THEORIE DES JEUX ALGORITHMIQUE

BNb_R[a >\b_cR` N[Q IaRSN[\ C_RaaV

B8CI8;<" :DHI#FN_V` ;Nb]UV[R

`aRSN[\$Z_RaaV 7QNb]UV[R$S_

JRY$2&'**&+*).'

Theorem 1 (Shapley 1953)

There is a unique map ϕ defined on **G**^{N} that satisfies EFF, SYM, NPP, ADD. Moreover, for any i∈N we have that

$$¤_i(v) = \frac{1}{n!}\sum_{\pm \in \Pi} m_i^{\pm}(v)$$

Here Π is the set of all permutations €:N •N of N, while $m_i^{€}(v)$ is the marginal contribution of player i according to the permutation €, which is defined as:

v({€(1), €(2), . . . , € (j)}), v({€(1), €(2), . . . , € (j ,1)}),

where j is the unique element of N s.t. i = €(j).

Unanimity games (1)

- DEF Let $T \in 2^N \setminus \{\emptyset\}$. The *unanimity game* on T is defined as the TU-game (N, u_T) such that

$$u_T(S) = \begin{cases} 1 \text{ if } T \subseteq S \\ 0 \text{ otherwise} \end{cases}$$

- Note that the class $\mathbf{G}^N$ of all n-person TU-games is a vector space (obvious what we mean for v+w and αv for $v, w \in \mathbf{G}^N$ and $\alpha \in \mathbb{R}$).
- the dimension of the vector space $\mathbf{G}^N$ is 2^n-1
- $\{u_T | T \in 2^N \setminus \{\emptyset\}\}$ is an interesting basis for the vector space $\mathbf{G}^N$.

Unanimity games (2)

Every coalitional game (N, v) can be written as a linear combination of unanimity games in a unique way, i.e.,

$$v = \sum_{S \subseteq :N\ :\ S \neq \emptyset} f_S(v) u_S \ .$$

The coefficients $f_S(v)$, for each $S \in 2^N$, are called unanimity coefficients of the game (N, v) and are given by the formula: $f_S(v) = \sum_{T \subseteq S : T \neq \emptyset} (,1)^{s,t}\ v(T)$.

.EXAMPLE Two TU-games v and w on N={1,2,3}

$$f_S(v) = \sum_{T\subseteq S:T\neq\emptyset} (,1)^{s,t}\ v(T)$$

v(1) =3	$\lambda_1(v) = 3$
v(2) =4	$\lambda_2(v) = 4$
v(3) = 1	$\lambda_3(v) = 1$
v(1, 2) =8	$\lambda_{\{1,2\}}(v) = -3-4+8=1$
v(1, 3) = 4	$\lambda_{\{1,3\}}(v) = -3-1+4=0$
v(2, 3) = 6	$\lambda_{\{2,3\}}(v) = -4-1+6=1$
v(1, 2, 3) = 10	$\lambda_{\{1,2,3\}}(v) = -3-4-1+8+4+6-10=0$

$$v=3u_{\{1\}}(v)+4u_{\{2\}}(v)+u_{\{3\}}(v)+u_{\{1,2\}}(v)+u_{\{2,3\}}$$

Sketch of the Proof of Theorem1

- Shapley value satisfies the four properties (easy).
- Properties EFF, SYM, NPP determine ϕ on the class of all games „v, with v a unanimity game and „$\in$IR.
  - Let $S \in 2^N$. The Shapley value of the unanimity game (N,u_S) is given by

$$\phi_i(„u_S) = \begin{cases} \alpha/|S| & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

- Since the class of unanimity games is a basis for the vector space, ADD allows to extend ϕ in a unique way to $\mathbf{G}^N$.

An alternative formulation of the Shapley value

- Let m€·i(v)=v({€…(1),€…(2),†,€…(j)}), v({€…(1),€…(2),†, €…(j ,1)}) where j is the unique element of N s.t. i = €…(j).
- Let S={€…(1), €…(2), . . . , €…(j-1)}.
- **Q**: How many other orderings $\sigma \in \Pi$ do we have in which {€(1), €(2), . . . , € (j-1)}=S and i = €…(j)?
- **A**: they are precisely $(s)! \times (n-s-1)!$
- Where s! Is the number of orderings of S and (n-s-1)! Is the number of orderings of $N \setminus (S \cup \{i\})$
- We can rewrite the formula of the Shapley value as the following:

$$\phi_i(v)=\sum_{S \subseteq N : i \notin S} s!(n-s-1)!/n! \, [v(S \cup \{i\})-v(S)]$$

Power indices: a general formulation (2)

$$\psi_i(v)=\sum_{S\subseteq N:i\notin S} p_i(S)\,[v(S\cup\{i\})-v(S)]$$

- According to the Banzhaf power index, every coalitions has the same probability to form: $p_i(S)=1/(2^{n-1})$, for each $S\in 2^N\setminus\{\emptyset\}$, $i\notin S$

- According to the Shapley-Shubick power index, compute $p_i(S)$ according to the following procedure to create at random from N a subset S to which i does not belong:
  - Draw at random a number out of the urn consisting of possible sizes 0,1,2 1 where each number has probability 1/n to be drawn
  - If size s is chosen, draw a set out of the urn consisting of subsets of $N\setminus\{i\}$ of size s, where each set has the same probability, i.e. 1/combinations(n-1,s)
  - indeed, $p_i(S)=(s!\,(n-s-1)!)/n!$

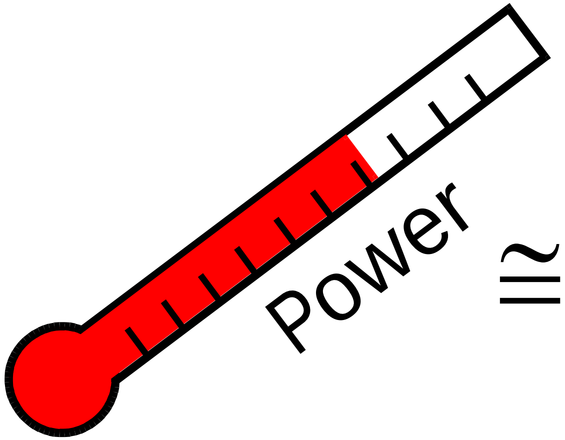
KD IRPb_Vaf :\b[PVY N` N dRVTUaRQ ZNW_Vaf TNZR

- @[aUR YN`a `Veaf fRN_`" TNZR aUR\_f UN` ORR[N]]YVRQ a\ ]\YVaVPNY N[Q `\PVNY ]_\OYRZ` a\ N``R`` aUR]\dR_ \S V[aR_NPaV[T NTR[a` V[ S\_PV[T N QRPV`V\[
- :YN``VPNY ReNZ]YR2 KD IRPb_Vaf :\b[PVY2 '+ ZRZOR_ `aNaR`" 5 Permanent members (China, France, Russian Federation, United Kingdom, USA) and 10 temporary seats (held for two-year terms)

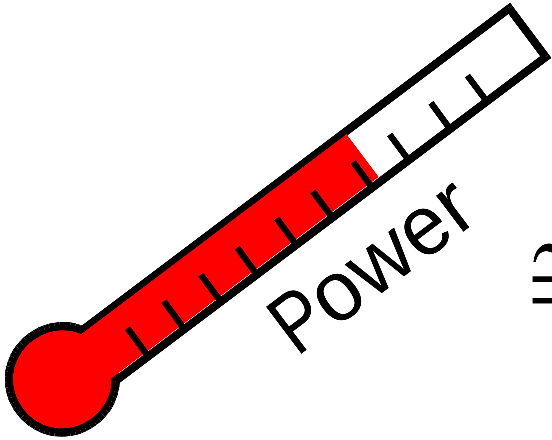
Decision Rule: **decisions** on all substantive matters need the positive vote of **at least nine Nations but** it is sufficient the **negative vote of one** among the permanent members **to reject** the decision.

