THEORIE DES JEUX ALGORITHMIQUE

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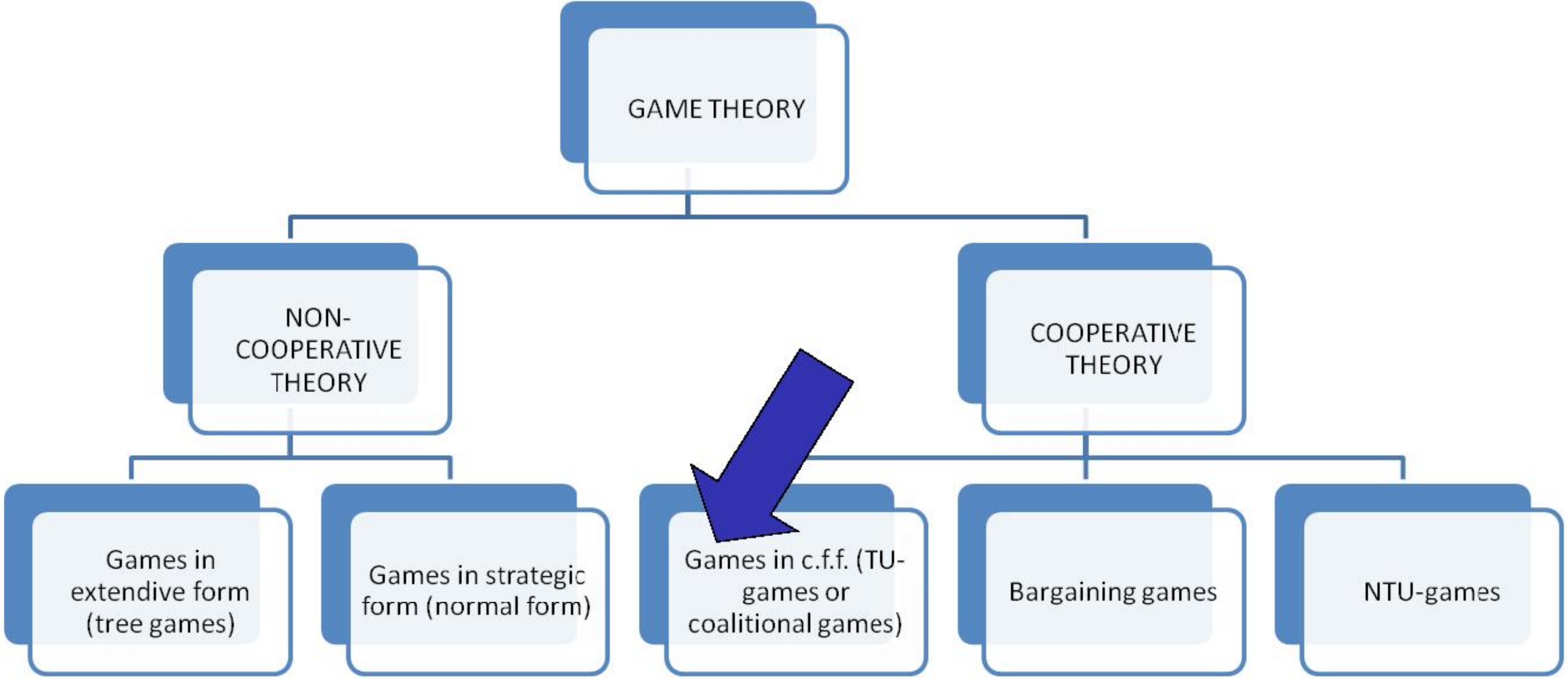


- Dominant strategies
- Nash eq. (NE)
- Subgame perfect NE
- NE & refinements

…

- Core
- Shapley value
- Nucleolus
- τ-value
- PMAS

….

- Nash sol.
- Kalai-Smorodinsky

….

