Théorie de la décision et théorie des jeux
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An introduction to cooperative games

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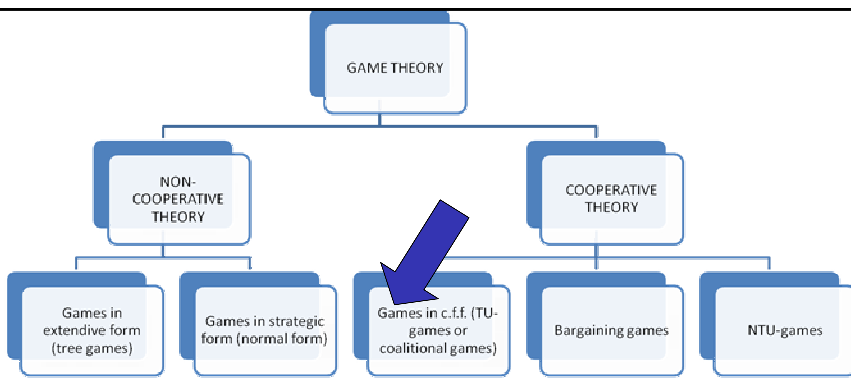


Non-cooperative theory:
- Dominant strategies
- Nash eq. (NE)
- Subgame perfect NE
- NE & refinements
- …

Games in c.f.f.:
- Core
- Shapley value
- Nucleolus
- τ -value
- PMAS
- ….

Bargaining games:
- Nash sol.
- Kalai-Smorodinsky
- ….

NTU-games:
- CORE
- NTU-value
- Compromise value
- …

No binding agreements
No side payments
Q: Optimal behaviour in conflict situations

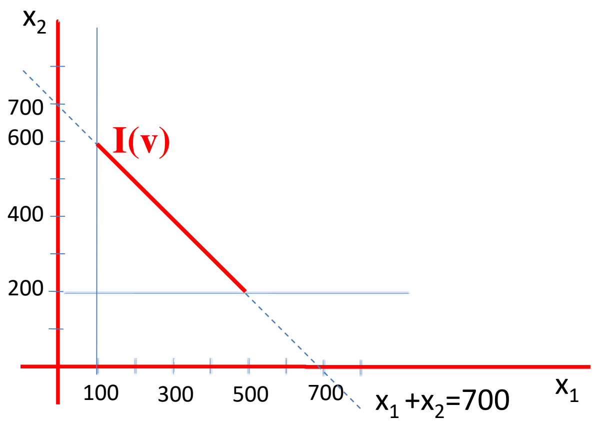
binding agreements
side payments are possible (sometimes)
Q: Reasonable (cost, reward)-sharing

Cooperative games: a simple example

Alone, player 1 (singer) and 2 (pianist) can earn 100€ 200€ respect.

Together (duo) 700€

How to divide the (extra) earnings?



Imputation set: $I(v)=\{x\in IR^2 | x_1\geq 100,\ x_2\geq 200,\ x_1+x_2=700\}$

Example: ice-cream game

(N,v) such that N={1,2,3}

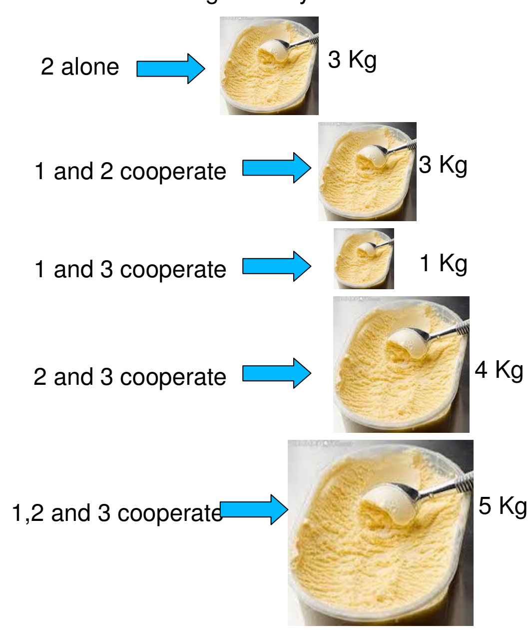
$v(1)=v(3)=0$
$v(2)=3$
$v(1,2)=3$
$v(1,3)=1$
$v(2,3)=4$
$v(1,2,3)=5$

Q.1: will do they join their money In order to buy more ice-cream?

Q.2: if yes, how do they share the Ice-cream?

Interpretation: kids want to buy ice-cream

1 and 3 not enough money → no ice-cream



COOPERATIVE GAME THEORY

Games in coalitional form

TU-game: (N,v)

N={1, 2, …, n}   set of players

$S \subset N$ coalition

2^N set of coalitions

v: $2^N \rightarrow$ IR   characteristic function

v(S) is the value (worth) of coalition S

DEF. (N,v) with $v(\emptyset)=0$ is a Transferable Utility (TU)-game with player set N.

NB: if N is fixed, $(N,v) \leftrightarrow v$

NB: if n=|N|, it is also called *n-person TU-game*…

Other names: *game in coalitional form*, *coalitional game*, *cooperative game with side payments*...

Q.1: which coalitions form?

DEF. (N,v) is a superadditive game iff

$v(S \cup T) \geq v(S)+v(T)$ for all S,T with $S \cap T=\emptyset$

Q.2: If the grand coalition N forms, how to divide v(N)?
(how to allocate the utility or the cost of the grand coalition?)

Many answers! (solution concepts)

One-point concepts:
- Shapley value (Shapley 1953)
- nucleolus (Schmeidler 1969)
- τ -value (Tijs, 1981)
- …

Subset concepts:
- Core (Gillies, 1954)
- stable sets (von Neumann, Morgenstern, ’44)
- kernel (Davis, Maschler)
- bargaining set (Aumann, Maschler)
- …..

Example: ice-cream game

(N,v) such that N={1,2,3}

v(1)=v(3)=0

v(2)=3

v(1,2)=3

v(1,3)=1

v(2,3)=4

v(1,2,3)=5

Claim: (N,v) is superadditive

We show that $v(S \cup T) \geq v(S)+v(T)$ for all $S,T \in 2^N \setminus \{\emptyset\}$ with $S \cap T = \emptyset$

$3=v(1,2) \geq v(1)+v(2)=0+3$

$1=v(1,3) \geq v(1)+v(3)=0+0$

$4=v(2,3) \geq v(2)+v(3)=3+0$

$5=v(1,2,3) \geq v(1)+v(2,3)=0+4$

$5=v(1,2,3) \geq v(2)+v(1,3)=3+1$

$5=v(1,2,3) \geq v(3)+v(1,2)=0+3$

The imputation set

DEF. Let (N,v) be a n-persons TU-game.