KD IRPb_Vaf :\b[PVY N` N dRVTUaRQ ZNW_Vaf TNZR

- BRa D4F∪J" dUR_R F4h'"(")"*"+j V` aUR `Ra \S FR_ZN[R[a ZRZOR_` N[Q J4h,"."/"1"'&"' '"'("')"'*"'+j V` aUR `Ra \S aRZ\_N_f `RNa`
- 8 `VZ]YR TNZR D"c! `a c I!4' VS i I i≥1 N[Q F⊂I$
- D"c! V` N dRVTUaRQ ZNW_Vaf TNZR" dUR_R
  - d$_V$4. S_ RNPU V∈F
  - d$_V$4' S_ RNPU V∈J
  - c I!4' ⇔ $\Sigma_{V\in I}$ d$_V$5) /



$\cong$ 19.6%



$\cong$ 0.2%

temporary seats since January 1st 2007
until January 1st 2009

Reformulations

Other axiomatic approaches have been provided for the Shapley value, of which we shall briefly describe those by Young and and by Hart and Mas-Colell.

PROPERTY 8 (Marginalism, MARG) A map ‡ :$\mathbf{G}^{N}$• ^{N} satisfies MARG if, given v,w$\in G^{N}$, for any player i$\in$N s.t.

v(S$\cup${i}) ,v(S) = w(S $\cup${i}) ,w(S) for each S$\in 2^{N}$,

the following is true:

‡ $_{i}$(v) = ‡ $_{i}$(w).

EXAMPLE Two TU-games v and w on N={1,2,3}

v	w
v(1) =3	w(1) =2
v(2) =4	w(2) =3
v(3) = 1	w(3) = 1
v(1, 2) =8	w(1, 2) =2
v(1, 3) = 4	w(1, 3) = 3
v(2, 3) = 6	w(2, 3) = 5
v(1, 2, 3) = 10	w(1, 2, 3) = 4

$w(\emptyset\cup\{3\})- w(\emptyset) = v(\emptyset\cup\{3\})- v(\emptyset)=1$

$w(\{1\}\cup\{3\})- w(\{1\}) = v(\{1\}\cup\{3\})- v(\{1\})=1$

$w(\{2\}\cup\{3\})- w(\emptyset) = v(\{2\}\cup\{3\})- v(\emptyset)=1$

$w(\{1,2\}\cup\{3\})- w(\{1,2\}) = v(\{1,2\}\cup\{3\})- v(\{1,2)=1$

⇓

$\Psi_3(v) = \Psi_3(w).$

(Young 1988)

Theorem 2

There is a unique map ‡ defined on G(N) tha satisfies EFF, SYM, and MARG. Such a ‡ coincides with the Shapley value.

Potential

- A quite different approach was pursued by Hart and Mas-Colell (1987).
- To each game (N, v) one can associate a real number P(N,v) (or, simply, P(v)), its *potential*.
- The ˆpartial derivative‰ of P is defined as

$$D^i(P)(N, v) = P(N,v), P(N\backslash\{i\}, v_{|N\backslash\{i\}})$$

Theorem 3 (Hart and Mas-Colell 1987) There is a unique map P, defined on the set of all finite games, that satisfies:

1) $P(\emptyset, v_0) = 0$,

2) $\Sigma_{i\in N} D^iP(N,v) = v(N)$.

Moreover, $D^i(P)(N, v) = \phi_i(v)$. [$\phi(v)$ is the Shapley value of v]

- there are formulas for the calculation of the potential.
- For example, $P(N,v)=\sum_{S\in 2^N} \lambda_S/|S|$ (*Harsanyi dividends*!

Example

$v(1)=3$

$v(2)=4$

$v(3)=1$

$v(1,2)=8$

$v(1,3)=4$

$v(2,3)=6$

$v(1,2,3)=10$

→

$\lambda_1(v)=3$

$\lambda_2(v)=4$

$\lambda_3(v)=1$

$\lambda_{\{1,2\}}(v)=1$

$\lambda_{\{1,3\}}(v)=0$

$\lambda_{\{2,3\}}(v)=1$

$\lambda_{\{1,2,3\}}(v)=0$

→

$P(\{1,2,3\},v)=3+4+1+1/2+1/2=9$

$P(\{1,2\},v_{|\{1,2\}})=3+4+1/2=15/2$

$P(\{1,3\},v_{|\{1,3\}})=3+1=4$

$P(\{2,3\},v_{|\{2,3\}})=4+1+1/2=11/2$

$$\phi_1(v)=P(\{1,2,3\},v)-P(\{2,3\},v_{|\{2,3\}})=9-11/2=7/2$$
$$\phi_2(v)=P(\{1,2,3\},v)-P(\{1,3\},v_{|\{1,3\}})=9-4=5$$
$$\phi_3(v)=P(\{1,2,3\},v)-P(\{1,2\},v_{|\{1,2\}})=9-15/2=3/2$$

:\ZZb[VPNaV\[[Rad_X`

- A cooperative game describes a situation in which all players can freely communicate with each other.
- Drop this assumption and assume that communication between players is restricted to a set of communication possibilities between players.

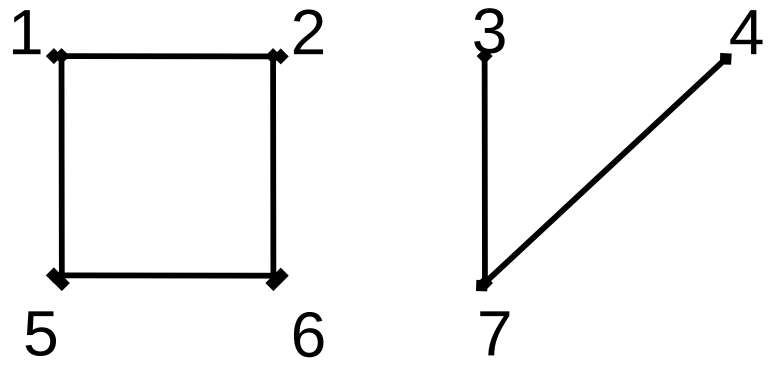
Communication networks as undirected graphs:

8[*undirected graph* V` N]NV_ D"B! dUR_R

- D V` N `Ra \S *vertices* YNaR_" *agents* _ *players*!
- B4h hV"Wj i hV"Wj⊆D" V≠W j V` aUR `Ra \S *edges* OVYNaR_NY *communication links*!
- 8 P\ZZb[VPNaV\[T_N]U D"B! `U\bYQ OR V[aR_]_RaRQ N` N dNf a\ Z\QRY _R`a_VPaRQ P\\]R_NaV\[2
  - FYNfR_` PN[P\\]R_NaR dVaU RNPU \aUR_ VS aURf N_R P\[[RPaRQ *directly,* _ *indirectly* cVN N]NaU!
  - @[QV_RPa P\ZZb[VPNaV\[ORadRR[ad\ ]YNfR_` _R^bV_R` aUR P\\]R_NaV\[\S]YNfR_` \[N P\[[RPaV[T]NaU$

<eNZ]YR

Consider the undirected graph (N,L) with N={1,2,3,4,5,6,7}
and L={{1,2}, {2,6}, {5,6}, {1,5}, {3,7}, {4,7}}



I\ZR [\aNaV\[`2

DMB4hh'"("+",j"h)"*".jj

`Ra \SP\Z]\[R[a`

B#(4hh+",j"h'"+j"h)".j"h*".jj   DMB#(4hh'"+",j"h)"*".j"h(jj

Communication situations (Myerson (1977))

