THEORIE DES JEUX ALGORITHMIQUE

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Theorem 1 (Shapley 1953)

There is a unique map ϕ defined on **G**^N that satisfies EFF, SYM, NPP, ADD. Moreover, for any i∈N we have that

$$W_{i}(v) = \frac{1}{n!} \sum_{\dagger \in \Pi} m_{i}^{\dagger}(v)$$

Here Π is the set of all permutations $:N \quad N \text{ of } N,$ while m $_i(v)$ is the marginal contribution of player i according to the permutation $\ ,$ which is defined as:

 $v(\{ (1), (2), \ldots, (j)\}) - v(\{ (1), (2), \ldots, (j-1)\}),$ where j is the unique element of N s.t. i = (j).

Unanimity games (1)

► <u>DEF</u> Let $T \in 2^N \setminus \{\emptyset\}$. The *unanimity game* on T is defined as the TU-game (N,u_T) such that

 $u_{T}(S) = \begin{cases} 1 \text{ if } T \subseteq S \\ 0 \text{ otherwise} \end{cases}$

- Note that the class \mathbf{G}^{N} of all n-person TU-games is a vector space (obvious what we mean for v+w and αv for v, $w \in \mathbf{G}^{N}$ and $\alpha \in IR$).
- \succ the dimension of the vector space \mathbf{G}^{N} is 2^{n} -1
- > {u_T|T∈2^N\{Ø}} is an interesting basis for the vector space **G**^N.

Unanimity games (2)

Every coalitional game (N, v) can be written as a linear combination of unanimity games in a unique way, i.e.,

$$v = \sum_{S \subseteq :N : S \neq \emptyset} s(v) u_S$$
.

The coefficients $_{S}(v)$, for each $S \in 2^{N}$, are called unanimity coefficients of the game (N, v) and are given by the formula: $_{S}(v) = \sum_{T \subseteq S:T \neq \emptyset} (-1)^{s-t} v(T)$.

. <u>EXAMPLE</u> T	wo TU-games v and w on N={1,2,3}
v(1) =3	$\lambda_1(v) = 3 \qquad s(v) = \sum_{T \subseteq S: T \neq \emptyset} (-1)^{s-t} v(T)$
v(2) =4	$\lambda_2(v) = 4$
v(3) = 1	$\lambda_3(v) = 1$
v(1, 2) = 8	$\lambda_{(1,2)}(v) = -3 - 4 + 8 = 1$
v(1, 3) = 4	$\lambda_{1,2}(v) = -3 - 1 + 4 = 0$
v(2, 3) = 6	$\lambda_{\{1,3\}}(v) = 3 + 4 = 0$
v(1, 2, 3) = 10	$\lambda_{\{2,3\}}(V) = -4 - 1 + 0 - 1$
$v = 3u_{\{1\}}(v)$	$\lambda_{\{1,2,3\}}(V) = -3 - 4 - 1 + 8 + 4 + 6 - 10 = 0$ + 4u _{2} (V) + u _{3} (V) + u _{1,2} (V) + u _{2,3}

Sketch of the Proof of Theorem1

- \succ Shapley value satisfies the four properties (easy).
- \blacktriangleright Properties EFF, SYM, NPP determine ϕ on the class of all games v, with v a unanimity game and \in IR.

 \blacktriangleright Let S $\in 2^{N}$. The Shapley value of the unanimity game (N,u_s) is given by

$$\phi_{i}(u_{s}) = \begin{cases} \alpha/|S| & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

otherwise

 \blacktriangleright Since the class of unanimity games is a basis for the vector space, ADD allows to extend ϕ in a unique way to \mathbf{G}^{N} .

