N.E.).

# Interpretation?

- Of course, there is no mathematical problem in the definition of ũ.
- But: f̃ and g̃ can still be interpreted as payoffs for the players?
- The answer is YES if the original f and g are vNM utility functions. Otherwise, we cannot attach a meaning to the operations that brought us from G to ũ.

# Mixed extension and equilibria for BoS

- The BoS is:

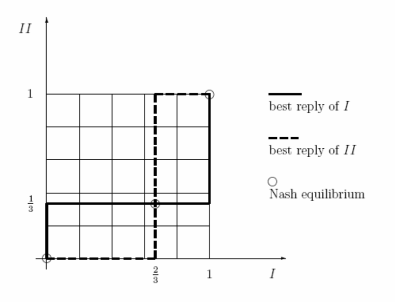
| | I \ II | L (q) | R (1-q) |
|---|---|---|---|
| p | T | (2,1) | (0,0) |
| 1-p | B | (0,0) | (1,2) |

- Instead of using $((p_1,p_2),(q_1,q_2))$ we use $((p,1-p),(q,1-q))$, with $p,q \in [0,1]$. So:
  $f_1((p,1-p),(q,1-q))=2pq+1(1-p)(1-q)=(3q-1)p+(1-q)$
- Given q, the best reply for player I to q is p* such that
  - $p^*=0$ if $0 \le q < 1/3$
  - $p^* \in [0,1]$ if $q=1/3$
  - $p^*=1$ if $1/3 < q \le 1$

# Mixed extension and equilibria for BoS (2)

- Given p, the best reply for player II to p* is
  $q^*=0$ if $0 \le p < 2/3$
  $q^* \in [0,1]$ if $p=2/3$
  $q^*=1$ if $2/3 < p \le 1$

- From the following picture we see there are 3 N.E.



## Nash existence theorem

- Theorem: Given a game (X, Y, f, g), where X and Y are finite sets. Then, the mixed extension (æ(X), æ(Y), f', g') possesses a Nash equilibrium.

  A proof of this theorem can be given using the fixed point thorem of Kakutani (1941):

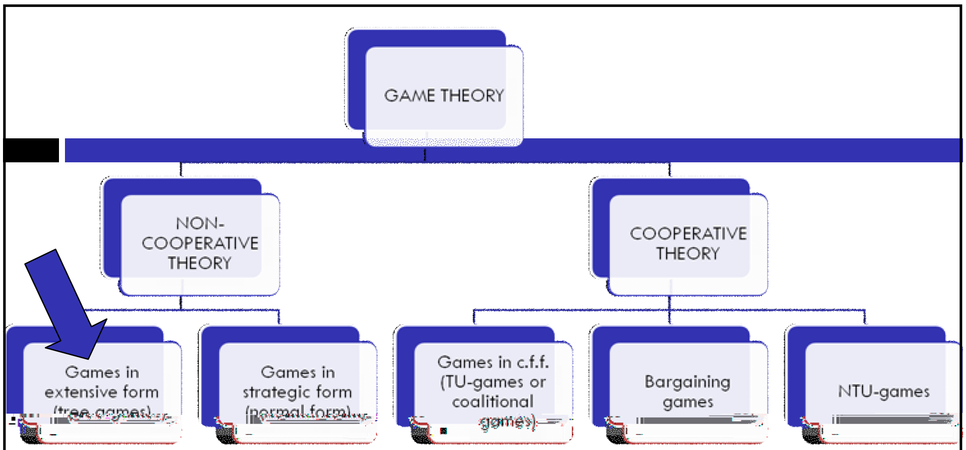
Theorem: Let X be a non-empty compact convex set of ℜ. Let f:X→X be a set-valued function for which

- for all x∈X the set f(x) is nonempty and convex;
- the graph of f is closed (i.e. for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ for all n, $x_n$Ÿ x, $y_n$Ÿ y, we have that y∈f(x)).

Then there exists a x*∈X such that x*∈f(x*).

## Nash existence theorem (2)

- Note that the mixed extension of a finite game satisfies the assumptions of the Kakutani's fixed point theorem: æ(X), æ(Y) are convex and compact
- Moreover we observed that a Nash eq. is a profile a* of actions in $A_i$ such that $a^*_i \in B_i(a^*_{-i})$ for each player i ∈N.
- Define the set valued function B:AŸ A (where A=$Ø_{i\in N}A_i$ is the cartesian product of $A_i$ over the set of players) such that B(a)=$Ø_{i\in N}B_i(a^*_{-i})$. Then a fixed point a*∈ B(a*) is a Nash eq.
- One can prove that by continuity of f' and g', the set valued function B has closed graph. Then by Kakutani's theorem it has a fixed point.