- 8 *communication situation* V` N a_V]YR D"c"B!
  - D"c! V` N [#]R_`\[JK#TNZR _R]_R`R[a` aUR RP\[\ZVP]\``VOVYVaVR` \S P\NYVaV\[`!
  - D"B! V` N P\ZZb[VPNaV\[ [Rad\_X _R]_R`R[a` _R`a_VPaRQ P\ZZb[VPNaV\[ ]\``VOVYVaVR`!
- JUR *graph-restricted game* D"c^B! V` QRSV[RQ N`

$$c^B \; J!4\sum_{: \in JMB} c \; :!$$

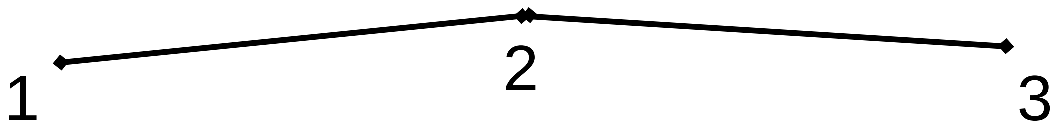
=_ RNPU I ∈ (^DMh∅j$

HRPNYY aUNa JMB V` aUR `Ra \S ZNeVZNY P\[[RPaRQ P\Z]\[R[a` V[ aUR R`a VPaV\[\S T N]U D"B! a\ J

Example

A weighted majority game ({1,2,3},v) with the winning quote fixed to 2/3 is considered. The votes of players 1, 2, and 3 are, respectively, 40%, 20%, and 40%. Then, v(1,3)=v(1,2,3)=1 and v(S)=0 for the remaining colitions.

The communication network is



Then,

$v^L(1,2,3)=1$, and $v^L(S)=0$ for the other coalitions.

Solutions for communication situations

- Myerson (1977) was the first to study solutions for communication situations.
- A solution ‡ is a map defined for each communication situation (N,v with value in N.

PROPERTY 1 Component Efficiency (CE)

For each communication situation (N,v,L) and all C∈N\L it holds that

$$\sum_{i \in C} ‡_i(N,v,L) = v(C).$$

- Property 1 is an ˆefficiency‰ condition that is assumed to hold only those coalitions whose players are able to communicate effectively among them and *are not connected to other players*. (maximal connected components)

Solutions for communication situations (2)

<u>PROPERTY 2</u> Fairness (F) For each communication situation (N,v,L) and all $\{i,j\} \in L$ it holds that

$$‡_i(N,v,L)\ ,‡_i(N,v,L\setminus\{\{i,j\}\}) = ‡_j(N,v,L),\ ‡_j(N,v,L\setminus\{\{i,j\}\}).$$

- Property 2 says that two players should gain or lose in exactly the same way, when a direct link between them is established (or deleted).

Myerson value

Theorem (Myerson (1977))

There exists a unique solution $\mu(N,v,L)$ which satisfies CE and F on the class of communication situations. Moreover,

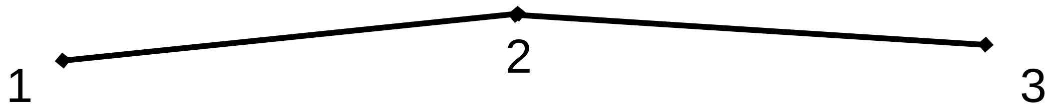
$$\mu(N,v,L)=\phi(v^L)$$

where $\phi(v^L)$ is the Shapley value of the graph-restricted game v^L.

Example

A weighted majority game ({1,2,3},v) with the winning quote fixed to 2/3 is considered. The votes of players 1, 2, and 3 are, respectively, 40%, 20%, and 40%. Then, v(1,3)=v(1,2,3)=1 and v(S)=0 for the remaining colitions.

The communication network is



Then,

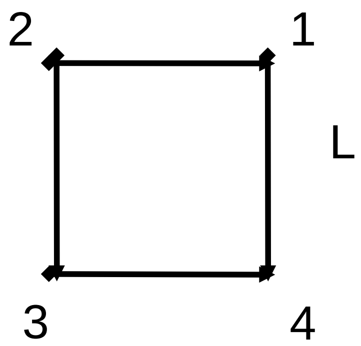
$v^L(1,2,3)=1$, and $v^L(S)=0$ for the other coalitions.

We have that

$\phi(v)=(1/2,0,1/2)$ and $\mu(N,v,L)= \phi(v^L)=(1/3,1/3,1/3)$.

Example

(N,v,L) communication situation such that L is the following network and $v=u_{\{2,4\}}$



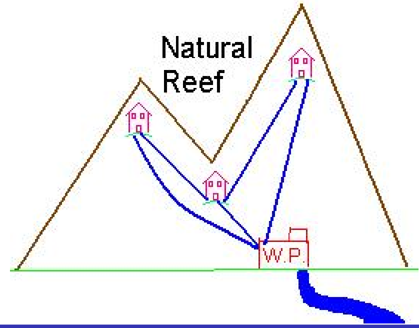
Note that, for instance, $v^L(2,4)=v(2)+v(4)=0$.

Easy to note that that $v^L=u_{\{1,2,4\}}+u_{\{2,3,4\}}-u_N$

Therefore,

$\mu(N,v,L)=\phi(vL)=(1/3,2/3,1/3,2/3)-(1/4,1/4,1/4,1/4)$

$=(1/12,5/12,1/12,5/12)$

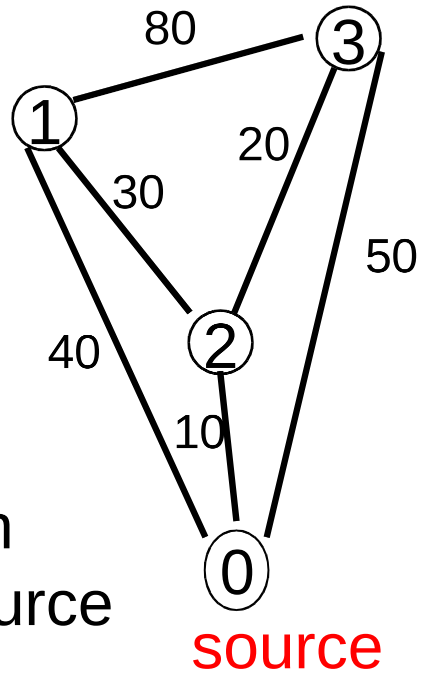


Section 2. Connection situations

- 8 P\[[RPaV\[`VabNaV\[ aNXR`]YNPR V[aUR]_R`R[PR \S N T_\b] \S NTR[a` D4h'"(" k"[j" RNPU \S dUVPU [RRQ` a\ OR P\[[RPaRQ QV_RPaYf \_ cVN \aUR_ NTR[a` a\ N `\b_PR$
- @S P\[[RPaV\[` NZ\[T NTR[a` N_R P\`aYf" aUR[ RNPU NTR[a dVYY RcNYbNaR aUR \]]\_ab[Vaf \S P\\]R_NaV[T dVaU \aUR_ NTR[a` V[_QR_ a\ _RQbPR P\`a`$
- @S N T_\b] \S NTR[a` QRPVQR` a\ P\\]R_NaR" N P\[SVTb_NaV\[\S YV[X` dUVPU ZV[VZVgR` aUR a\aNY P\`a \S P\[[RPaV\[ V`]_\cVQRQ Of N ZV[VZbZ P\`a `]N[[V[T a_RR ZP`a!$
- JUR]_\OYRZ \S SV[QV[T N ZP`a ZNf OR RN`VYf `\YcRQ aUN[X` a\ QVSSR_R[a NYT_VaUZ` ]_\]\`RQ V[YVaR_Nab_R 9_bcXN '1(,!" A_b`XNY '1+,!" F_VZ '1+.!" ;VWX`a_N '1+1!!