An alternative formulation of the Shapley value

- > Let $m'_i(v)=v(\{ '(1), '(2),..., '(j)\})-v(\{ '(1), '(2),..., '(j-1)\}),$ where j is the unique element of N s.t. i = '(j).
- > Let $S = \{ (1), (2), \ldots, (j-1) \}.$
- Q: How many other orderings $\sigma \in \Pi$ do we have in which
 { (1), (2),..., (j-1)}=S and i = '(j)?
- > A: they are precisely $(s)! \times (n-s-1)!$
- ➤ Where s! Is the number of orderings of S and (n-s-1)! Is the number of orderings of N\(S∪{i})
- We can rewrite the formula of the Shapley value as the following:

 $\phi_i(v) = \sum_{S \subseteq N: i \notin S} s!(n-s-1)!/n! [v(S \cup \{i\})-v(S)]$

Power indices: a general formulation (2)

$\psi_i(v) = \sum_{S \subseteq N: i \notin S} p_i(S) \left[v(S \cup \{i\}) - v(S) \right]$

- According to the Banzhaf power index, every coalitions has the same probability to form: p_i(S)=1/(2ⁿ⁻¹), for each S∈2^N\{Ø}, i∉S
- According to the Shapley-Shubick power index, compute p_i(S) according to the following procedure to create at random from N a subset S to which i does not belong:
 - Draw at random a number out of the urn consisting of possible sizes 0,1,2,...,n-1 where each number has probability 1/n to be drawn
 - If size s is chosen, draw a set out of the urn consisting of subsets of N\{i} of size s, where each set has the same probability, i.e.
 1/combinations(n-1,s)
 - > indeed, $p_i(S)=(s! (n-s-1)!)/n!$

UN Security Council as a weighted majority game

In the last sixty years, game theory has been applied to political and social problems to assess the power of interacting agents in forcing a decision

Classical example: UN Security Council: 15 member states, 5 Permanent members (China, France, Russian Federation, United Kingdom, USA) and 10 temporary seats (held for two-year terms)

Decision Rule: decisions on all substantive matters need the positive vote of at least nine Nations but it is sufficient the negative vote of one among the permanent members to reject the decision. UN Security Council as a weighted majority game

- ➤ Let N=P∪T, where P={1,2,3,4,5} is the set of Permanent members and T={6,7,8,9,10,11,12,13,14,15} is the set of temorary seats
- A simple game (N,v) s.t. v(S)=1 if $|S| \ge 9$ and P \subset S.
- (N,v) is a weighted majority game, where
 - \gg w_i=7 for each i \in P
 - \gg w_i=1 for each i \in T
 - $\succ v(S)=1 \iff \sum_{i\in S} w_i > 38$





temporary seats since January 1st 2007 until January 1st 2009

Reformulations

Other axiomatic approaches have been provided for the Shapley value, of which we shall briefly describe those by Young and and by Hart and Mas-Colell.

PROPERTY 8 (Marginalism, MARG) A map $: \mathbb{G}^{\mathbb{N}} \mathbb{R}^{\mathbb{N}}$ satisfies MARG if, given $v, w \in \mathbb{G}^{\mathbb{N}}$, for any player $i \in \mathbb{N}$ s.t. $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for each $S \in 2^{\mathbb{N}}$, the following is true:

$$_{i}(\mathbf{v}) = _{i}(\mathbf{w}).$$

EXAMPLE Two TU-games v and w on N={1,2,3}

$$v(1) = 3$$
 $w(1) = 2$ $w(\emptyset \cup \{3\}) - w(W(\{1\} \cup \{3\})) - w(W(\{1\} \cup \{3\})) - W(W(\{1\} \cup \{3\})) - W(\{1\} \cup \{3\})) - W(\{2\} \cup \{3\}) - W(\{1\} \cup \{3\})) - W(\{1\} \cup \{3\}) - W(\{1\} \cup \{3\}) - W(\{1\} \cup \{3\})) - W(\{1\} \cup \{3\}) - W(\{1\} \cup \{3\}) - W(\{1\} \cup \{3\})) - W(\{1\} \cup \{3\}) -$

$$w(\emptyset \cup \{3\}) - w(\emptyset) = v(\emptyset \cup \{3\}) - v(\emptyset) = 1$$