- CORE
- NTU-value
- Compromise value

…

No binding agreements
No side payments
Q: Optimal behaviour in conflict situations

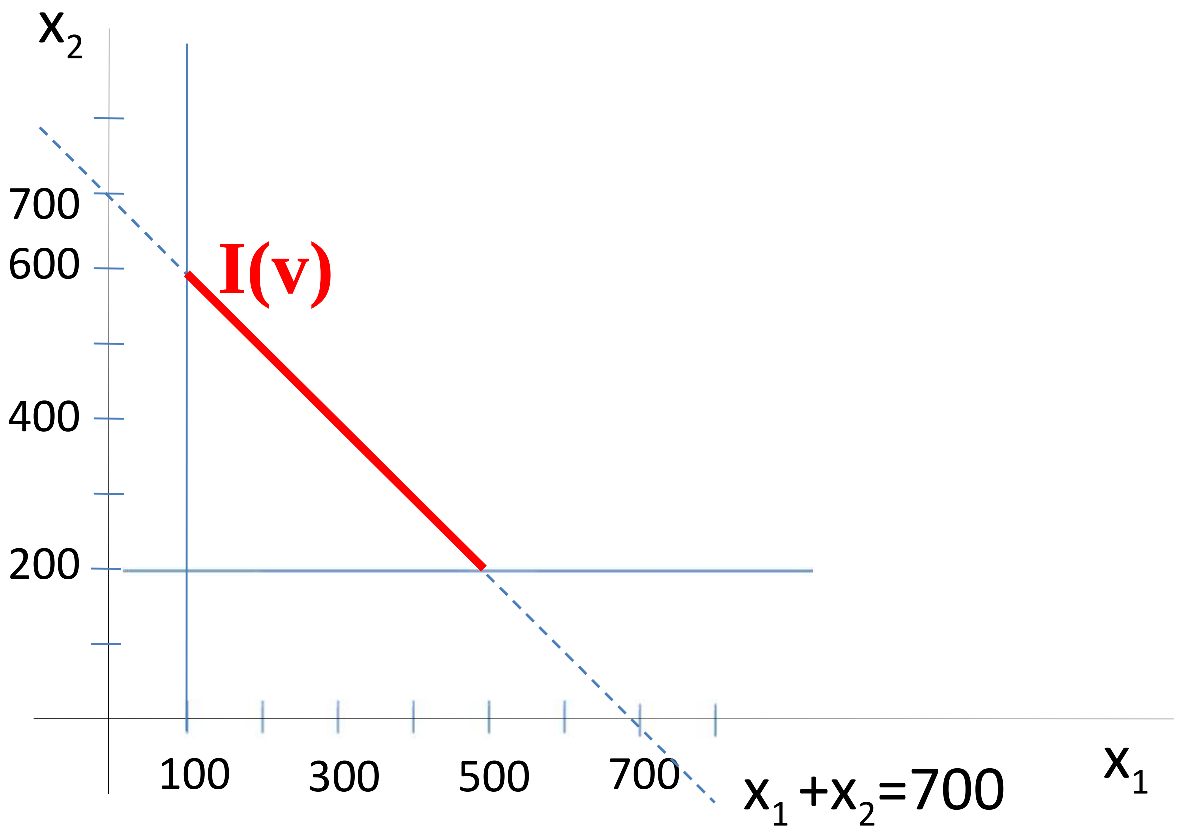
binding agreements
side payments are possible (sometimes)
Q: Reasonable (cost, reward)-sharing

Cooperative games: a simple example

Alone, player 1 (singer) and 2 (pianist) can earn 100€ 200€ respect.

Together (duo) 700€

How to divide the (extra) earnings?



Imputation set: $I(v)=\{x \in IR^2 | x_1 \geq 100,\ x_2 \geq 200,\ x_1 + x_2 = 700\}$

COOPERATIVE GAME THEORY

Games in coalitional form

TU-game: (N,v) or v

$N=\{1, 2, \ldots, n\}$	set of players
$S\subset N$	coalition
2^N	set of coalitions

DEF. $v: 2^N \rightarrow \mathbb{R}$ with $v(\emptyset)=0$ is a Transferable Utility (TU)-game with player set N.

