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# An introduction to cooperative games

#### Stefano Moretti

UMR 7243 CNRS Laboratoire d'Analyse et Modélisation de Systèmes pour l'Aide à la décision (LAMSADE) Université Paris-Dauphine

email: <u>stefano.moretti@dauphine.fr</u> Office room: P422







## COOPERATIVE GAME THEORY

#### **Games in coalitional form**

TU-game: (N,v)N={1, 2, ..., n}set of playersS $\subset$ Ncoalition $2^N$ set of coalitionsv:  $2^N \rightarrow$ IRcharacteristic functionv(S) is the value (worth) of coalition S

DEF. (N,v) with  $v(\emptyset)=0$  is a Transferable Utility (TU)-game with player set N. NB: if N is fixed,  $(N,v)\leftrightarrow v$ NB: if n=|N|, it is also called *n*-person TU-game... Other names: game in coalitional form, coalitional game, cooperative game with side payments...

<b>Q</b> .1: which coalitions form?
<b><u>DEF.</u></b> $(N,v)$ is a <u>superadditive game</u> iff
$v(S \cup T) \ge v(S) + v(T)$ for all S,T with $S \cap T = \emptyset$
Q.2: If the grand coalition N forms, how to divide v(N)? (how to allocate the utility or the cost of the grand coalition?)
Many answers! (solution concepts)
One-point concepts: - Shapley value (Shapley 1953)
- nucleolus (Schmeidler 1969)
- τ -value (Tijs, 1981)
Subset concepts: - Core (Gillies, 1954)
- stable sets (von Neumann, Morgenstern, '44)
- kernel (Davis, Maschler)
- bargaining set (Aumann, Maschler)

Example: ice-cream game (N,v) such that  $N = \{1,2,3\}$ v(1)=v(3)=0v(2)=3v(1,2)=3v(1,3)=1v(2,3)=4v(1,2,3)=5Claim: (N,v) is superadditive We show that  $v(S \cup T) \ge v(S) + v(T)$  for all  $S, T \in 2^N \setminus \{\emptyset\}$  with  $S \cap T = \emptyset$  $3=v(1,2)\geq v(1)+v(2)=0+3$  $1=v(1,3)\geq v(1)+v(3)=0+0$  $4=v(2,3)\geq v(2)+v(3)=3+0$  $5=v(1,2,3) \ge v(1)+v(2,3)=0+4$  $5=v(1,2,3) \ge v(2)+v(1,3)=3+1$  $5=v(1,2,3) \ge v(3)+v(1,2)=0+3$ 

#### The imputation set

**DEF.** Let (N,v) be a n-persons TU-game. A vector  $x=(x_1, x_2, ..., x_n) \in IR^N$  is called an <u>imputation</u> iff

> (1) x is <u>individual rational</u> i.e.  $x_i \ge v(i)$  for all  $i \in N$

(2) x is <u>efficient</u>  $\Sigma_{i \in N} x_i = v(N)$ 

[interpretation x<sub>i</sub>: payoff to player i]

 $I(v) = \{x \in IR^{N} \mid \sum_{i \in N} x_{i} = v(N), x_{i} \ge v(i) \text{ for all } i \in N \}$ <u>Set of imputations</u>



The core of a gameDEF. Let (N,v) be a TU-game. The core C(v) of (N,v) is thesetC(v)={x \in I(v) |  $\Sigma_{i \in S} x_i \ge v(S)$  for all  $S \in 2^N \setminus \{\emptyset\}$ }stability conditionsno coalition S has the incentive to split offif x is proposedNote:  $x \in C(v)$  iff(1)  $\Sigma_{i \in N} x_i = v(N)$  efficiency(2)  $\Sigma_{i \in S} x_i \ge v(S)$  for all  $S \in 2N \setminus \{\emptyset\}$  stabilityBad news: C(v) can be emptyGood news: many interesting classes of games have a non-empty core.





Example (Game of pirates) Three pirates 1,2, and 3. On the other side of the river there is a treasure (10€). At least two pirates are needed to wade the river... (N,v), N={1,2,3}, v(1)=v(2)=v(3)=0, v(1,2)=v(1,3)=v(2,3)=v(1,2,3)=10 Suppose  $(x_1, x_2, x_3) \in C(v)$ . Then efficiency  $x_1 + x_2 + x_3 = 10$ stability\_ $x_1 + x_2 \ge 10$  $x_2 + x_3 \ge 10$  $20=2(x_1 + x_2 + x_3) \ge 30$  Impossible. So  $C(v)=\emptyset$ . Note that (N,v) is superadditive.



# How to share v(N)...

- The Core of a game can be used to exclude those allocations which are not stable.
- But the core of a game can be a bit "extreme" (see for instance the glove game)
- Sometimes the core is *empty* (pirates)
- And if it is not empty, there can be many allocations in the core (which is the best?)