C\[\ZbZ :\`a l]N[[\[T J_RR l\abNa\\ [

Consider a complete weighted graph



– whose vertices represent agents

– vertex 0 is the source

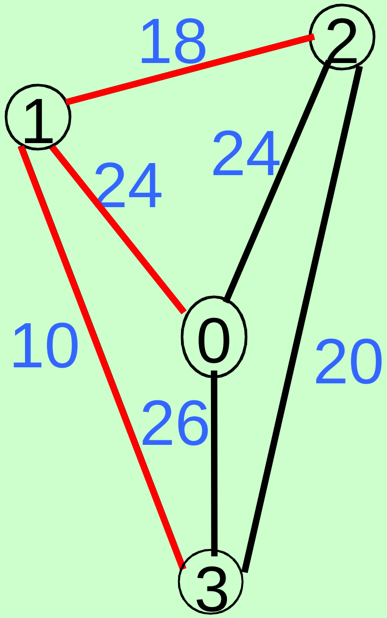
– edges represent connections between agents or between an agent and the source

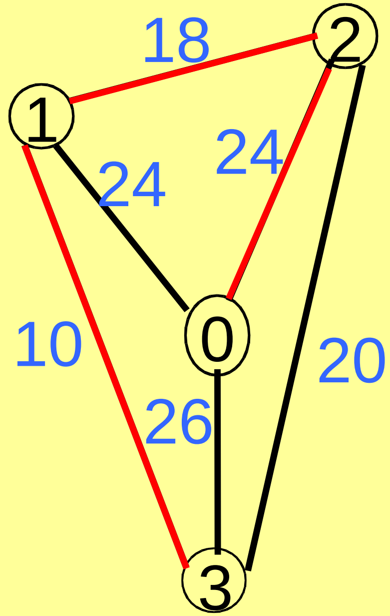
– numbers close to edges are connection costs

C\[\ZbZ P\`a `]N[[\[T a_RR ZP`a!]_\OYRZ

E]a\Z\gNa\\[ ]_\OYRZ2

?\d a\ P\[[RPa RNPU [\QR a\ aUR `\b_PR &\[ `bPU N dNf aUNa aUR P\`a \S P\[`a_bPa\\[\S N `]N[[\[T [Rad\_X dU\PU P\[ [RPa` RcR_f [\QR Q_RPaYf _ \[Q_RPaYf a\ aUR `\b_PR &! \` Z\[\ZbZ6





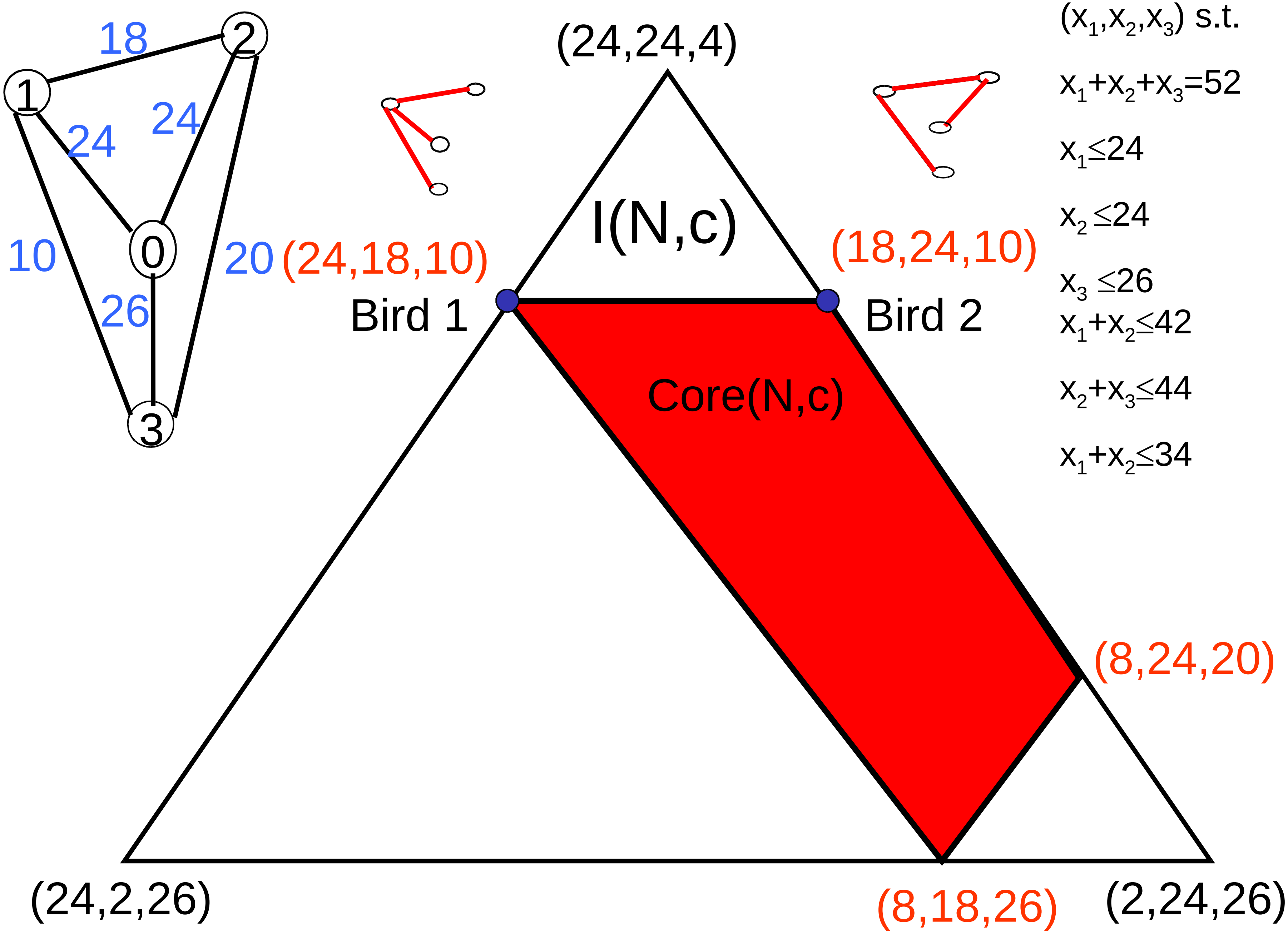
The Bird allocation w.r.t .this mcst is

$(x_1, x_2, x_3)=(24, 18, 10)$

The Bird allocation w.r.t. this mcst is

$(x_1, x_2, x_3)=(18, 24, 10)$

Both allocations belong to the core of the mcst game (and also their convex combination).



18
24
24
10
20
26
1
2
0
3
(24,24,4)
I(N,c)
(24,18,10)
(18,24,10)
Bird 1
Bird 2
Core(N,c)
(8,24,20)
(24,2,26)
(8,18,26)
(2,24,26)
(x_1,x_2,x_3) s.t.
$x_1+x_2+x_3=52$
$x_1 \leq 24$
$x_2 \leq 24$
$x_3 \leq 26$
$x_1+x_2 \leq 42$
$x_2+x_3 \leq 44$
$x_1+x_2 \leq 34$