({1}\\\\{3}) - w({1}) = v({1}\\\\{3}) - v({1}) = 1
w({2}\\\\{3\}) - w(\emptyset) = v({2}\\\\{3\}) - v(\emptyset) = 1

$$w(\{1,2\}\cup\{3\})-w(\{1,2\})=v(\{1,2\}\cup\{3\})-v(\{1,2\})=1$$

$$v(3) = 1$$
 $w(3) = 1$ $v(1, 2) = 8$ $w(1, 2) = 2$ $v(1, 3) = 4$ $w(1, 3) = 3$ $v(2, 3) = 6$ $w(2, 3) = 5$ $v(1, 2, 3) = 10$ $w(1, 2, 3) = 4$

v(1)

 $_{3}(v) = _{3}(w).$

(Young 1988)

Theorem 2

There is a unique map defined on G(N) that satisfies EFF, SYM, and MARG. Such a coincides with the Shapley value.

Potential

- A quite different approach was pursued by Hart and Mas-Colell (1987).
- To each game (N, v) one can associate a real number P(N,v) (or, simply, P(v)), its *potential*.
- > The "partial derivative" of P is defined as

 $D^{i}(P)(N, v) = P(N,v) - P(N \setminus \{i\}, v_{|N \setminus \{i\}})$

Theorem 3 (Hart and Mas-Colell 1987) There is a unique map P, defined on the set of all finite games, that satisfies:

1) $P(\emptyset, v_0) = 0$,

2) $\Sigma_{i \in \mathbb{N}} D^i P(N, v) = v(N).$

Moreover, $D^{i}(P)(N, v) = \phi_{i}(v)$. [$\phi(v)$ is the Shapley value of v]

- there are formulas for the calculation of the potential.
- ≻ For example, $P(N,v) = \sum_{S \in 2^N} \lambda_S / |S|$ (*Harsanyi dividends*)



 $\phi_1(v) = P(\{1,2,3\},v) - P(\{2,3\},v_{|\{2,3\}}) = 9 - 11/2 = 7/2$ $\phi_2(v) = P(\{1,2,3\},v) - P(\{1,3\},v_{|\{1,3\}}) = 9 - 4 = 5$ $\phi_3(v) = P(\{1,2,3\},v) - P(\{1,2\},v_{|\{1,2\}}) = 9 - 15/2 = 3/2$

Communication networks

- A cooperative game describes a situation in which all players can freely communicate with each other.
- Drop this assumption and assume that communication between players is restricted to a set of communication possibilities between players.

Communication networks as undirected graphs:

An *undirected graph* is a pair (N,L) where

- N is a set of vertices (later, agents or players)
- L={ {i,j} | {i,j}⊆N, i≠j } is the set of edges (bilateral communication links)
- A communication graph (N,L) should be interpreted as a way to model restricted cooperation:
 - Players can cooperate with each other if they are connected (*directly*, or *indirectly* via a path)
 - Indirect communication between two players requires the cooperation of players on a connecting path.

Example

Consider the undirected graph (N,L) with N= $\{1,2,3,4,5,6,7\}$ and L= $\{\{1,2\}, \{2,6\}, \{5,6\}, \{1,5\}, \{3,7\}, \{4,7\}\}$



Some notations:

N\L={{1,2,5,6},{3,4,7}} set of components

 $L_{-2} = \{\{5,6\}, \{1,5\}, \{3,7\}, \{4,7\}\}$ $N \setminus L_{-2} = \{\{1,5,6\}, \{3,4,7\}, \{2\}\}$

Communication situations (Myerson (1977))

- A communication situation is a triple (N,v,L)
 - (N,v) is a n-person TU-game (represents the economic possibilities of coalitions)
 - (N,L) is a communication network (represents restricted communication possibilities)
- The graph-restricted game (N,v^L) is defined as

$$v^{L}(T) = \sum_{C \in T \setminus L} v(C)$$

For each $S \in 2^{\mathbb{N}} \setminus \{\emptyset\}$.

Recall that T\L is the set of maximal connected components in the restriction of graph (N.L) to T

Example

A weighted majority game ($\{1,2,3\},v$) with the winning quote fixed to 2/3 is considered. The votes of players 1, 2, and 3 are, respectively, 40%, 20%, and 40%. Then, v(1,3)=v(1,2,3)=1 and v(S)=0 for the remaining colitions.