## An axiomatic approach (Shapley (1953)

- Similar to the approach of Nash in bargaining: which properties an allocation method should satisfy in order to divide v(N) in a reasonable way?
- Given a subset C of  $G^N$  (class of all TU-games with N as the set of players) a *(point map) solution* on C is a map  $\Phi: C \rightarrow IR^N$ .
- For a solution  $\Phi$  we shall be interested in various properties...

## Symmetry

PROPERTY 1(SYM) Let  $v \in G^N$  be a TU-game. Let i,  $j \in N$ . If  $v(S \cup \{i\}) = v(S \cup \{j\})$  for all  $S \in 2^{N \setminus \{i,j\}}$ , then  $\Phi_i(v) = \Phi_j(v)$ . EXAMPLE We have a TU-game  $(\{1,2,3\},v)$  s.t. v(1) = v(2) = v(3) = 0, v(1, 2) = v(1, 3) = 4, v(2, 3) = 6, v(1, 2, 3) = 20. Players 2 and 3 are symmetric. In fact:  $v(\emptyset \cup \{2\}) = v(\emptyset \cup \{3\}) = 0$  and  $v(\{1\} \cup \{2\}) = v(\{1\} \cup \{3\}) = 4$ If  $\Phi$  satisfies SYM, then  $\Phi_2(v) = \Phi_3(v)$ 

## Efficiency

**<u>PROPERTY 2 (EFF)</u>** Let  $v \in G^N$  be a TU-game.

 $\sum_{i \in N} \Phi_i(v) = v(N)$ , i.e.,  $\Phi(v)$  is a pre-imputation.

#### **Null Player Property**

<u>DEF.</u> Given a game  $v \in \mathbf{G}^N$ , a player  $i \in N$  s.t.

 $v(S \cup i) = v(S)$  for all  $S \in 2^N$  will be said to be a null player.

**PROPERTY 3 (NPP)** Let  $v \in G^N$  be a TU-game. If  $i \in N$  is a null player, then  $\Phi_i(v) = 0$ .

**EXAMPLE** We have a TU-game ( $\{1,2,3\}$ ,v) such that v(1) =0, v(2) = v(3) = 2, v(1, 2) = v(1, 3) = 2, v(2, 3) = 6, v(1, 2, 3)

3) = 6. Player 1 is null. Then  $\Phi_1(v) = 0$ 



Additivity					
<b><u>PROPERTY 4 (ADD)</u></b> Given $v, w \in \mathbf{G}^N$ ,					
$\Phi(\mathbf{v}) + \Phi(\mathbf{w}) = \Phi(\mathbf{v} + \mathbf{w}).$					
. <u>EXAMPLE</u> Two TU-games v and w on N={1,2,3}					
v(1) =3	Φ	w(1) =1	Φ	v+w(1) =4	
v(2) =4		w(2) =0		v+w(2) = 4	
v(3) = 1		w(3) = 1		v+w(3) = 2	
v(1, 2) =8	╋	w(1, 2) =2	=	v+w(1, 2) = 10	
v(1, 3) = 4		w(1, 3) = 2		v+w(1, 3) = 6	
v(2, 3) = 6		w(2, 3) = 3		v+w(2, 3) = 9	
v(1, 2, 3) = 10		w(1, 2, 3) = 4		v+w(1, 2, 3) = 14	

#### Theorem 1 (Shapley 1953)

There is a unique map  $\phi$  defined on **G**<sup>N</sup> that satisfies EFF, SYM, NPP, ADD. Moreover, for any i  $\in$  N we have that

$$\phi_i(v) = \frac{1}{n!} \sum_{\sigma \in \Pi} m_i^{\sigma}(v)$$

Here  $\Pi$  is the set of all permutations  $\sigma: N \rightarrow N$  of N, while  $m^{\sigma}{}_{i}(v)$  is the marginal contribution of player i according to the permutation  $\sigma$ , which is defined as:

v({  $\sigma$  (1),  $\sigma$  (2), ...,  $\sigma$  (j)})-v({  $\sigma$  (1),  $\sigma$  (2), ...,  $\sigma$  (j-1)}), where j is the unique element of N s.t. i =  $\sigma$  (j).

<ul> <li>Probabilistic interpretation: (the "room parable")</li> <li>Players gather one by one in a room to create the "grand coalition", and each one who enters gets his marginal contribution.</li> <li>Assuming that all the different orders in which they enter are equiprobable, the Shapley value gives to each player her/his expected payoff.</li> </ul>				
Example	Permutation	1	2	3
(N,v) such that $N-(1,2,3)$	1,2,3	0	3	2
v(1)=v(3)=0, v(2)=3,	1,3,2	0	4	1
	2,1,3	0	3	2
v(1,2)=3, v(1,2)=1	2,3,1	1	3	1
v(1,3)=1, v(2,3)=4,	3,2,1	1	4	0
v(1,2,3)=5.	3,1,2	1	4	0
	Sum	3	21	6
	φ(ν)	3/6	21/6	6/6





# Unanimity games (2)

► Every coalitional game (N, v) can be written as a linear combination of unanimity games in a unique way, i.e.,  $v = \sum_{s \in 2^N} \lambda_s(v)u_s$ .