NB: $(N,v)\leftrightarrow v$

NB2: if $n=|N|$, it is also called n-person TU-game, game in coalitional form, coalitional game, cooperative game with side payments...

v(S) is the value (worth) of coalition S

Example

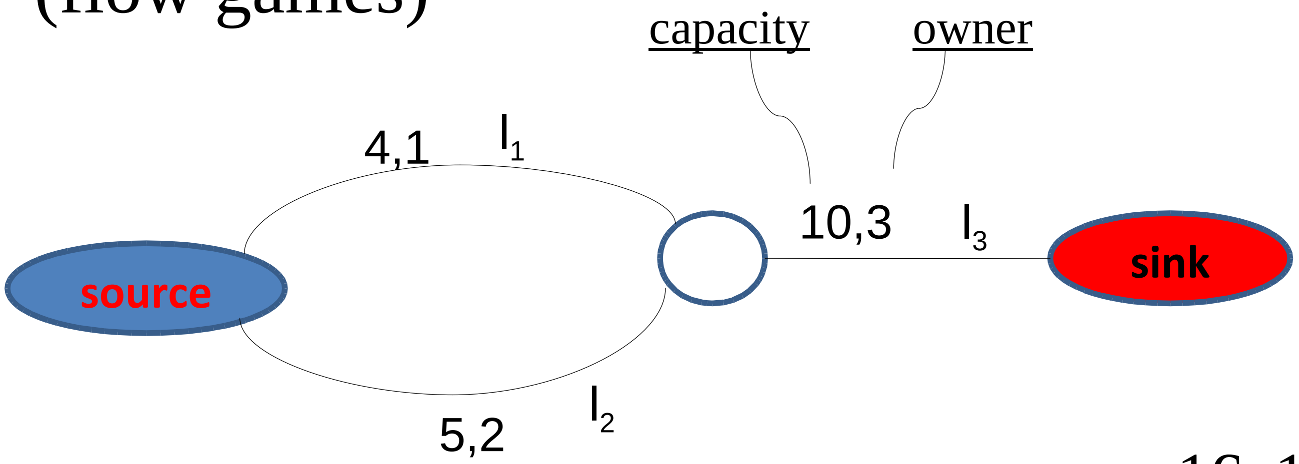
(Glove game) $N=L\cup R$, $L\cap R=\emptyset$

$i\in L$ ($i\in R$) possesses 1 left (right) hand glove

Value of a pair: 1€

$v(S)=\min\{|L\cap S|, |R\cap S|\}$ for each coalition $S\in 2^N\setminus\{\emptyset\}$.

Example
(flow games)



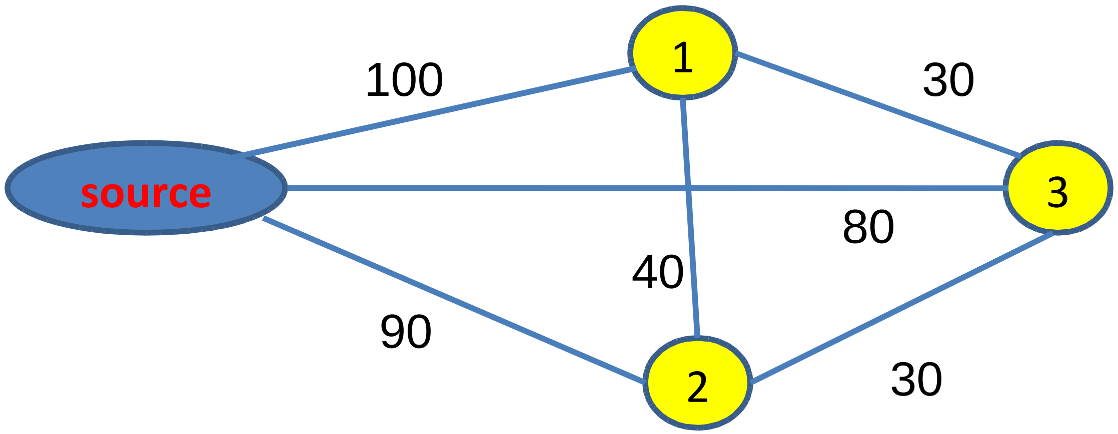
N={1,2,3}

1€: 1 unit source → sink

S=	∅	{1}	{2}	{3}	{1,2}	{1.3}	{2,3}	{1,2,3}
v(S)	0	0	0	0	0	4	5	9

Example

(Three cooperating communities)



$N=\{1,2,3\}$

S=	Ą	{1}	{2}	{3}	{1,2}	{1.3}	{2,3}	{1,2,3}
c(S)	0	100	90	80	130	110	110	140
v(S)	0	0	0	0	60	70	60	130

$$v(S)=\Sigma_{i\in S}c(i) - c(S)$$

DEF. (N,v) is a <u>superadditive game</u> iff

$v(S \cup T) \geq v(S)+v(T)$ for all S,T with $S \cap T = \emptyset$

Q.1: which coalitions form?