A vector $x=(x_1, x_2, \ldots, x_n) \in IR^N$ is called an imputation iff

(1) x is individual rational i.e.

$x_i \geq v(i)$ for all $i \in N$

(2) x is efficient

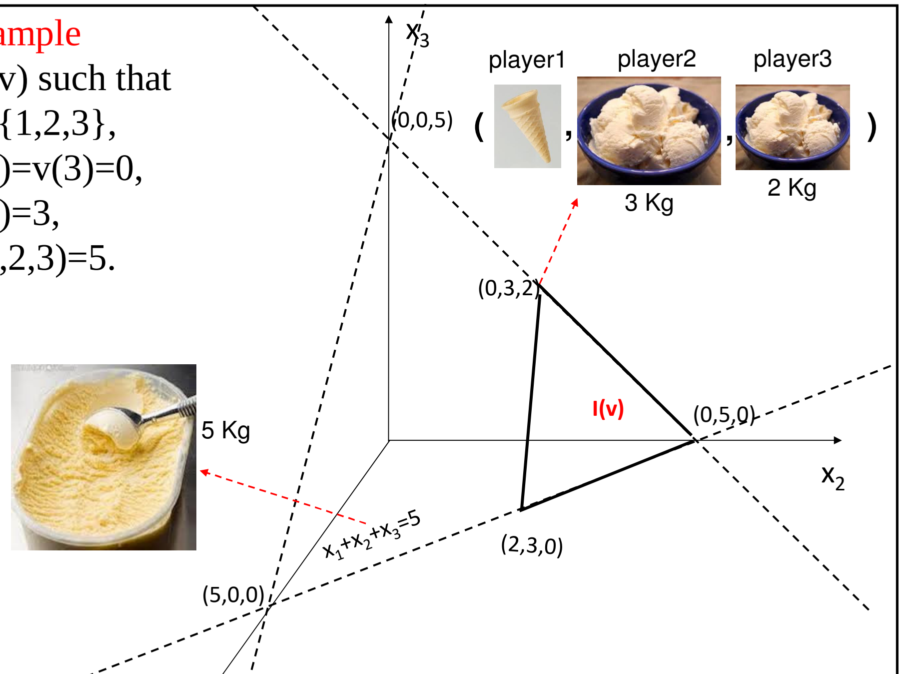
$\Sigma_{i \in N} x_i = v(N)$

[interpretation x_i: payoff to player i]

$I(v)=\{x \in IR^N \mid \Sigma_{i \in N} x_i = v(N),\ x_i \geq v(i) \text{ for all } i \in N\}$

Set of imputations

Example
(N,v) such that
N={1,2,3},
v(1)=v(3)=0,
v(2)=3,
v(1,2,3)=5.



$I(v)=\{x\in IR^3 \mid x_1,x_3\geq 0,\ x_2\geq 3,\ x_1+x_2+x_3=5\}$

The core of a game

DEF. Let (N,v) be a TU-game. The core C(v) of (N,v) is the set

$C(v)=\{x\in I(v) \mid \Sigma_{i\in S}\, x_i \geq v(S) \text{ for all } S\in 2^N\backslash\{\varnothing\}\}$

stability conditions

no coalition S has the incentive to split off if x is proposed

Note: $x \in C(v)$ iff

(1) $\Sigma_{i\in N}\, x_i = v(N)$ *efficiency*

(2) $\Sigma_{i\in S}\, x_i \geq v(S)$ for all $S\in 2N\backslash\{\varnothing\}$ *stability*

Bad news: C(v) can be empty

Good news: many interesting classes of games have a non-empty core.

Example
(N,v) such that
N={1,2,3},
v(1)=v(3)=0,
v(2)=3,
v(1,2)=3,
v(1,3)=1
v(2,3)=4
v(1,2,3)=5.

Core elements satisfy the following conditions:

$x_1, x_3 \geq 0$, $x_2 \geq 3$, $x_1+x_2+x_3=5$

$x_1+x_2 \geq 3$, $x_1+x_3 \geq 1$, $x_2+x_3 \geq 4$

We have that

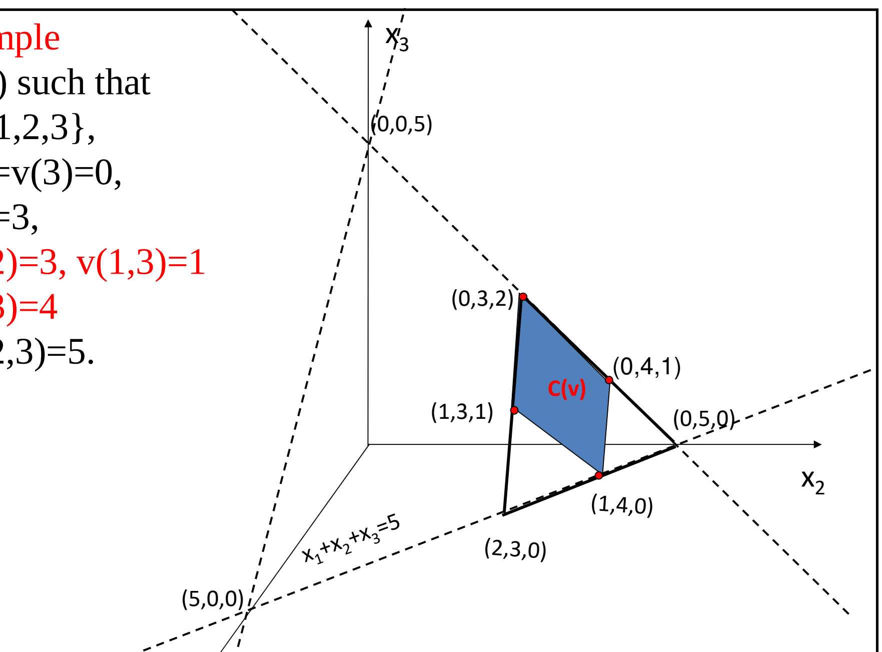
$5-x_3 \geq 3 \Leftrightarrow x_3 \leq 2$

$5-x_2 \geq 1 \Leftrightarrow x_2 \leq 4$

$5-x_1 \geq 4 \Leftrightarrow x_1 \leq 1$

$C(v)=\{x \in IR^3 \mid 1 \geq x_1 \geq 0, 2 \geq x_3 \geq 0, 4 \geq x_2 \geq 3, x_1+x_2+x_3=5\}$

Example
(N,v) such that
N={1,2,3},
v(1)=v(3)=0,
v(2)=3,
v(1,2)=3, v(1,3)=1
v(2,3)=4
v(1,2,3)=5.