➤ The coefficients  $\lambda_{s}(v)$ , for each  $S \in 2^{N}$ , are called unanimity coefficients of the game (N, v) and are given by the formula:  $\lambda_{s}(v) = \sum_{T \in 2^{s}} (-1)^{s-t} v(T)$ .



#### An alternative formulation

Let  $m^{\sigma'}(v)=v(\{\sigma'(1),\sigma'(2),...,\sigma'(j)\})-v(\{\sigma'(1),\sigma'(2),...,\sigma'(j-1)\}),$ where j is the unique element of N s.t.  $i = \sigma'(j)$ .

► Let S={  $\sigma'(1), \sigma'(2), \ldots, \sigma'(j)$  }.

- > Q: How many other orderings  $\sigma \in \Pi$  do we have in which {  $\sigma$  (1),  $\sigma$  (2), ...,  $\sigma$  (j)}=S and i =  $\sigma$  '(j)?
- A: they are precisely (|S|-1)!×(|N|-|S|)!
- Where (|S|-1)! Is the number of orderings of S\{i} and (|N|-|S|)! Is the number of orderings of N\S
- We can rewrite the formula of the Shapley value as the following:

$$\phi_i(v) = \sum_{S \in 2^N : i \in S} \frac{(s-1)!(n-s)!}{n!} (v(S) - v(S \setminus \{i\}))$$

## Convex games (1)

<u>DEF.</u> An n-persons TU-game (N,v) is convex iff  $v(S)+v(T) \le v(S \cup T)+v(S \cap T)$  for each  $S,T \in 2^N$ .

This condition is also known as *supermodularity*. It can be rewritten as

 $v(T)-v(S \cap T) \le v(S \cup T)-v(S)$  for each  $S,T \in 2^N$ 

For each  $S,T \in 2^N$ , let  $C = (S \cup T) \setminus S$ . Then we have:  $v(C \cup (S \cap T)) - v(S \cap T) \le v(C \cup S) - v(S)$ 

Interpretation: the marginal contribution of a coalition C to a disjoint coalition S does not increase if S becomes smaller

## Convex games (2)

The searce of the supermodularity is equivalent to  $v(S \cup \{i\})-v(S) \le v(T \cup \{i\})-v(T)$ 

for all  $i \in N$  and all  $S,T \in 2^N$  such that  $S \subseteq T \subseteq N \setminus \{i\}$ > interpretation: player's marginal contribution to a large coalition is not smaller than her/his marginal contribution to a smaller coalition (which is stronger than superadditivity)

Clearly all convex games are superadditive ( $S \cap T = \emptyset$ ...)

➢A superadditive game can be not convex (try to find one)
➢An important property of convex games is that they are (*totally*) balanced, and it is "easy" to determine the core (coincides with the Weber set, i.e. the convex hull of all marginal vectors...)





Minimum Cost Spanning Tree Situations
 Consider a complete weighted graph
 whose vertices represent agents
 vertex 0 is the source
 edges represent connections between agents or between an agent and the source
 numbers close to edges are connection costs

































# P-value

>Always provides a unique allocation (given a most situation).

>It is in the core of the corresponding most game.

>Satisfies cost monotonicity.

>Satisfies population monotonicity.

>on a subclass of connection problems it coincides with the Shapley value of mcst games

≻...

# **Weighted Majority Games**

Suppose that four parties receive these vote shares: Party A, 27%; Party B, 25%; Party C, 24%; Party 24%.

Seats are apportioned in a 100-seat parliament according some apportionment formula. In this case, the apportionment of seats is straight-forward:

Party A:	27 seats	Party C:	24 seats
Party B:	25 seats	Party D:	24 seats

Suppose a simple majority is required (at least 51 seats) to be winning















United States presidential election (3)				
A game (N,v), where  N =51 (the number of states plus Washington D.C.), each one with a weight given by the number of electors				
$\succ v(S)=1 \iff \sum_{i \in S} w_i > 270$				
➢ In 1977, weights are between 3 (smallest states				
and Washington D.C.) and 45 (California)				
State	Electors	Shapley	Banzhaf	
California	45	0.08831	0.38694	
Washington DC	3	0.005412	0.02402	
Florida	17	0.03147	0.13736	
Montaa	4	0.00723	0.03202	



# Power indices: a general formulation

Let (N,v) be a simple game (assume v is *monotone*: for each  $S,T \in 2^N$ .  $S \subseteq T \Rightarrow v(S) \leq v(T)$ )

Let  $p_i(S)$ , for each  $S \in 2^N \setminus \{\emptyset\}$ ,  $i \notin S$ , be the probability of coalition  $S \cup \{i\}$  to form (of course  $\sum_{S \subseteq N: i \notin S} p_i(S)=1$ )

A power index  $\psi_i(v)$  is defined as the probability of player i to be pivotal in v according to p:

 $\psi_i^{p}(v) = \sum_{S \subseteq N: i \notin S} p_i(S) \left[ v(S \cup \{i\}) - v(S) \right]$