The communication network is

Then, $v^{L}(1,2,3)=1$, and $v^{L}(S)=0$ for the other coalitions.

Solutions for communication situations

- Myerson (1977) was the first to study solutions for communication situations.
- A solution is a map defined for each communication situation (N,v,L)
 with value in R^{N.}

PROPERTY 1 Component Efficiency (CE)

For each communication situation (N,v,L) and all $C \in N \setminus L$ it holds that

$$\sum_{i\in C} i(N,v,L) = v(C).$$

 Property 1 is an "efficiency" condition that is assumed to hold only for those coalitions whose players are able to communicate effectively among them and *are not connected to other players*. (maximal connected components) <u>PROPERTY 2</u> Fairness (F) For each communication situation (N,v,L) and all $\{i,j\} \in L$ it holds that $_{i}(N,v,L) - _{i}(N,v,L \setminus \{\{i,j\}\}) = _{j}(N,v,L) - _{j}(N,v,L \setminus \{\{i,j\}\}).$

Property 2 says that two players should gain or lose in exactly the same way, when a direct link between them is established (or deleted).

Myerson value

Theorem (Myerson (1977))

There exists a unique solution $\mu(N,v,L)$ which satisfies CE and F on the class of communication situations. Moreover,

 $\mu(N,v,L) = \phi(v^L)$

where $\phi(v^L)$ is the Shapley value of the graph-restricted game v^L .

Example

A weighted majority game ($\{1,2,3\},v$) with the winning quote fixed to 2/3 is considered. The votes of players 1, 2, and 3 are, respectively, 40%, 20%, and 40%. Then, v(1,3)=v(1,2,3)=1 and v(S)=0 for the remaining colitions.

The communication network is

Then, $v^{L}(1,2,3)=1$, and $v^{L}(S)=0$ for the other coalitions. We have that $\phi(y)=(1/2,0,1/2)$ and $\psi(N|y|L)=\phi(yL)=(1/2,1/2,1/2)$

 $\phi(v)=(1/2,0,1/2)$ and $\mu(N,v,L)=\phi(v^L)=(1/3,1/3,1/3)$.

Example

(N,v,L) communication situation such that L is the following network and $v=u_{\{2,4\}}$ 2 1

3

4

Note that, for instance, $v^{L}(2,4)=v(2)+v(4)=0$.

Easy to note that that $v^{L}=u_{\{1,2,4\}}+u_{\{2,3,4\}}-u_{N}$

Therefore,

 $\mu(N,v,L) = \phi(vL) = (1/3,2/3,1/3,2/3) - (1/4,1/4,1/4,1/4)$ = (1/12,5/12,1/12,5/12)



Section 2. Connection situations

- A connection situation takes place in the presence of a group of agents N={1,2, ...,n}, each of which needs to be connected directly or via other agents to a source.
- If connections among agents are costly, then each agent will evaluate the opportunity of cooperating with other agents in order to reduce costs.
- If a group of agents decides to cooperate, a configuration of links which minimizes the total cost of connection is provided by a minimum cost spanning tree (mcst).
- The problem of finding a mcst may be easily solved thanks to different algorithms proposed in literature (Boruvka (1926), Kruskal (1956), Prim (1957), Dijkstra (1959))

Minimum Cost Spanning Tree Situation

Consider a complete weighted graph

- whose vertices represent agents
- vertex 0 is the source
- edges represent connections between agents or between an agent and the source
- numbers close to edges are connection costs



Optimization problem:

How to connect each node to the source 0 in such a way that the cost of construction of a <u>spanning network</u> (which connects every node directly or indirectly to the source 0) <u>is</u> <u>minimum</u>? **Example**: The cost game ({1,2,3},c) is defined on the following connection situation:



The game ({1,2,3}, c) is said mcst game (Bird (1976))

How to divide the total cost? (Bird 1976)



- The predecessor of 1 is 0: the Bird allocation gives to player 1 the cost of {0,1}.
- •The predecessor of 2 is 1: the Bird allocation gives to player 2 the cost of {1,2};
- The predecessor of 3 is 1: the Bird allocation gives to player 3 the cost of {1,3}.

w(**Γ**)=52

Bird allocation w.r.t. to Γ , $(x_1, x_2, x_3) = (24, 18, 10)$ is in the core of ({1,2,3},c).