$C(v)=\{x \in IR^3 \mid 1 \geq x_1 \geq 0, 2 \geq x_3 \geq 0, 4 \geq x_2 \geq 3, x_1+x_2+x_3=5\}$

Example (Game of pirates) Three pirates 1,2, and 3. On the other side of the river there is a treasure (10€). At least two pirates are needed to wade the river…

(N,v), $N=\{1,2,3\}$, $v(1)=v(2)=v(3)=0$,

$v(1,2)=v(1,3)=v(2,3)=v(1,2,3)=10$

Suppose $(x_1, x_2, x_3) \in C(v)$. Then

efficiency $x_1+x_2+x_3=10$

$$\text{stability} \begin{cases} x_1+x_2 \geq 10 \\ x_1+x_3 \geq 10 \\ x_2+x_3 \geq 10 \end{cases}$$

$20=2(x_1+x_2+x_3) \geq 30$ Impossible. So $C(v)=\varnothing$.

Note that (N,v) is superadditive.

Example

(Glove game with $L=\{1,2\}$, $R=\{3\}$)

$v(1,3)=v(2,3)=v(1,2,3)=1$, $v(S)=0$ otherwise

Suppose $(x_1, x_2, x_3) \in C(v)$. Then

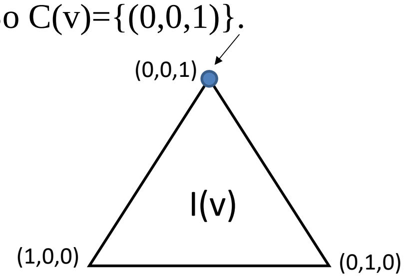
$x_1+x_2+x_3=1$ $\quad x_2=0$

$x_1+x_3 \geq 1$ $\quad x_1+x_3=1$

$x_2 \geq 0$

$x_2+x_3 \geq 1$ $\quad x_1=0$ and $x_3=1$

So $C(v)=\{(0,0,1)\}$.



How to share v(N)…

- The Core of a game can be used to exclude those allocations which are *not stable*.
- But the core of a game can be a bit "*extreme*" (see for instance the glove game)
- Sometimes the core is *empty* (pirates)
- And if it is not empty, there can be many allocations in the core (*which is the best*?)

An axiomatic approach (Shapley (1953)

- Similar to the approach of Nash in bargaining: *which properties an allocation method should satisfy in order to divide v(N) in a reasonable way*?
- Given a subset **C** of $\mathbf{G}^N$ (class of all TU-games with N as the set of players) a *(point map) solution* on **C** is a map $\Phi:\mathbf{C} \to IR^N$.
- For a solution Φ we shall be interested in various properties…

Symmetry

PROPERTY 1(SYM) Let $v \in G^N$ be a TU-game.

Let $i, j \in N$. If $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \in 2^{N \setminus \{i,j\}}$, then $\Phi_i(v) = \Phi_j(v)$.

EXAMPLE

We have a TU-game $(\{1,2,3\},v)$ s.t. $v(1) = v(2) = v(3) = 0$, $v(1, 2) = v(1, 3) = 4$, $v(2, 3) = 6$, $v(1, 2, 3) = 20$.

Players 2 and 3 are symmetric. In fact:

$v(\emptyset \cup \{2\}) = v(\emptyset \cup \{3\}) = 0$ and $v(\{1\} \cup \{2\}) = v(\{1\} \cup \{3\}) = 4$

If Φ satisfies SYM, then $\Phi_2(v) = \Phi_3(v)$

Efficiency

PROPERTY 2 (EFF) Let $v \in G^N$ be a TU-game.

$\sum_{i \in N} \Phi_i(v) = v(N)$, i.e., $\Phi(v)$ is a *pre-imputation*.

Null Player Property

DEF. Given a game $v \in \mathbf{G}^N$, a player $i \in N$ s.t. $v(S \cup i) = v(S)$ for all $S \in 2^N$ will be said to be a null player.

PROPERTY 3 (NPP) Let $v \in G^N$ be a TU-game. If $i \in N$ is a null player, then $\Phi_i(v) = 0$.

EXAMPLE We have a TU-game $(\{1,2,3\},v)$ such that $v(1) = 0$, $v(2) = v(3) = 2$, $v(1, 2) = v(1, 3) = 2$, $v(2, 3) = 6$, $v(1, 2, 3) = 6$. Player 1 is null. Then $\Phi_1(v) = 0$

EXAMPLE We have a TU-game ({1,2,3},v) such that v(1) =0, v(2) = v(3) = 2, v(1, 2) = v(1, 3) = 2, v(2, 3) = 6, v(1, 2, 3) = 6. On this particular example, if Φ satisfies NPP, SYM and EFF we have that

$\Phi_1(v) = 0$ by NPP

$\Phi_2(v) = \Phi_3(v)$ by SYM

$\Phi_1(v)+\Phi_2(v)+\Phi_3(v)=6$ by EFF

So $\Phi=(0,3,3)$

But our goal is to characterize Φ on $\mathbf{G}^N$**.** One more property is needed.

Additivity

PROPERTY 4 (ADD) Given $v,w \in \mathbf{G}^N$,

$$\Phi(v)+\Phi(w)=\Phi(v+w).$$

.EXAMPLE Two TU-games v and w on N={1,2,3}

v		w		v+w
v(1) =3	Φ	w(1) =1	Φ	v+w(1) =4 Φ
v(2) =4		w(2) =0		v+w(2) =4
v(3) = 1		w(3) = 1		v+w(3) = 2
v(1, 2) =8	+	w(1, 2) =2	=	v+w(1, 2) =10
v(1, 3) = 4		w(1, 3) = 2		v+w(1, 3) = 6
v(2, 3) = 6		w(2, 3) = 3		v+w(2, 3) = 9
v(1, 2, 3) = 10		w(1, 2, 3) = 4		v+w(1, 2, 3) = 14

Theorem 1 (Shapley 1953)

There is a unique map ϕ defined on $\mathbf{G}^N$ that satisfies EFF, SYM, NPP, ADD. Moreover, for any $i \in N$ we have that

$$\phi_i(v) = \frac{1}{n!} \sum_{\sigma \in \Pi} m_i^{\sigma}(v)$$

Here Π is the set of all permutations $\sigma : N \rightarrow N$ of N, while $m^{\sigma}{}_i(v)$ is the marginal contribution of player i according to the permutation σ, which is defined as:

$v(\{\sigma(1), \sigma(2), \ldots, \sigma(j)\}) - v(\{\sigma(1), \sigma(2), \ldots, \sigma(j-1)\})$,

where j is the unique element of N s.t. $i = \sigma(j)$.