Bird allocation rule

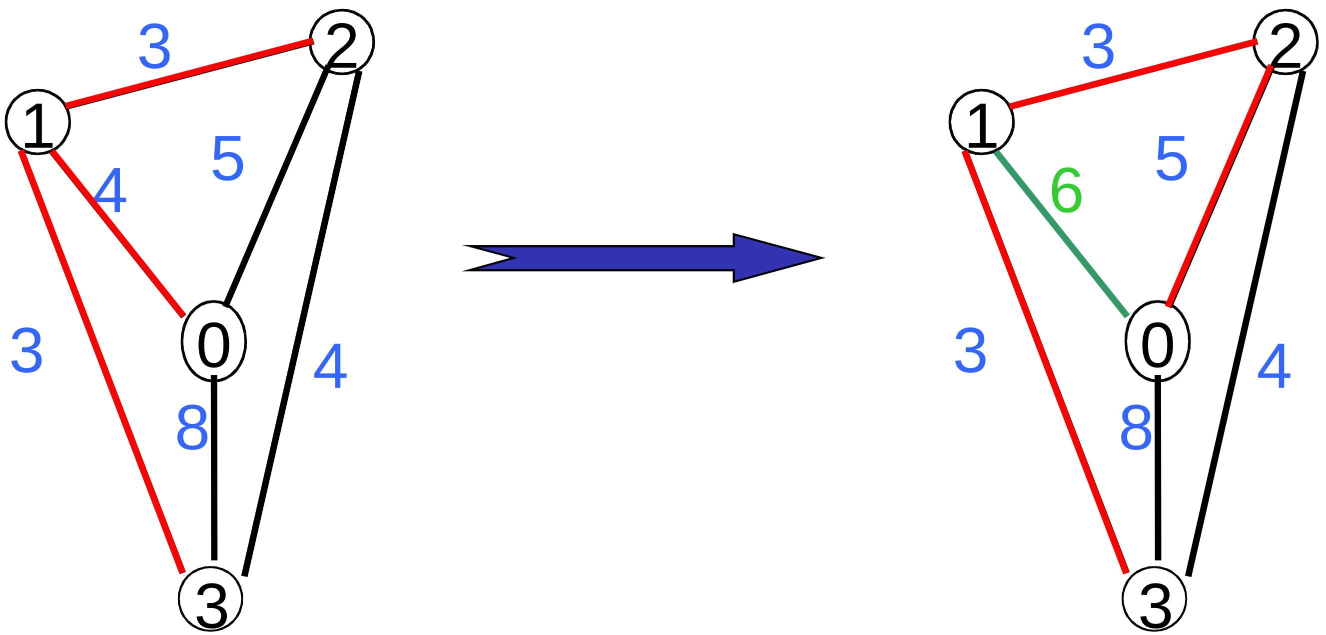
- It always provides an allocation (given a connection situation).
- In general, not a unique allocation (each mcst determines a corresponding Bird allocation...).
- Bird allocations are in the core of mcst games (but are extreme points)

What happens when the structure of the network changes?

Imagine to use a certain rule to allocate costs.

- The cost of edges may increase; if the cost of an edge increases, nobody should be better off, according to such a rule (*cost monotonicity*).
- One or more players may leave the connection situation; nobody of the remaining players should be better off (*population monotonicity*).

: \`a Z\ [\a\ [VPVaf2 9V_Q NYY\PNaV\ [ORUNcV\b_

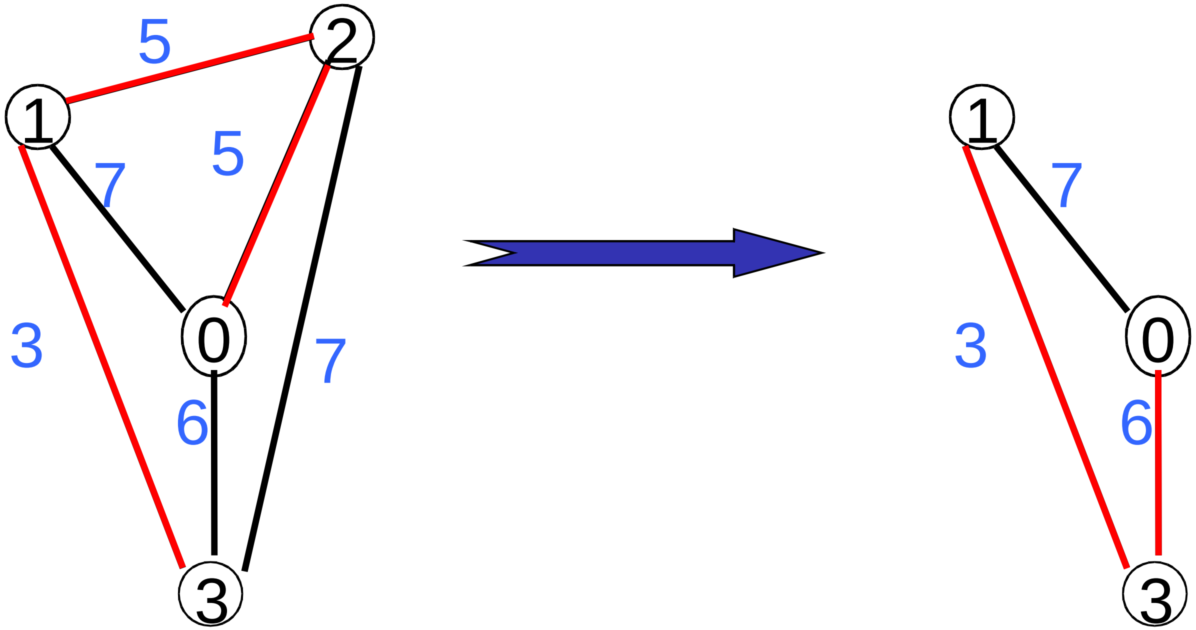


Bird allocation: (4, 3 ,3)

Bird allocation: (3, 5 ,3)

→ Bird rule does not satisfy cost monotonicity.

Population monotonicity: Bird allocation behaviour



Bird allocation: (5, 5 ,3)

Bird allocation: (3, * ,6)

→ Bird rule does not satisfy population monotonicity

: \ [`a_bPa :UN_TR _bYR`

8_R ON`RQ \[ aUR S\YY\dV[T TR[R_NY P\`a NYY\PNaV\[ ]_\a\P\Y2

- 8` `\\[N` NYV[X V` P\[`a_bPaRQ V[ aUR A_b`XNY NYT_VaUZ]_\PRQb_R2

'! Va Zb`a OR a\aNYYf PUN_TRQ NZ\[T NTR[a` dUVPU N_R [\a fRa P\[[RPaRQ dVaU aUR `\b_PR *connection property*!