The Bird allocation w.r.t .this most is

The Bird allocation w.r.t. this most is

 $(x_1, x_2, x_3) = (24, 18, 10)$

 $(x_1, x_2, x_3) = (18, 24, 10)$

Both allocations belong to the core of the mcst game (and also their convex combination).



- It always provides an allocation (given a connection situation).
- In general, not a unique allocation (each mcst determines a corresponding Bird allocation...).
- Bird allocations are in the core of mcst games (but are extreme points)

What happens when the structure of the network changes?

Imagine to use a certain rule to allocate costs.

- The cost of edges may increase: if the cost of an edge increases, nobody should be better off, according to such a rule (*cost monotonicity*);

 One or more players may leave the connection situation: nobody of the remaining players should be better off (*population monotonicity*).

Cost monotonicity: Bird allocation behaviour



Bird allocation: (4, 3, 3)

Bird allocation: (3, 5, 3)

Bird rule does not satisfy cost monotonicity.

Population monotonicity: Bird allocation behaviour



Bird allocation: (5, 5, 3) Bird allocation: (3, *, 6)

Bird rule does not satisfy population monotonicity

Construct & Charge rules

Are based on the following general cost allocation protocol:

- As soon as a link is constructed in the Kruskal algorithm procedure:
 - 1) it must be totally charged among agents which are not yet connected with the source (*connection property*)
 - 2) Only agents that are on some path containing the new edge may be charged (*involvement property*)
- when the construction of a mcst is completed, each agent has been charged for a total amount of fractions equal to 1 (*total aggregation property*).

P-value: Feltkamp (1994), Branzei et al. (2004), Moretti (2008)





Make the sum of all edge-by-edge allocations:

(0, 0, 0) +(5, 0, 5) +(3,12,3) +(0, 0, 0) +(8, 8, 8) =

P-value = (16, 20, 16)

Algorithm to calculate the P-value

IDEA: charge the cost of an edge constructed during the Kruskal algorithm only between agents involved, keeping into account the cardinality of the connected components at that step and at the previous step of the algorithm:

- At any step of the Kruskal algorithm where a component is connected to some agents, charge the cost of that edge among these agents in the following way:
 - Proportionally to the cardinality_current_step⁻¹ if an agent is connected to a component which is connected to the source,
 - Otherwise, charge it proportionally to the difference: cardinality_previous_step⁻¹ - cardinality_current_step⁻¹

P-value

Always provides a unique allocation (given a most situation).

- >It is in the core of the corresponding mcst game.
- Satisfies cost monotonicity.

Satisfies population monotonicity.

>on a subclass of connection problems it coincides with the Shapley value of mcst games

Axiomatic characterization (4 independent axioms)

A solution for most situations $F: W^{N'} \to \Re^{N}$

Property 1. The solution *F* is *efficient* (EFF) if for each $w \in W^{N'}$

$$\sum_{i\in N} F_i(w) = w(\Gamma),$$

where Γ is a minimum cost spanning network on *N*'.

Example:

Property 2. The solution *F* has the *Equal Treatment* (ET) property if for each $w \in W^{N'}$ and for each $i, j \in N$ with $C_i(w) = C_j(w)$

$$F_i(w) = F_j(w).$$

Example:

 $P(w) = (2, 2, 6)^{t}$

Property 3. The solution *F* has the *Upper Bounded Contribution* (UBC) property if for each $w \in W^{N'}$ and every (w,N')-component $C \neq \{0\}$

$$\sum_{i \in C \setminus \{0\}} F_i(w) \le \min_{i \in C \setminus \{0\}} w(\{i, 0\})$$

Example:

$$P(w) = (0, 4, 6)^{t}$$

Note that 1 is dummy in the corresponding mcst game **Property 4**. The solution *F* has the *Cone-wise Positive Linearity* (CPL) property if for each $\dagger \in \Sigma_{E_N}$, for each pair of most situations $w, \hat{w} \in K^{\dagger}$ and for each pair $r, \hat{r} \ge 0$, we have

Theorem 1. The P-value is the unique solution which satisfies the properties EFF, ET, UBC and CPL on the class $W^{N'}$ of mcst situations.