Probabilistic interpretation: (the "room parable")

- Players gather one by one in a room to create the "grand coalition", and each one who enters gets his marginal contribution.
- Assuming that all the different orders in which they enter are equiprobable, the Shapley value gives to each player her/his expected payoff.

Example

(N,v) such that
N={1,2,3},
v(1)=v(3)=0,
v(2)=3,
v(1,2)=3,
v(1,3)=1,
v(2,3)=4,
v(1,2,3)=5.

Permutation	1	2	3
1,2,3	0	3	2
1,3,2	0	4	1
2,1,3	0	3	2
2,3,1	1	3	1
3,2,1	1	4	0
3,1,2	1	4	0
Sum	3	21	6
$\phi(v)$	3/6	21/6	6/6

Example

(N,v) such that

N={1,2,3},

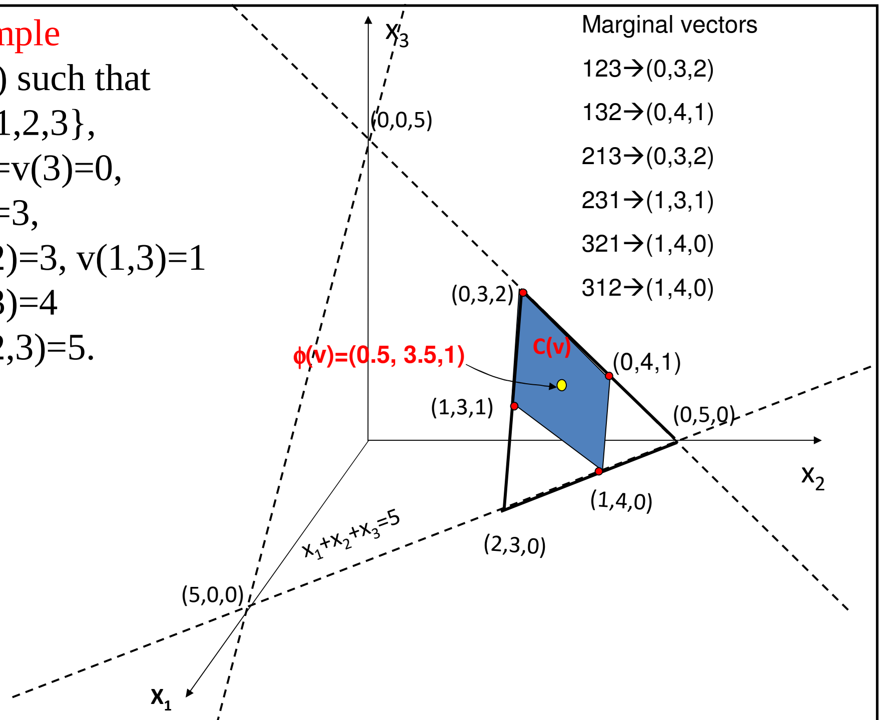
v(1)=v(3)=0,

v(2)=3,

v(1,2)=3, v(1,3)=1

v(2,3)=4

v(1,2,3)=5.



Unanimity games (1)

- DEF Let $T\in 2^N\backslash\{\varnothing\}$. The *unanimity game* on T is defined as the TU-game (N,u_T) such that

$$u_T(S)=\begin{cases} 1 \text{ is } T\subseteq S \\ 0 \text{ otherwise} \end{cases}$$

- Note that the class $\mathbf{G}^N$ of all n-person TU-games is a vector space (obvious what we mean for v+w and αv for $v,w\in \mathbf{G}^N$ and $\alpha\in$ IR).
- the dimension of the vector space $\mathbf{G}^N$ is 2^n-1
- $\{u_T|T\in 2^N\backslash\{\varnothing\}\}$ is an interesting basis for the vector space $\mathbf{G}^N$.

Unanimity games (2)

- Every coalitional game (N, v) can be written as a linear combination of unanimity games in a unique way, i.e., $v = \sum_{S \in 2^N} \lambda_S(v) u_S$.
- The coefficients $\lambda_S(v)$, for each $S \in 2^N$, are called unanimity coefficients of the game (N, v) and are given by the formula: $\lambda_S(v) = \sum_{T \in 2^S} (-1)^{s-t} v(T)$.

Sketch of the Proof of Theorem1

- Shapley value satisfies the four properties (easy).
- Properties EFF, SYM, NPP determine ϕ on the class of all games α v, with v a unanimity game and $\alpha \in IR$.
  - Let $S \in 2^N$. The Shapley value of the unanimity game (N, u_S) is given by

$$\phi_i(\alpha u_S) = \begin{cases} \alpha/|S| & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

- Since the class of unanimity games is a basis for the vector space, ADD allows to extend ϕ in a unique way to $\mathbf{G}^N$.

An alternative formulation

- Let $m^{\sigma'}_i(v)=v(\{\sigma'(1),\sigma'(2),\ldots,\sigma'(j)\})-v(\{\sigma'(1),\sigma'(2),\ldots,\sigma'(j-1)\})$, where j is the unique element of N s.t. $i=\sigma'(j)$.
- Let $S=\{\sigma'(1),\ \sigma'(2),\ldots,\ \sigma'(j)\}$.
- **Q**: How many other orderings $\sigma\in\Pi$ do we have in which $\{\sigma(1),\ \sigma(2),\ldots,\ \sigma(j)\}=S$ and $i=\sigma'(j)$?
- **A**: they are precisely $(|S|-1)!\times(|N|-|S|)!$
- Where $(|S|-1)!$ Is the number of orderings of $S\backslash\{i\}$ and $(|N|-|S|)!$ Is the number of orderings of $N\backslash S$
- We can rewrite the formula of the Shapley value as the following:

$$\phi_i(v)=\sum_{S\in 2^N:i\in S}\frac{(s-1)!(n-s)!}{n!}\big(v(S)-v(S\setminus\{i\})\big)$$