**No binding agreements**
**No side payments**
**Goal: Optimal behaviour in conflict situations**

**binding agreements**
**side payments are possible (sometimes)**
**Goal: Reasonable sharing**

# Matching pennies revisited

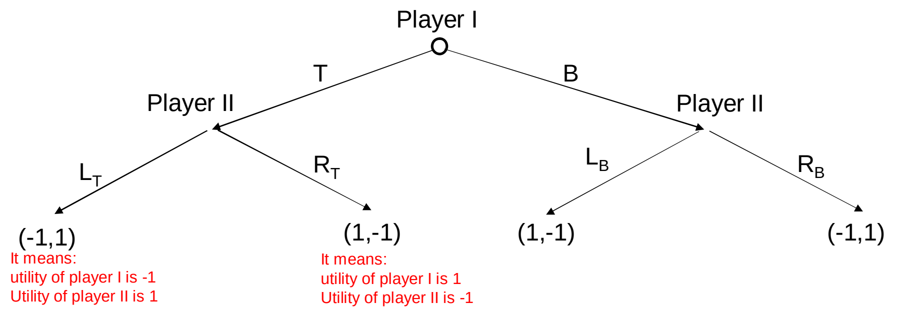
- Let us redefine the rules
  first you choose, L or R
  then I choose, T or B
- Is it ok? Is it the same game?
- It depends.
- Essential is not the chronological (physical) time, but the information that I have when I must decide. So, can I see (know) your choice before deciding?

## Trees

- We have introduced two important aspects:
  - the dynamic structure of the interaction
  - the role of info on past events (on the history)
- To represent them it is appropriate a tree structure

## Information sets (1)

- First chooses player I . Then, being informed of the choice made by player I, player II makes his choice.



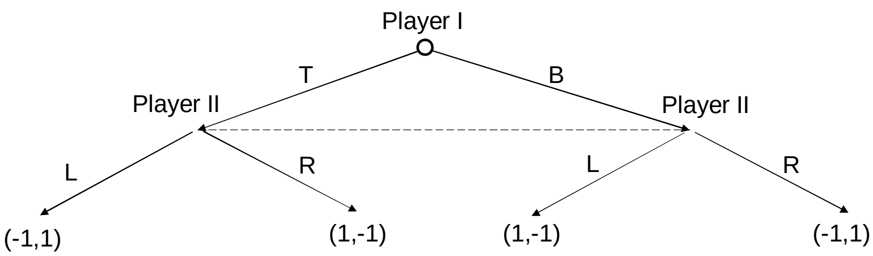
# Strategy

- Consider the tree game on the previous slide
- Player I must begin choosing between T or B .
- Player II must choose among: $L_T\,L_B$, $L_T\,R_B$, $R_T\,L_B$, $R_T\,R_B$
  - $L_T\,L_B$ means: chose $L_T$ if player I has chosen T at the first step and $L_B$ if player I has chosen B at the first step
  - $L_T\,R_B$ means: chose $L_T$ if player I has chosen T at the first step and $R_B$ if player I has chosen B at the first step
  - ....
- So, we have a game in strategic form. Payoffs? Follow the path!

| I \ II | $L_T\,L_B$ | $L_T\,R_B$ | $R_T\,L_B$ | $R_T\,R_B$ |
|---|---|---|---|---|
| T | (-1,1) | (-1,1) | (1,-1) | (1,-1) |
| B | (1,-1) | (-1,1) | (1,-1) | (-1,1) |

# Information sets (2)

- A άtrické to take note of what player II does not know about the past (the history).
- Two nodes are connected with a dashed line. The meaning is that a player will know that has to make a choice knowing that:
  - he knows he is in one of these nodes
  - but he does not know in which.
- These sets of nodes connected with dashed lines are called άinformation setsέ.



## Strategy

- Consider the tree game on the previous slide
- Player I must begin choosing between T or B .
- Player II must choose between L and R.
- we have the following game in strategic form:

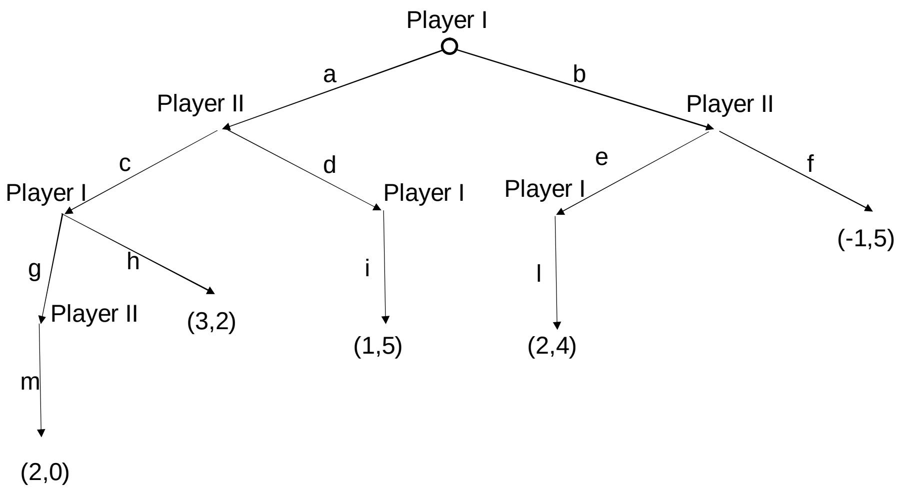
| I \ II | L | R |
|---|---|---|
| T | (-1,1) | (1,-1) |
| B | (1,-1) | (-1,1) |