It is possible to prove that the P-value satisfies the four properties EFF, ET, UBC and CPL.

To prove the uniqueness consider a solution for mcst situation F which satisfies EFF, ET, UBC and CPL:

•first look at the simple mcst situations (0-1 cost of edges): on such simple situation, EFF, ET and UBC imply F=P-value;

it is possible to decompose each mcst situation as a linear combination of simple mcst problems;

by CPL it follows that the F=P-value on each mcst situation.

Genes interaction and centrality

- Classical centrality measures are appropriate under the assumption that nodes failures occur independently...
- ...and the system is sensible to the failure of each single node.
- On the contrary, in biological complex networks, assuming that the failure of the nodes (genes/ proteins) is independent is not realistic and the consequence on the system can be appreciated only if many nodes fail.

Jeong, Mason, Barabasi, Oltvai. Lethality and centrality in protein networks. Nature 2001;411:41-42.

Lethality and centrality in protein networks

The most highly connected proteins in the cell are the most important for its survival.

roteins are traditionally identified on the basis of their individual actions as catalysts, signalling molecules, or building blocks in cells and microorganisms. But our post-genomic view is expanding the protein's role into an element in a network of protein-protein interactions as well, in which it has a contextual or cellular function within functional modules^{1,2}. Here we provide quantitative support for this idea by demonstrating that the phenotypic consequence of a single gene deletion in the yeast Saccharomyces cerevisiae is affected to a large extent by the topological position of its protein product in the complex hierarchical web of molecular interactions.

The *S. cerevisiae* protein–protein interaction network we investigate has 1,870 proteins as nodes, connected by 2,240 identified direct physical interactions, and is derived from combined, non-overlapping data^{3,4}, obtained mostly by systematic twohybrid analyses³. Owing to its size, a complete map of the network (Fig. 1a), although informative, in itself offers little insight into its large-scale characteristics.

Figure 1 Characteristics of the yeast proteome. **a**, Map of protein–protein interactions. The largest cluster, which contains -78% of all proteins, is shown. The colour of a node signifies the phenotypic effect of removing the corresponding protein (red, lethal; green, non-lethal; orange, slow growth; yellow, unknown). **b**, Connectivity distribution *P*(*k*) of interacting yeast proteins, giving the probability that a

Addressed questions

Q1: can we use the same game theoretic models to measure the relevance of interacting genes (or proteins) in response to certain biological conditions?

Q2: how to combine the information provided by biological networks with the attribution of relevance, keeping into account the interaction of genes?

The ingredients of the model

Players are genes

- A decision rule is created on the data-set to establish which groups (coalitions) of genes are winning.
- Example: First we define a criterion to establish which genes are abnormally expressed on each array

A group of genes is *winning* on a single array if all genes that have abnormal expressions belong to that group

	array1
gene1	0
gene2	1
gene3	1

Both groups {gene2, gene3} and group {gene1, gene2, gene3} are winning.

coalition {gene2, gene3} is winning two times out of three;
coalition {gene1, gene2} is winning one time out of three;

•And so on for each coalition...

Example

The corresponding *microarray game* $<\{g_1, g_2, g_3\}, v>$ such that $v(\emptyset) = v(\{g_1\}) = v(\{g_2\}) = 0$ $v(\{g_1, g_3\}) = v(\{g_1, g_2\}) = v(\{g_3\}) = 1/3$ $v(\{g_2, g_3\}) = 2/3$ $v(\{g_1, g_2, g_3\}) = 1.$

We look for an index which is able to resume all the information about coalitions in a single attribution of relevance for each individual gene

Axioms for a relevance index on microarray games

Property 1: Null Gene (NG)

A gene which does not contribute to change the worth of any coalition of genes, should receive zero power.