Convex games (1)

DEF. An n-persons TU-game (N,v) is convex iff

$$v(S)+v(T)\leq v(S\cup T)+v(S\cap T)\quad \text{for each } S,T\in 2^N.$$

This condition is also known as *supermodularity*. It can be rewritten as

$$v(T)-v(S\cap T)\leq v(S\cup T)-v(S)\quad \text{for each } S,T\in 2^N$$

For each $S,T\in 2^N$, let $C=(S\cup T)\backslash S$. Then we have:

$$v(C\cup(S\cap T))-v(S\cap T)\leq v(C\cup S)-v(S)$$

Interpretation: the marginal contribution of a coalition C to a disjoint coalition S does not increase if S becomes smaller

Convex games (2)

- It is easy to show that supermodularity is equivalent to

$$v(S \cup \{i\})-v(S) \leq v(T \cup \{i\})-v(T)$$

for all $i \in N$ and all $S,T \in 2^N$ such that $S \subseteq T \subseteq N \setminus \{i\}$

- interpretation: player's marginal contribution to a large coalition is not smaller than her/his marginal contribution to a smaller coalition (which is stronger than superadditivity)
  - Clearly all convex games are superadditive ($S \cap T = \emptyset \ldots$)
  - A superadditive game can be not convex (try to find one)
- An important property of convex games is that they are (*totally*) balanced, and it is "easy" to determine the core (coincides with the Weber set, i.e. the convex hull of all marginal vectors…)

Example

(N,v) such that

$N=\{1,2,3\}$,

$v(1)=v(3)=0$,

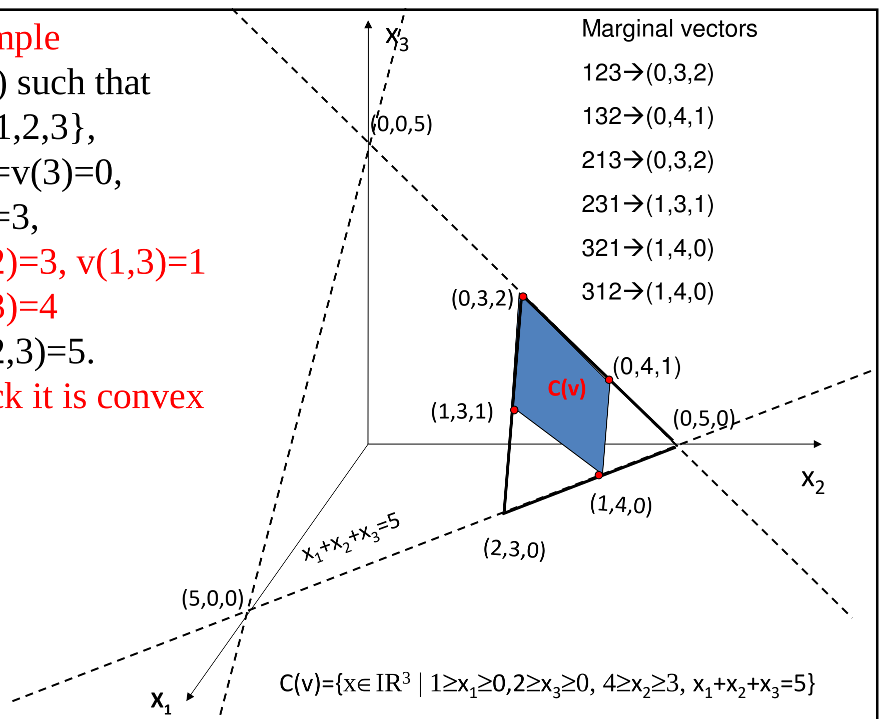
$v(2)=3$,

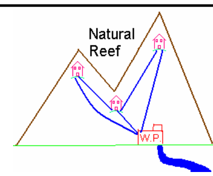
$v(1,2)=3$, $v(1,3)=1$

$v(2,3)=4$

$v(1,2,3)=5$.

Check it is convex

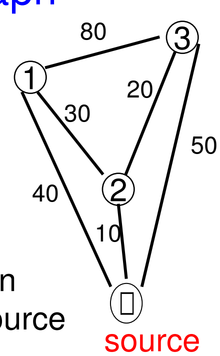




- A connection situation takes place in the presence of a group of agents N={1,2, …,n}, each of which needs to be connected directly or via other agents to a source.
- If connections among agents are costly, then each agent will evaluate the opportunity of cooperating with other agents in order to reduce costs.
- If a group of agents decides to cooperate, a configuration of links which minimizes the total cost of connection is provided by a minimum cost spanning tree (mcst).
- The problem of finding a mcst may be easily solved thanks to different algorithms proposed in literature (Boruvka (1926), Kruskal (1956), Prim (1957), Dijkstra (1959))

Minimum Cost Spanning Tree Situations

Consider a complete weighted graph



– whose vertices represent agents

– vertex 0 is the source

– edges represent connections between agents or between an agent and the source

– numbers close to edges are connection costs

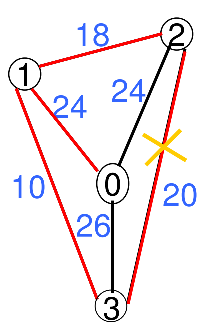
Minimum cost spanning tree (mcst) problem

Optimization problem:

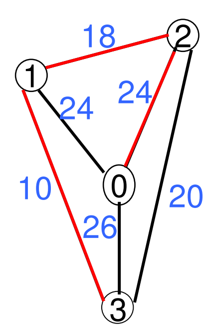
How to connect each node to the source 0 in such a way that the cost of construction of a spanning network (which connects every node directly or indirectly to the source 0) is minimum?