Q.2: If the grand coalition N forms, how to divide v(N)?
(how to allocate costs?)

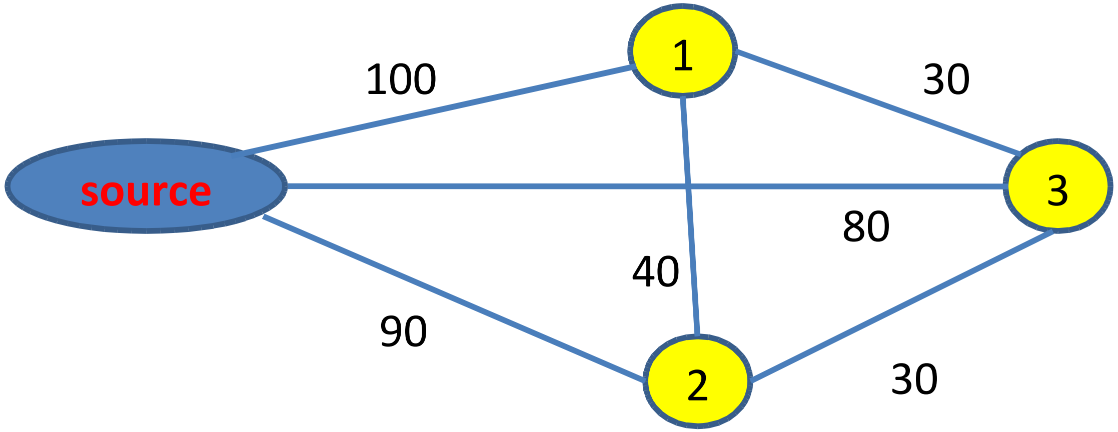
Many answers! (solution concepts)

One-point concepts:
- Shapley value (Shapley 1953)
- nucleolus (Schmeidler 1969)
- τ-value (Tijs, 1981)
- …

Subset concepts:
- Core (Gillies, 1954)
- stable sets (von Neumann, Morgenstern, ’44)
- kernel (Davis, Maschler)
- bargaining set (Aumann, Maschler)
- …..

Example

(Three cooperating communities)



$N=\{1,2,3\}$

S=	∅	{1}	{2}	{3}	{1,2}	{1.3}	{2,3}	{1,2,3}
c(S)	0	100	90	80	130	110	110	140
v(S)	0	0	0	0	60	70	60	130

$$v(S)=\Sigma_{i\in S}c(i) - c(S)$$

Show that v is superadditive and c is subadditive.

Claim 1: (N,v) is superadditive

We show that $v(S\cup T)\geq v(S)+v(T)$ for all $S,T\in 2^N\backslash\{\emptyset\}$ with $S\cap T=\emptyset$

60=v(1,2)≥v(1)+v(2)=0+0

70=v(1,3)≥v(1)+v(3)=0+0

60=v(2,3)≥v(2)+v(3)=0+0

60=v(1,2)≥v(1)+v(2)=0+0

130=v(1,2,3) ≥v(1)+v(2,3)=0+60

130=v(1,2,3) ≥v(2)+v(1,3)=0+70

130=v(1,2,3) ≥v(3)+v(1,2)=0+60

Claim 2: (N,c) is *subadditive*

We show that $c(S\cup T)\leq c(S)+c(T)$ for all $S,T\in 2^N\backslash\{\emptyset\}$ with $S\cap T=\emptyset$

130=c(1,2) ≤ c(1)+c(2)=100+90

110=c(2,3) ≤ c(2)+v(3)=100+80

110=c(1,2) ≤ c(1)+v(2)=90+80

140=c(1,2,3) ≤ c(1)+c(2,3)=100+110

140=c(1,2,3) ≤ c(2)+c(1,3)=90+110

140=c(1,2,3) ≤ c(3)+c(1,2)=80+130

Example

(Glove game) (N,v) such that $N=L\cup R$, $L\cap R=\emptyset$

$v(S)=\min\{|L\cap S|, |R\cap S|\}$ for all $S\in 2N\backslash\{\emptyset\}$

Claim: the glove game is superadditive.