Prop.2:Equal Splitting (ES)

Each sample should receive the same level of reliability. So the power of a gene on two samples should be equal to the sum of the power on each sample divided by two.

A group of genes S such that does not exist a proper (\subset) subset of S which contributes in changing the worth of genes outside S.

Example

These two sets are partnerships of genes in the corresponding Microarray game

		s1	s2	s3
	g1	0	1	1
	g2	0	1	1
/	•g3)	1	0	1

Property 3: Partnership Monotonicity (PM)

(N,v) a microarray game. If two partnerships of genes S and T, with $|T| \ge |S|$ are such that they are *-disjoint* (S \cap T=Ø),

-equivalent (v(S)=v(T))

-*exhaustive* (v(S \cup T)=v(N)),

then genes in the smaller *partnership* S must receive more relevance then genes in T.

Example

		S ₁	S ₂
Ψ_1	g1	0	1
Ψ_2	g2	0	1
Ψ_3	g3	1	0
Ψ_4	g4	1	0
Ψ_5	g5	1	0

 $\psi_i \quad \psi_k$

For each $i \in \{1,2\}$ $k \in \{3,4,5\}$ Other properties concerning partnership

Property 4: Partnership Rationality (PR) The total amount of power index received from players of a partnership S should not be smaller than v(S)

Property 5: Partnership Feasibility (PF) The total amount of power index received from players of a partnership S should not be greater than v(N)

Theorem (Moretti, Patrone, Bonassi (2007)):

The Shapley value is the unique solution which satisfies NG, ES, PM, PR, PF on the class of microarray games.

Players gather one by one in a room to create the "grand coalition", and each one who enters gets his marginal contribution.

Assuming that all the different orders in which they enter are equiprobable, the Shapley value gives to each player her/his expected payoff.

BMC Bioinformatics

Methodology article

Combining Shapley value and statistics to the analysis of gene expression data in children exposed to air pollution

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Identification of low intratumoral gene expression heterogeneity in doi:10.1186/147 neuroblastic tumors by genome-wide expression analysis and game theory^T

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Example

	Array1	Array2	Array3
g_1	0	1	0
g ₂	1	1	0
g ₃	1	0	1

The corresponding *microarray game* $\{g_1, g_2, g_3\}, v > such that$ $v(\emptyset) = v(\{g_1\}) = v(\{g_2\}) = 0$ $v(\{g_1, g_3\}) = v(\{g1, g2\}) = v(\{g_3\}) = 1/3$ $v(\{g_2, g_3\}) = 2/3$ $v(\{g_1, g_2, g_3\}) = 1.$

The Shapley value is

$$Sh_{g1}=1/6$$
 $Sh_{g2}=1/3$ $Sh_{g3}=1/2$

Data		
44	array1	
gl	0	
g2	1	
g3	1	

Restrict the interaction possibilities

A priori game Graph-restricted g		d game	X Centrality			
S	v	φ(ν)	S	w	φ(w)	φ(w)-φ(v)
{g1}	0	0	{g1}	0	1/3	1/3
{g2}	0	1/2	{g2}	0	1/3	-1/6
{g3}	0	1/2	{g3}	0	1/3	-1/6
{g1,g2}	0		{g1,g2}	0		
{g1,g3}	0		{g1,g3}	0		
{g2,g3}	1		{g2,g3}	0		
{g1,g2,g3}	1		{g1,g2,g3}	1		

all genes between 2 and 7 are
needed to connect the abnormally
expressed genes 1 and 8, γ centrality
behaves similar to degree centrality

node	Х	degree	Betws.
1	-0.38	1	0
2	0.13	2	6
3	0.13	2	10
4	0.13	2	12
5	0.13	2	12
6	0.13	2	10
7	0.13	2	6
8	-0.38	1	0

genes 3 and 5 are intermediary genes not necessary to connect abnormal genes 1 and 8, and therefore they receive a null level of centrality both from x and betweenness centrality

node	Х	degree	Betws.
1	-0.33	1	0
2	0.17	3	6
3	0	2	0
4	0.17	4	12
5	0	2	0
6	0.17	3	10
7	0.17	2	6
8	-0.33	1	0