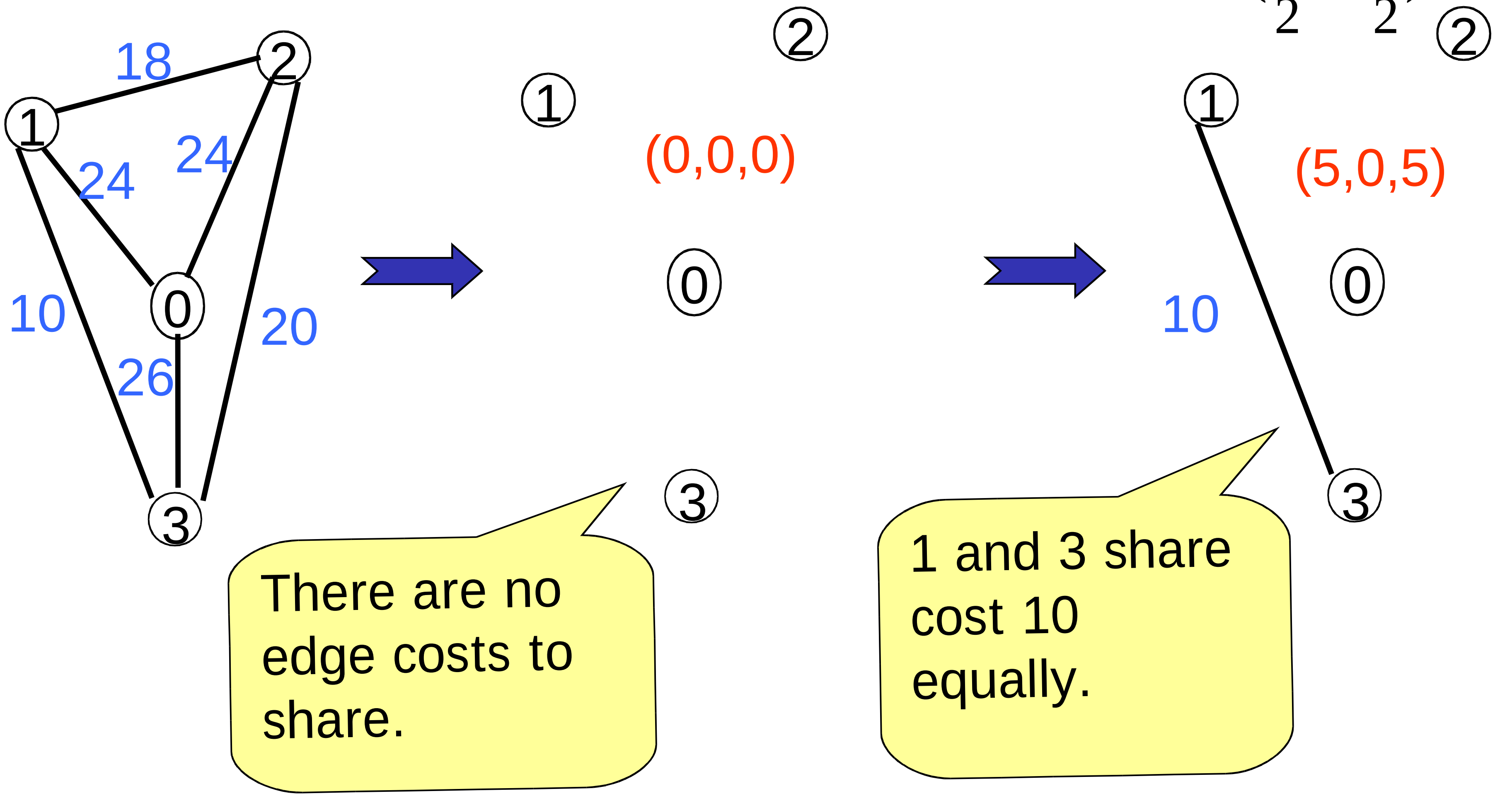
(! E[Yf NTR[a` aUNa N_R \[ `\ZR]NaU P\[aNV[V[T aUR [Rd RQTR ZNf OR PUN_TRQ *involvement property*!

- dUR[aUR P\[`a_bPaV\[ \S N ZP`a V` P\Z]YRaRQ" RNPU NTR[a UN` ORR[PUN_TRQ S_ N a\aNY NZ\b[a \S S_NPaV\[` R^bNY a\ ' *total aggregation property*!$

P-value: Feltkamp (1994), Branzei et al. (2004), Moretti (2008)

$$b^{\square,0} = (1,1,1)^t$$

$$b^{\square,1} = (\frac{1}{2},1,\frac{1}{2})^t$$



[redacted]

2

0

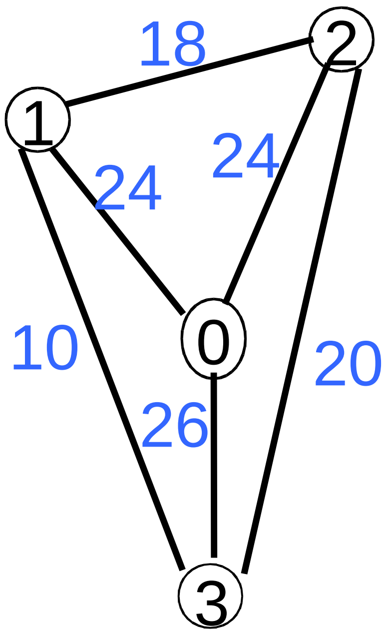
[redacted]

3 wh y
con ys 2/3
of s the
re s shared
e between 1 and

[redacted]

a cycle: nobody
want it.

P-value



Make the sum of all edge-by-edge allocations:

(0, 0, 0) +
(5, 0, 5) +
(3,12,3) +
(0, 0, 0) +
(8, 8, 8) =

P-value = (16,20,16)

8YT_VaUZ a\ PNYPbYNaR aUR F#cNYbR

IDEA: charge the cost of an edge constructed during the Kruskal algorithm only between agents involved, keeping into account the cardinality of the connected components at that step and at the previous step of the algorithm:

- At any step of the Kruskal algorithm where a component is connected to some agents, charge the cost of that edge among these agents in the following way:
  - Proportionally to the cardinality_current_step^{-1} if an agent is connected to a component which is connected to the source,
  - Otherwise, charge it proportionally to the difference: cardinality_previous_step^{-1} - cardinality_current_step^{-1}

F#cNYbR

- Always provides a unique allocation (given a mcst situation).
- It is in the core of the corresponding mcst game.
- Satisfies cost monotonicity.
- Satisfies population monotonicity.
- on a subclass of connection problems it coincides with the Shapley value of mcst games
- …

Axiomatic characterization (4 independent axioms)

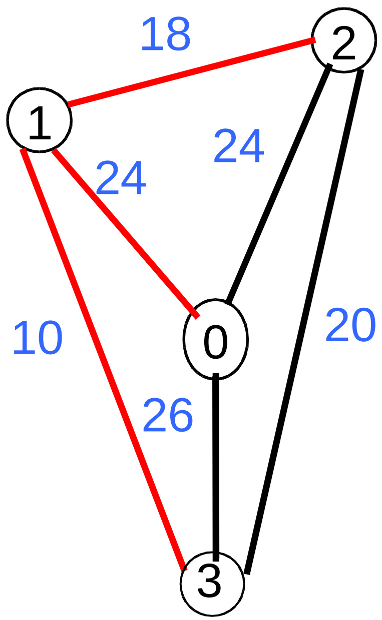
A solution for mcst situations $F : W^{N'} \to \Re^N$

Property 1. The solution F is *efficient* (EFF) if for each $w \in W^{N'}$

$$\sum_{i \in N} F_i(w) = w(\Gamma),$$

where Γ is a minimum cost spanning network on N'.

Example:



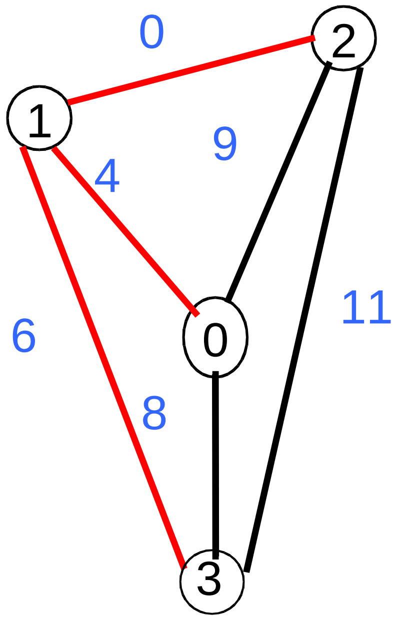
$w(\Gamma)=52$

$$P(w) = M^{\square} w^{\square} = (16, 20, 16)^t$$

Property 2. The solution F has the *Equal Treatment* (ET) property if for each $w \in W^{N'}$ and for each $i,j \in N$ with $C_i(w) = C_j(w)$

$$F_i(w) = F_j(w).$$

Example:

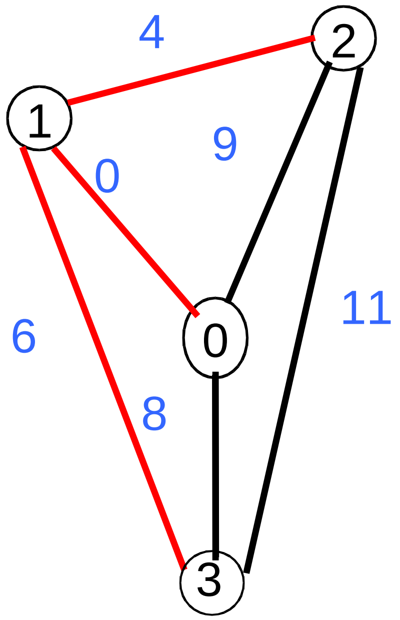


$P(w)=(2,2,6)^t$

Property 3. The solution F has the *Upper Bounded Contribution* (UBC) property if for each $w \in W^{N'}$ and every (w,N')-component $C \neq \{0\}$

$$\sum_{i \in C \setminus \{0\}} F_i(w) \leq \min_{i \in C \setminus \{0\}} w(\{i,0\}).$$

Example:



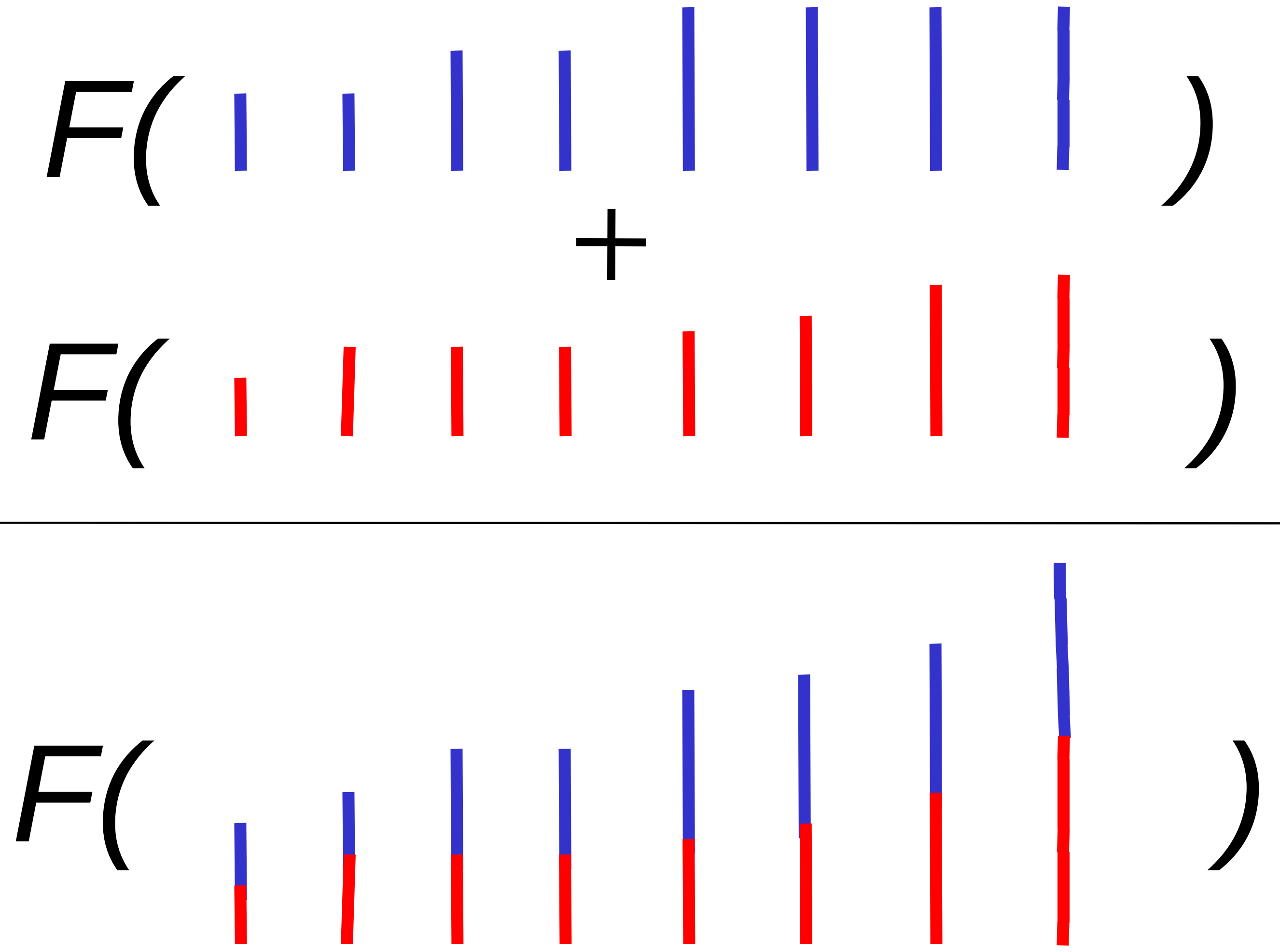
$P(w)=(0,4,6)^t$

Note that 1 is dummy in the corresponding mcst game

Property 4. The solution F has the *Cone-wise Positive Linearity* (CPL) property if for each $\pm \in \Sigma_{E_{N'}}$, for each pair of mcst situations $w, w\check{S} \in K^{\pm}$ and for each pair $\sim, \sim\check{S} \geq 0$, we have

$$F(\sim w + \sim\check{S} w\check{S}) = \sim F(w) + \sim\check{S} F(w\check{S}).$$

Example:



Theorem 1. *The P-value is the unique solution which satisfies the properties EFF, ET, UBC and CPL on the class $W^{N'}$ of mcst situations.*

- It is possible to prove that the P-value satisfies the four properties EFF, ET, UBC and CPL.

- To prove the uniqueness consider a solution for mcst situation F which satisfies EFF, ET, UBC and CPL:

  - first look at the simple mcst situations (0-1 cost of edges): on such simple situation, EFF, ET and UBC imply F=P-value;

  - it is possible to decompose each mcst situation as a linear combination of simple mcst problems;

  - by CPL it follows that the F=P-value on each mcst situation.

Genes interaction and centrality

- Classical centrality measures are appropriate under the assumption that nodes failures occur independently...

...and the system is sensible to the failure of each single node.

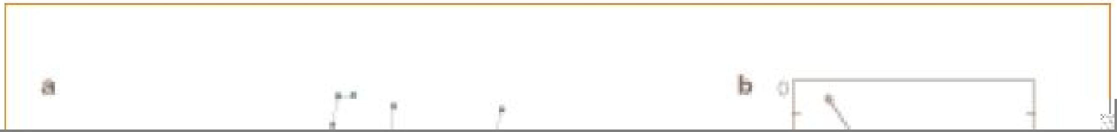
On the contrary, in biological complex networks, assuming that the failure of the nodes (genes/ proteins) is independent is not realistic and the consequence on the system can be appreciated only if many nodes fail.

Jeong, Mason, Barabasi, Oltvai. Lethality and centrality in protein networks. Nature 2001;411:41-42.

Lethality and centrality in protein networks

The most highly connected proteins in the cell are the most important for its survival.