Suppose $S,T\in 2^N\backslash\{\emptyset\}$ with $S\cap T=\emptyset$. Then

$v(S)+v(T)= \min\{|L\cap S|, |R\cap S|\} + \min\{|L\cap T|, |R\cap T|\}$

$=\min\{|L\cap S|+|L\cap T|,|L\cap S|+|R\cap T|,|R\cap S|+|L\cap T|,|R\cap S|+|R\cap T|\}$

$\leq\min\{|L\cap S|+|L\cap T|, |R\cap S|+|R\cap T|\}$

since $S\cap T=\emptyset$

$=\min\{|L\cap(S \cup T)|, |R \cap (S\cup T)|\}$

$=v(S \cup T)$.

The imputation set

DEF. Let (N,v) be a n-persons TU-game.

A vector $x=(x_1, x_2, \ldots, x_n) \in IR^N$ is called an imputation iff

(1) x is individual rational i.e.

$$x_i \geq v(i) \text{ for all } i \in N$$

(2) x is efficient

$$\Sigma_{i \in N} x_i = v(N)$$

[interpretation x_i: payoff to player i]

$$I(v)=\{x \in IR^N \mid \Sigma_{i \in N} x_i = v(N),\ x_i \geq v(i) \text{ for all } i \in N\}$$

Set of imputations

Example

(Glove game) (N,v) such that $N=L\cup R$, $L\cap R=\emptyset$

$v(S)=\min\{|L\cap S|, |R\cap S|\}$ for all $S\in 2N\backslash\{\emptyset\}$

Claim: the glove game is superadditive.

Suppose $S,T\in 2^N\backslash\{\emptyset\}$ with $S\cap T=\emptyset$. Then

$v(S)+v(T)= \min\{|L\cap S|, |R\cap S|\} + \min\{|L\cap T|, |R\cap T|\}$

$=\min\{|L\cap S|+|L\cap T|,|L\cap S|+|R\cap T|,|R\cap S|+|L\cap T|,|R\cap S|+|R\cap T|\}$

$\leq\min\{|L\cap S|+|L\cap T|, |R\cap S|+|R\cap T|\}$

since $S\cap T=\emptyset$

$=\min\{|L\cap(S \cup T)|, |R \cap (S\cup T)|\}$

$=v(S \cup T)$.