## Nothing newX

- Every game in extensive form can be converted into a game in strategic form, using the (natural) idea of strategy we have seen.
- So, we can say that the strategic form is, somehow, fundamental, at least for non-cooperative games.
- We can use the Nash equilibrium also as a solution for games in extensive form.
- It seems that everything is so easyX

## Backward induction

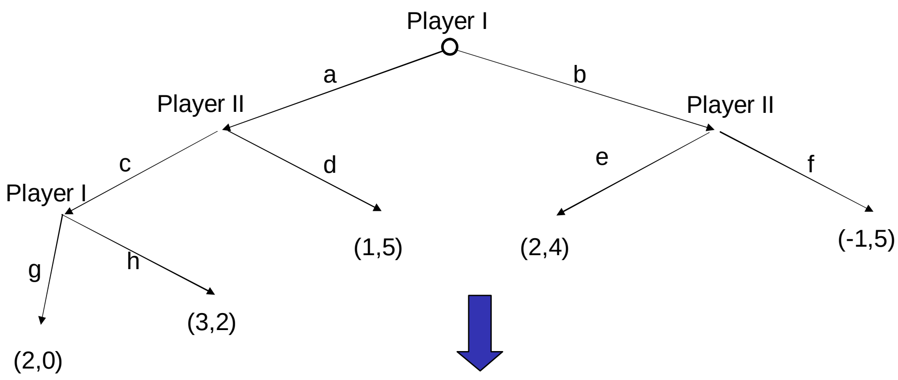
- Consider the very simple game depicted in the following slide
- Player I must begin choosing between a or b . But there is nothing that obliges him to think locally.
- He knows that he could be called to play again. So, before the game starts, he can decide his strategy.
- It means, choose among: agil, ahil, bgil, bhil .
- Similarly, II can choose among:cem, cfm, dem, dfm.
- So, we have a game in strategic form.



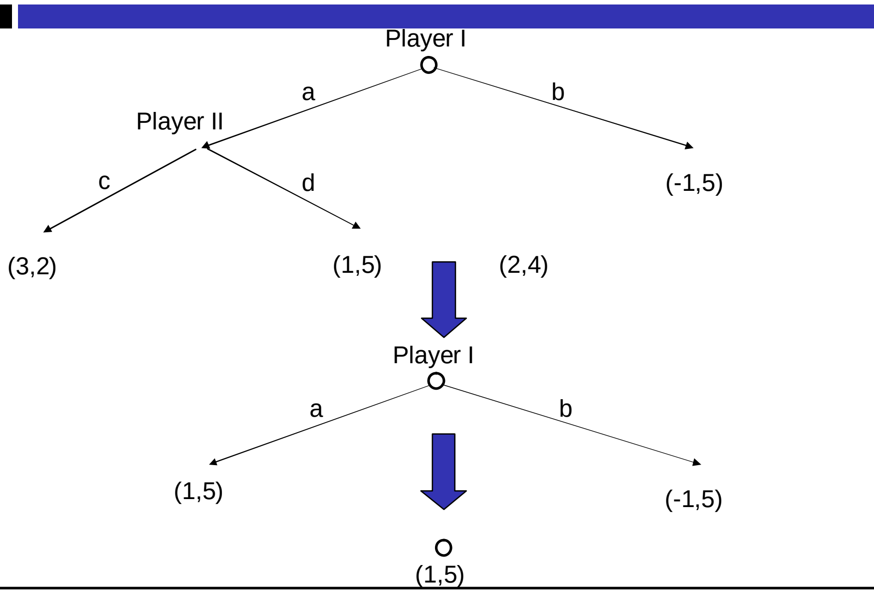
## Backward induction (2)

- Look at the άpenultimate" nodes. There the choice is easy, it is a single DM that has all of the power to enforce the outcome he prefers.
- having done this, look at the pre-penultimate nodes. Taking into account the choices that will be made by the player who follows, the choice for the player at the pre-penultimate node becomes obvious too.
- And so on, till we reach the root of the tree.
- The method we followed is called backward induction and works for games with perfect information (i.e.: information sets are all singletons).
- Well, we get a strategy profile that has good chances to be considered a solution! Actually, it can be proved that it is a Nash equilibrium.