Example $N=\{1,2,3\}, E_{N'}=\{\{1,0\},\{2,0\},\{2,1\},\{3,0\},\{3,1\},\{3,2\}\}$ cost function shown on graphs

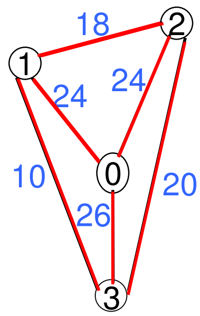
Kruskal algorithm



Prim algorithm



Example: mcst cost game ({1,2,3},c) defined on the following connection situation:



c(1)=24

c(2)=24

c(3)=26

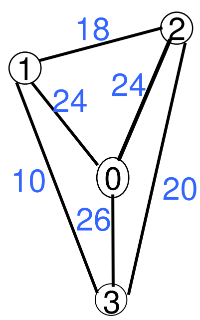
c(1,3)=34

c(1,2)=42

c(2,3)=44

c(1,2,3)=52

Example: The cost game ({1,2,3},c) is defined on the following connection situation:



c(1)=24

c(2)=24

c(3)=26

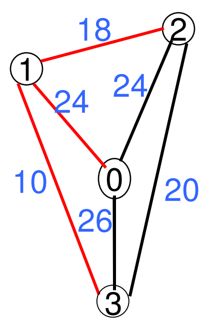
c(1,3)=34

c(1,2)=42

c(2,3)=44

c(1,2,3)=52

[illegible]



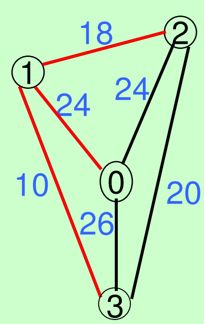
- The predecessor of 1 is 0: the Bird allocation gives to player 1 the cost of {0,1}.

[illegible] 2 is 1: [illegible] 2 [illegible] {1,2};

- [illegible] 1: [illegible] 3 [illegible] {1,3}.

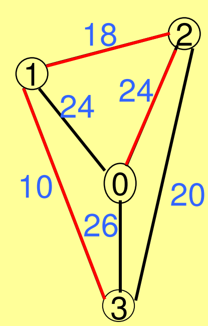
w(Γ)=52

Bird allocation w.r.t. to Γ [illegible] (x_1, x_2, x_3)=(24, 18 ,10) is in the core of ({1,2,3},c).



The Bird allocation w.r.t .this mcst is

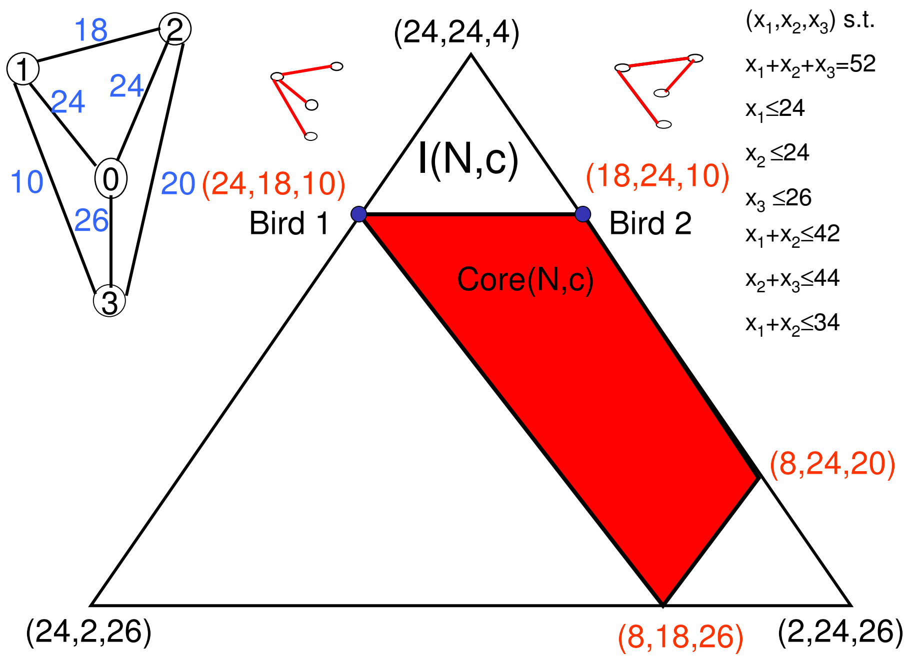
(x_1, x_2, x_3)=(24, 18 ,10)



[illegible]

(x_1, x_2, x_3)=(18, 24 ,10)

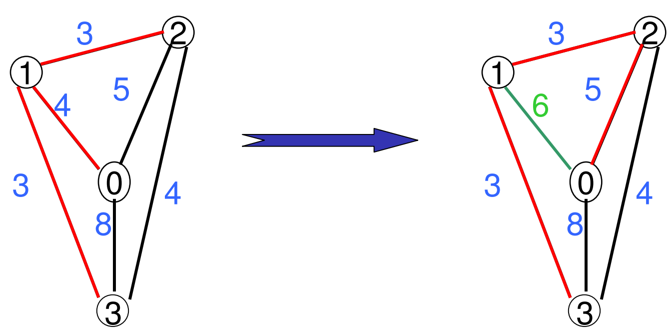
Both allocations belong to the core of the mcst game (and also their convex combination).


18
24
24
10
26
20
1
2
0
3
(24,24,4)
I(N,c)
(24,18,10)
Bird 1
(18,24,10)
Bird 2
Core(N,c)
(8,24,20)
(24,2,26)
(8,18,26)
(2,24,26)
(x_1,x_2,x_3) s.t.
$x_1+x_2+x_3=52$
$x_1\leq24$
$x_2 \leq24$
$x_3 \leq26$
$x_1+x_2\leq42$
$x_2+x_3\leq44$
$x_1+x_2\leq34$

What happens when the structure of the network changes?

- Imagine to use a certain rule to allocate costs.
  - The cost of edges may increase: if the cost of an edge increases, nobody should be better off, according to such a rule (*cost monotonicity*);
  - One or more players may leave the connection situation: nobody of the remaining players should be better off (*population monotonicity*).

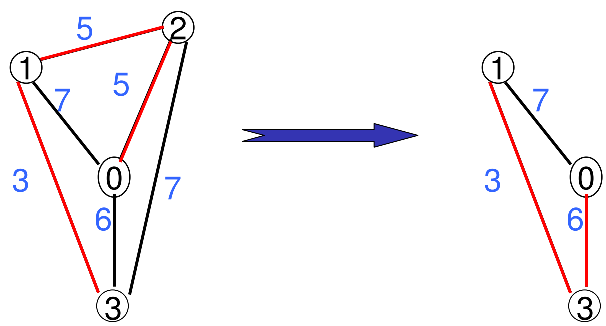
Cost monotonicity: Bird allocation behaviour



Bird allocation: (4, 3 ,3) □□□□ □□□□□□□□□□: (3, 5 ,3)

Bird rule does not satisfy cost monotonicity.