Proteins are traditionally identified on the basis of their individual actions as catalysts, signalling molecules, or building blocks in cells and microorgan-



Addressed questions

G'2PN[dR b`R aUR `NZR TNZR aUR_RaVP Z\QRY` a\ ZRN`b_R aUR _RYRcN[PR \S V[aR_NPaV[T TR[R` \_ ]_\aRV[`! V[_R`]\[`R a\ PR_aNV[OV\Y\TVPNY P\[QVaV\[`6

G(2 U\d a\ P\ZOV[R aUR V[S_ZNaV\[ ]_\cVQRQ Of OV\Y\TVPNY [Rad_X` dVaU aUR Naa_VObaV\[ \S _RYRcN[PR" XRR]V[T V[a\ NPP\b[a aUR V[aR_NPaV\[ \S TR[R`6

JUR V[T_RQVR[a` \S aUR Z\QRY

- FYNfR_` N_R <span>TR[R`
- 8 **decision rule** V` P_RNaRQ \[ aUR QNaN#`Ra a\ R`aNOYV`U dUVPU T_\b]` P\NYVaV\[`! \S TR[R` N_R dV[[V[T$
- <eNZ]YR2 =V_`a dR QRSV[R N P_VaR_V\[ a\ R`aNOYV`U dUVPU TR[R` N_R NO[_ZNYYf Re]_R``RQ \[RNPU N_Nf

	array1
gene1	1.121
gene2	2.453
gene3	3.586

→

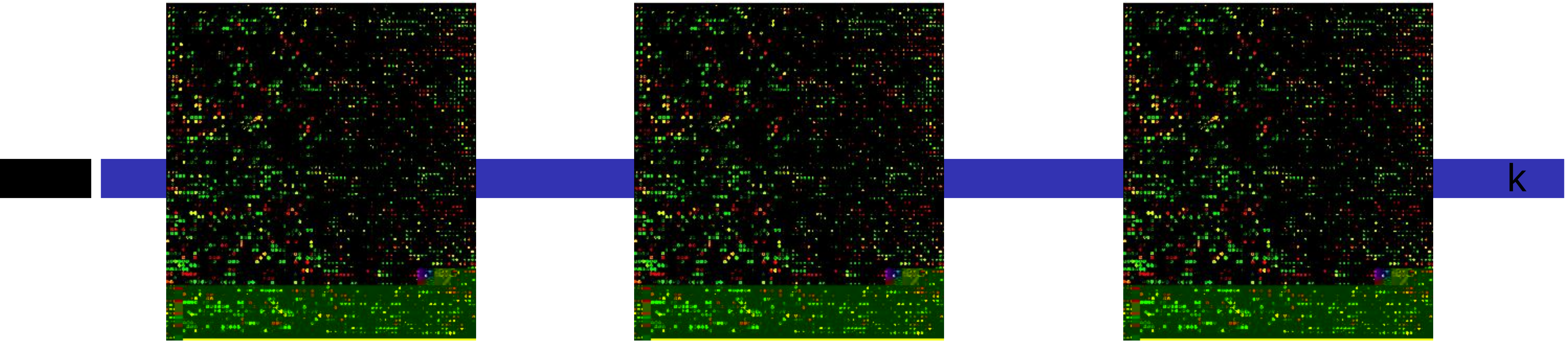
	array1
gene1	0
gene2	1
gene3	1

Decision rule

8 T_\b] \S TR[R` V` *winning* \[N `V[TYR N_Nf VS NYY TR[R` aUNa UNcR NO[_ZNY Re]_R``V\[` ORY\[T a\ aUNa T_\b]

	array1
gene1	0
gene2	1
gene3	1

Both groups {gene2, gene3} and group {gene1, gene2, gene3} are winning.



	array1	array2	array3
gene1	0	1	0
gene2	1	1	0
gene3	1	0	1

‹coalition {gene2, gene3} is winning two times out of three;

‹coalition {gene1, gene2} is winning one time out of three;

‹And so on for each coalition...

Example

	8_Nf'	8_Nf(	8_Nf)
T.	&	'	&
T(	'	'	&
T)	'	&	'

JUR P_R`]\[QV[T *microarray game*

<hT."T("T)j"c5 `bPU aUNa

c ∅!4c hT.j!4c hT(j!4&

c hT."T)j!4c hT."T(j!4c hT)j!4'%)

c hT("T)j!4(%)

c hT."T("T)j!4'$

We look for an index which is able to resume all the information about coalitions in a single attribution of relevance for each individual gene

Axioms for a relevance index on microarray games

Property 1: Null Gene (NG)

A gene which does not contribute to change the worth of any coalition of genes, should receive zero power.

Property 2: Equal Splitting (ES)

Each sample should receive the same level of reliability. So the power of a gene on two samples should be equal to the sum of the power on each sample divided by two.

	s_1	
g_1	0	ψ_1
g_2	0	ψ_2
g_3	1	ψ_3

+

s_2	
1	ψ'_1
1	ψ'_2
0	ψ'_3

=

	s_1	s_2	
g_1	0	1	$(\psi_1+\psi'_1)/2$
g_2	0	1	$(\psi_2+\psi'_2)/2$
g_3	1	0	$(\psi_3+\psi'_3)/2$

FN_a[R_`UV] \S TR[R`

A group of genes S such that does not exist a proper (⊂) subset of S which contributes in changing the worth of genes outside S.

<eNZ]YR

These two sets are partnerships of genes in the corresponding Microarray game

	`'	`(	`)
T'	&	'	'
T(	&	'	'
T)	'	&	'

Property 2: Partnership Monotonicity (PM)

(N,v) a microarray game. If two partnerships of genes S and T, with |T|≥|S| are such that they are

-*disjoint* ($S \cap T = \emptyset$),

-*equivalent* ($v(S)=v(T)$)

-*exhaustive* ($v(S \cup T)=v(N)$),

then genes in the smaller *partnership* S must receive more relevance then genes in T.

Example

		$`_{'}$	$`_{(}$
ψ_1	T'	&	'
ψ_2	T(	&	'
ψ_3	T)	'	&
ψ_4	T*	'	&
ψ_5	T+	'	&

ψ_i Œ ψ_k

For each

$i \in \{1,2\}$

$k \in \{3,4,5\}$

EaUR_]_\]R_aVR` P\[PR_[V[T ]N_a[R_`UV]

F_\]R_af *2 FN_a[R_`UV] HNaV\[NYVaf FH!

The total amount of power index received from players of a partnership S should not be smaller than v(S)

F_\]R_af +2 FN_a[R_`UV] =RN`VOVYVaf F=!

The total amount of power index received from players of a partnership S should not be greater than v(N)

Theorem C_RaaV" FNa_\[R" 9\[N``V (&&.!!:

JUR IUN]YRf cNYbR V` aUR b[V^bR `\YbaV\[dUVPU `NaV`SVR` D>" <I" FC" FH" F= \[aUR PYN`` \S ZVP_\N_Nf TNZR`$

Players gather one by one in a room to create the “grand coalition”, and each one who

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Methodology article

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Combining Shapley value and statistics to the analysis of gene expression data in children exposed to air pollution

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Cancer

Original Article

Identification of low intratumoral gene expression heterogeneity in neuroblastic tumors by genome-wide expression analysis and game theory†

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Issue

Cancer

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Example

	8_Nf'	8_Nf(	8_Nf)
T.	&	'	&
T(	'	'	&
T)	'	&	'

JUR P_R`] \ [QV[T *microarray game*

<hT."T("T)j"c5 `bPU aUNa

c ∅!4c hT.j!4c hT(j!4&

c hT."T)j!4c hT'"T(j!4c hT)j!4'%)

c hT("T)j!4(%)

c hT."T("T)j!4'$

JUR IUN]YRf cNYbR V`

IUT. 4'%, IUT(4'%) IUT) 4'%(

Data

	array1
g1	0
g2	1
g3	1

Restrict the interaction possibilities

(biological network)

A priori game

S	v	$\varphi(v)$
{g1}	0	0
{g2}	0	1/2
{g3}	0	1/2
{g1,g2}	0	
{g1,g3}	0	
{g2,g3}	1	
{g1,g2,g3}	1	

Graph-restricted game — ¥ *Centrality*

S	w	$\varphi(w)$	$\varphi(w)-\varphi(v)$
{g1}	0	1/3	1/3
{g2}	0	1/3	-1/6
{g3}	0	1/3	-1/6
{g1,g2}	0		
{g1,g3}	0		
{g2,g3}	**0**		
{g1,g2,g3}	1		

genes 3 a
necessar
and 8, a
level of
betwee