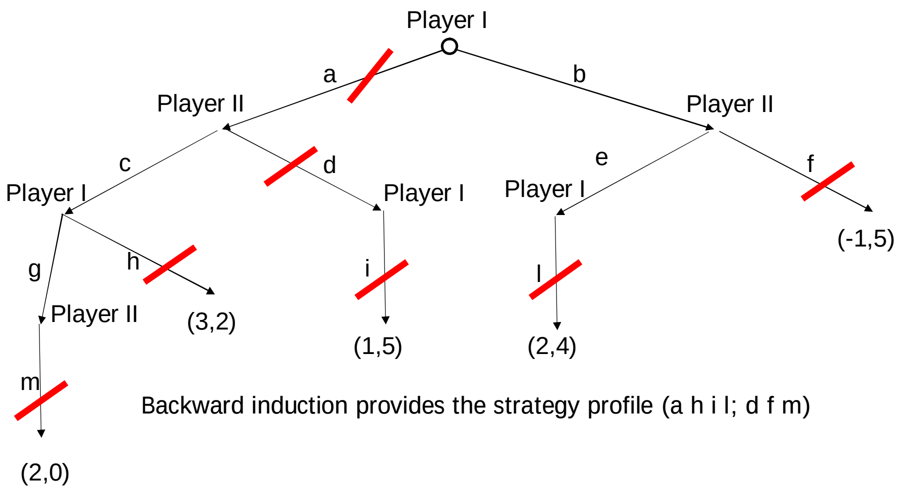
## Backward induction (3)



## Backward induction (4)




## Summing up



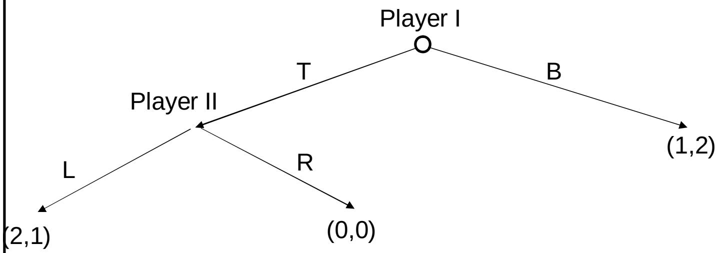

Backward induction provides the strategy profile (a h i l; d f m)

## Game in strategic form

| I \ II | c e m | c f m | d e m | d f m |
|---|---|---|---|---|
| a g i l | (2,0) | (2,0) | (1,5) | (1,5) |
| a h i l | (3,2) | (3,2) | (1,5) | (1,5) |
| b g i l | (2,4) | (-1,5) | (2,4) | (-1,5) |
| b h i l | (2,4) | (-1,5) | (2,4) | (-1,5) |

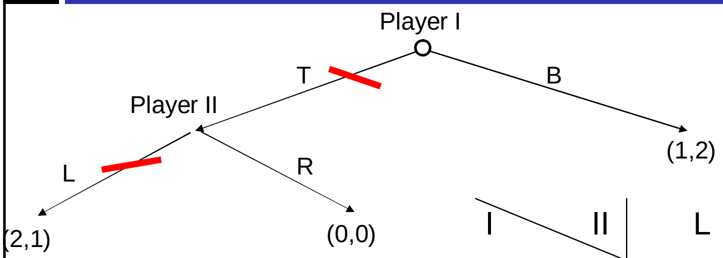
(a h i l; d f m) is a Nash equilibrium of the game (actually there is already another Nash equilibrium, but with an equivalent outcome é)

## Let's see a game, very small:




| I \ II | L | R |
|---|---|---|
| T | (2,1) | (0,0) |
| B | (1,2) | (1,2) |

# A small problem




| I \ II | L | R |
|---|---|---|
| T | (2,1) | (0,0) |
| B | (1,2) | (1,2) |

Backward induction gives (T; L). But the strategic form has two Nash equilibria: (T; L) and (B; R)! And so?

# Subgame Perfect Equilibrium (SPE)

- The key point is that the strategy profile determined by backward induction is more than a NE. It is a SPE. That is, it is not only a NE, but it is also a NE for all of the subgames.
- What are subgames? For games with perfect information they coincide with subtrees. More interesting the general case, but the idea is obvious: we want a subtree that can be seen sensibly as a game (subgame of the given game).
- Games with perfect information: there is coincidence between SPE and strategy profiles found by backward induction.
- SPE can be defined for any game in extensive form. It is the first example of refinement of NE.