Population monotonicity: Bird allocation behaviour



Bird allocation: (5, 5 ,3)   Bird allocation: (3, * ,6)

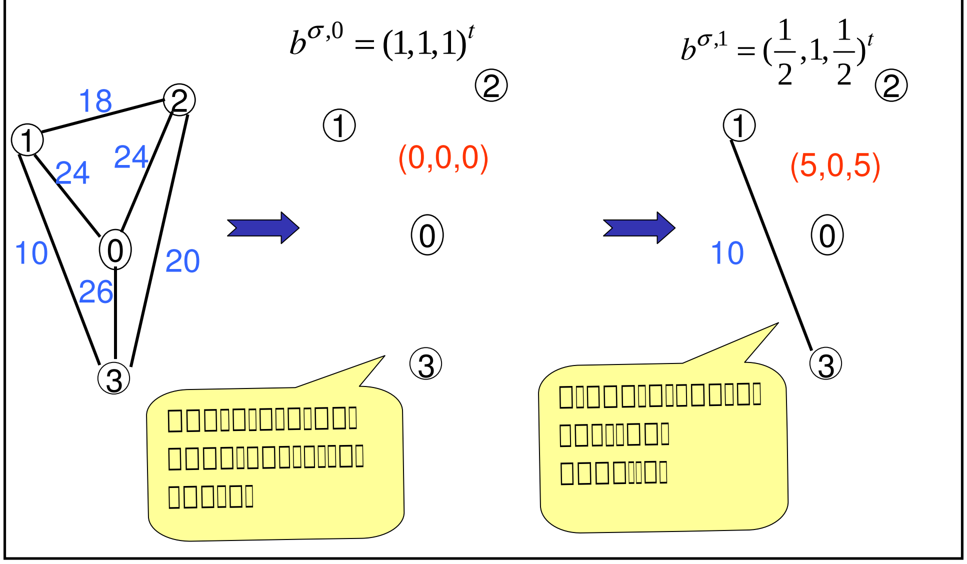
➡ Bird rule does not satisfy population monotonicity

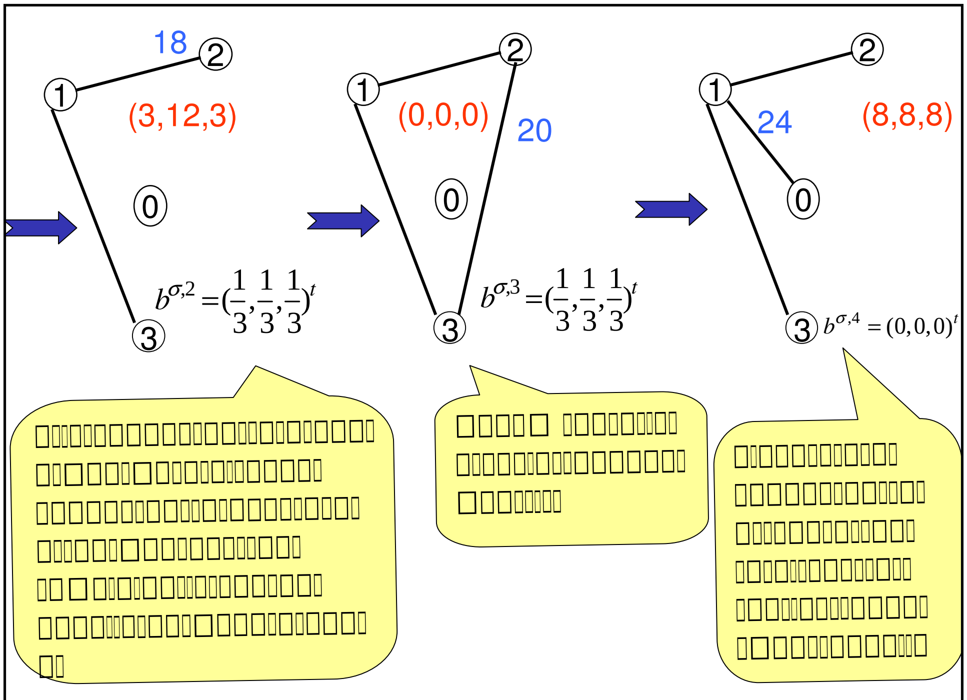
Construct & Charge rules

Are based on the following general cost allocation protocol:

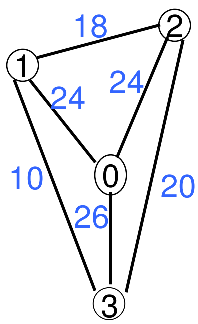
- As soon as a link is constructed in the Kruskal algorithm procedure:
  1) it must be totally charged among agents which are not yet connected with the source (*connection property*)
  2) Only agents that are on some path containing the new edge may be charged (*involvement property*)
- when the construction of a mcst is completed, each agent has been charged for a total amount of fractions equal to 1 (*total aggregation property*).

P-value: Feltkamp (1994), Branzei et al. (2004), Moretti (2008)





P-value



$$
\begin{array}{r}
(0,\ 0,\ 0)\ + \\
(5,\ 0,\ 5)\ + \\
(3,12,3)\ + \\
(0,\ 0,\ 0)\ + \\
(8,\ 8,\ 8)\ = \\
\hline
\end{array}
$$

P-value = (16,20,16)

Algorithm to calculate the P-value

IDEA: charge the cost of an edge constructed during the Kruskal algorithm only between agents involved, keeping into account the cardinality of the connected components at that step and at the previous step of the algorithm:

- At any step of the Kruskal algorithm where a component is connected to some agents, charge the cost of that edge among these agents in the following way:
  - Proportionally to the $\text{cardinality\_current\_step}^{-1}$ if an agent is connected to a component which is connected to the source,
  - Otherwise, charge it proportionally to the difference: $\text{cardinality\_previous\_step}^{-1} - \text{cardinality\_current\_step}^{-1}$

P-value

- Always provides a unique allocation (given a mcst situation).
- It is in the core of the corresponding mcst game.
- Satisfies cost monotonicity.
- Satisfies population monotonicity.
- on a subclass of connection problems it coincides with the Shapley value of mcst games
- …

Weighted Majority Games

Suppose that four parties receive these vote shares: Party A, 27%; Party B, 25%; Party C, 24%; Party 24%.

Seats are apportioned in a 100-seat parliament according some apportionment formula. In this case, the apportionment of seats is straight-forward:

Party A: 27 seats	Party C: 24 seats
Party B: 25 seats	Party D: 24 seats

Suppose a simple majority is required (at least 51 seats) to be winning

The Shapley-Shubik Index

Suppose coalition formation starts at the top of each ordering, moving downward to form coalitions of increasing size.

At some point a winning coalition formed.

The “grand coalition” {A,B,C,D} is certainly winning.

For each ordering, identify the *pivotal* player who, when added to the players already in the coalition, converts a losing coalition into a winning coalition.

Given the seat shares of parties A, B, C, and D before the election, the pivotal player in each ordering is identified by the arrow (<=).

A	A	A	A	A	A	B	B	B	B	B	B
B<=	B<=	C<=	D<=	C<=	D<=	A<=	A<=	C	D	C	D
C	D	B	B	D	C	C	D	A<=	A<=	D<=	C<=
D	C	D	C	B	B	D	C	D	C	A	A

C	C	C	C	C	C	D	D	D	D	D	D
D	D	A<=	B	A<=	B	C	C	A<=	B	A<=	B
A<=	B<=	D	D<=	B	A<=	A<=	B<=	C	C<=	B	A<=
B	A	B	A	D	D	B	A	B	A	C	C

The Shapley-Shubik Index (cont.)

Voter *i*’s *Shapley-Shubik power index value SS*(*i*) is simply:

$$SS(i) = \frac{\text{Number of orderings in which the voter } i \text{ is pivotal}}{\text{Total number of orderings}}$$

Clearly such power index values of all voters add up to 1.

Counting up, we see that A is pivotal in 12 orderings and each of B, C, and D is pivotal in 4 orderings. Thus:

Voter	*SS Power*
A	1/2 = .500
B	1/6 = .167
C	1/6 = .167
D	1/6 = .167

So according to the Shapley-Shubik index, Party A has 3 times the voting power of each other party.

Lloyd Shapley and Martin Shubik, *American Political Science Review*, 1955.

UN Security Council

- 15 member states:
  - 5 Permanent members: China, France, Russian Federation, United Kingdom, USA
  - 10 temporary seats (held for two-year terms) (http://www.un.org/)

UN Security Council decisions

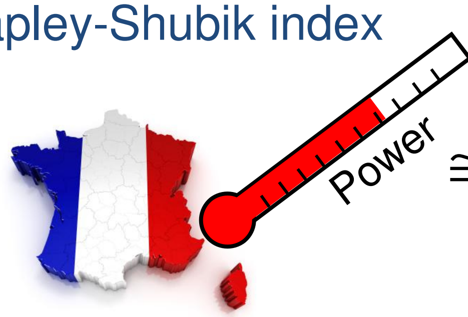
- Decision Rule: **substantive resolutions** need the positive vote of at least nine Nations but…

  …it is sufficient the negative vote of one among the permanent members to reject the decision.
- How much decision power each Nation inside the ONU council to force a substantive decision?
- Game Theory gives an answer using the Shapley-Shubik power index:

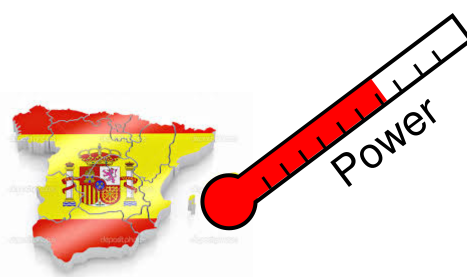
UN Security Council as a weighted majority game

- Let $N=P\cup T$, where $P=\{1,2,3,4,5\}$ is the set of Permanent members and $T=\{6,7,8,9,10,11,12,13,14,15\}$ is the set of temporary seats
- A simple game (N,v) s.t. $v(S)=1$ if $|S|\geq 9$ and $P\subset S$.
- (N,v) is a weighted majority game, where
  - $w_i=7$ for each $i\in P$
  - $w_i=1$ for each $i\in T$
  - $v(S)=1 \iff \sum_{i\in S} w_i \geqslant 39$

Shapley-Shubik index



$\cong$ 19.6%



$\cong$ 0.2%

temporary seat 2015-2016

United States presidential elections

- are indirect elections in which voters will select presidential electors who in turn will elect a new president and vice president through the electoral college.
- electors directly elect the President (and Vice President).
- Once chosen, the electors can vote for anyone, but – with rare exceptions– they vote for their designated candidates
- Each state is allocated a number of Electoral College electors equal to the number of its Senators and Representatives in the U.S. Congress (i.e. by population).
- Additionally, Washington, D.C. is given a number of electors equal to the number held by the smallest state.

United States presidential election (3)

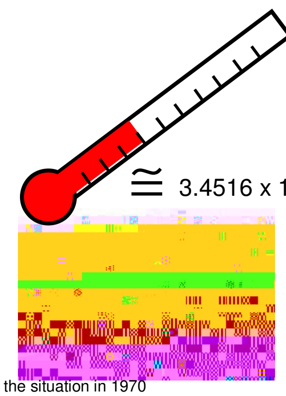
- A game (N,v), where |N|=51 (the number of states plus Washington D.C.), each one with a weight given by the number of electors
- $v(S)=1 \Leftrightarrow \sum_{i\in S} w_i > 270$
- In 1977, weights are between 3 (smallest states and Washington D.C.) and 45 (California)

State	**Electors**	**Shapley**	**Banzhaf**
California	45	0.08831	0.38694
Washington DC	3	0.005412	0.02402
Florida	17	0.03147	0.13736
Montaa	4	0.00723	0.03202

United States presidential elections

- are *indirect elections* in which *voters will select presidential electors* who in turn will *elect a new president and vice president* through the electoral college.
- In this indirect election the power of each voter in California and Montana is the following

$\cong$ 7.8476 x 10^{-9}



$\cong$ 3.4516 x 10^{-9}

Owen, G. (1995) Game Theory. 3rd edn. Academic Press. Calculations of indices for the situation in 1970

Power indices: a general formulation

Let (N,v) be a simple game (assume v is *monotone*: for each $S,T \in 2^N$. $S \subseteq T \Rightarrow v(S) \leq v(T)$)

Let $p_i(S)$, for each $S \in 2^N \setminus \{\emptyset\}$, $i \notin S$, be the probability of coalition $S \cup \{i\}$ to form (of course $\sum_{S \subseteq N: i \notin S} p_i(S)=1$)

A power index $\psi_i(v)$ is defined as the probability of player i to be pivotal in v according to p:

$$\psi_i^p(v)=\sum_{S \subseteq N: i \notin S} p_i(S)\ [v(S \cup \{i\})-v(S)]$$