Example

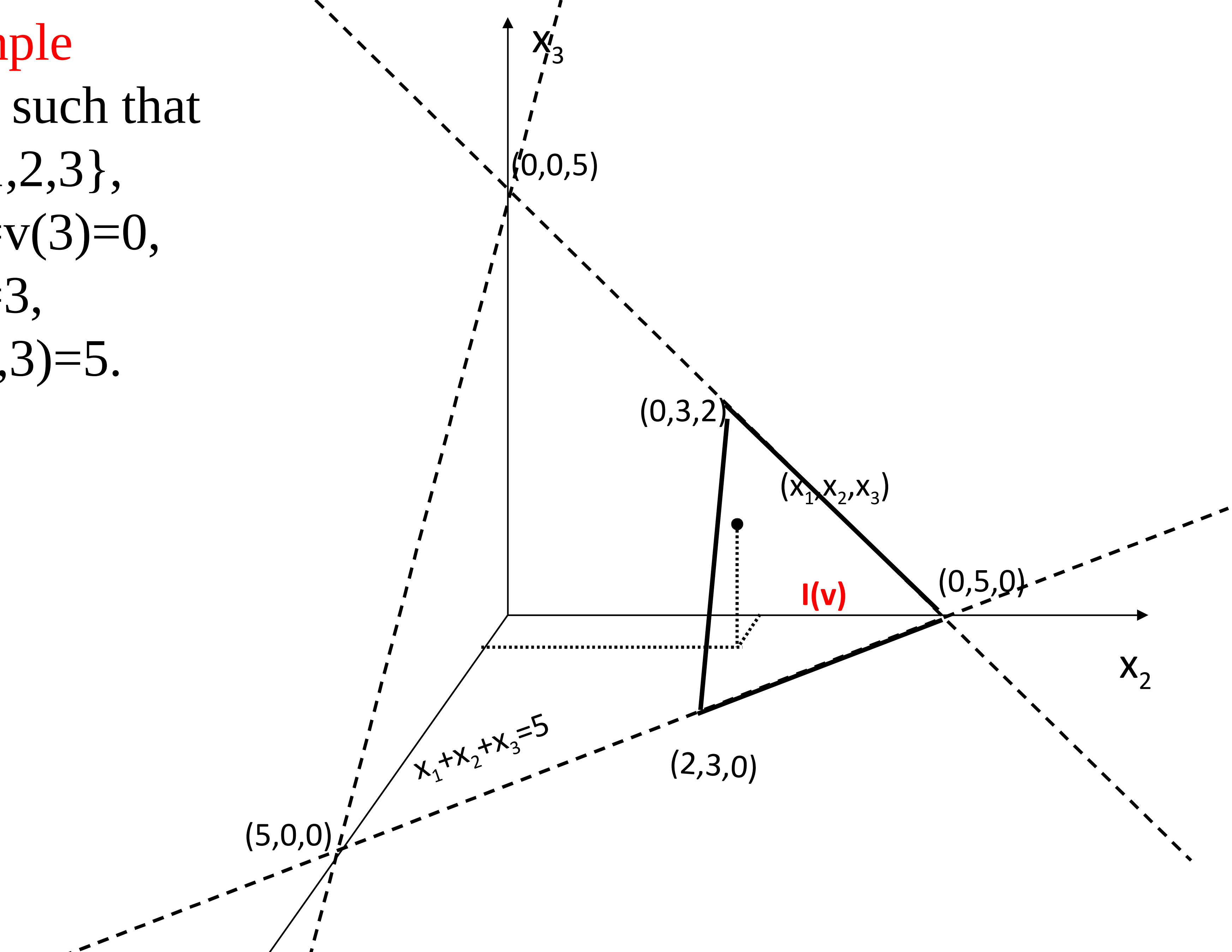
(N,v) such that

N={1,2,3},

v(1)=v(3)=0,

v(2)=3,

v(1,2,3)=5.



$I(v)=\{x$ČIR$^3 \mid x_1,x_3$ñ$0,\ x_2$ñ$3,\ x_1+x_2+x_3=5\}$

Claim: (N,v) a n-person (n=|N|) TU-game. Then

$$I(v)\neq\emptyset \qquad \Leftrightarrow v(N)\geq\sum_{i\in N}v(i)$$

Proof

($\Rightarrow$)

Suppose $x\in I(v)$. Then

$$v(N) \quad = \quad \sum_{i\in N}x_i \quad \geq \quad \sum_{i\in N}v(i)$$

EFF IR

($\Leftarrow$)

Suppose $v(N)\geq\sum_{i\in N}v(i)$. Then the vector

$$(v(1), v(2), \ldots, v(n-1), \underbrace{v(N)- \sum_{i\in\{1,2,\ldots,n-1\}}v(i)}_{\geq v(n)})$$

is an imputation.

The core of a game

DEF. Let (N,v) be a TU-game. The core C(v) of (N,v) is the set

$C(v)=\{x\in I(v) \mid \Sigma_{i\in S}\, x_i \geq v(S)$ for all $S\in 2N\backslash\{\varnothing\}\}$

stability conditions

no coalition S has the incentive to split off if x is proposed

Note: $x \in C(v)$ iff

(1) $\Sigma_{i\in N}\, x_i = v(N)$ *efficiency*

(2) $\Sigma_{i\in S}\, x_i \geq v(S)$ for all $S\in 2N\backslash\{\varnothing\}$ *stability*

Bad news: C(v) can be empty

Good news: many interesting classes of games have a non-empty core.

Example

(N,v) such that

$N=\{1,2,3\}$,

$v(1)=v(3)=0$,

$v(2)=3$,

$v(1,2)=3$,

$v(1,3)=1$

$v(2,3)=4$

$v(1,2,3)=5$.

Core elements satisfy the following conditions:

$x_1, x_3 \geq 0,\ x_2 \geq 3,\ x_1+x_2+x_3=5$

$x_1+x_2 \geq 3,\ x_1+x_3 \geq 1,\ x_2+x_3 \geq 4$

We have that

$5-x_3 \geq 3 \Leftrightarrow x_3 \leq 2$

$5-x_2 \geq 1 \Leftrightarrow x_2 \leq 4$

$5-x_1 \geq 4 \Leftrightarrow x_1 \leq 1$

$$C(v)=\{x \in IR^3 \mid 1 \geq x_1 \geq 0, 2 \geq x_3 \geq 0,\ 4 \geq x_2 \geq 3,\ x_1+x_2+x_3=5\}$$

Example

(N,v) such that

N={1,2,3},

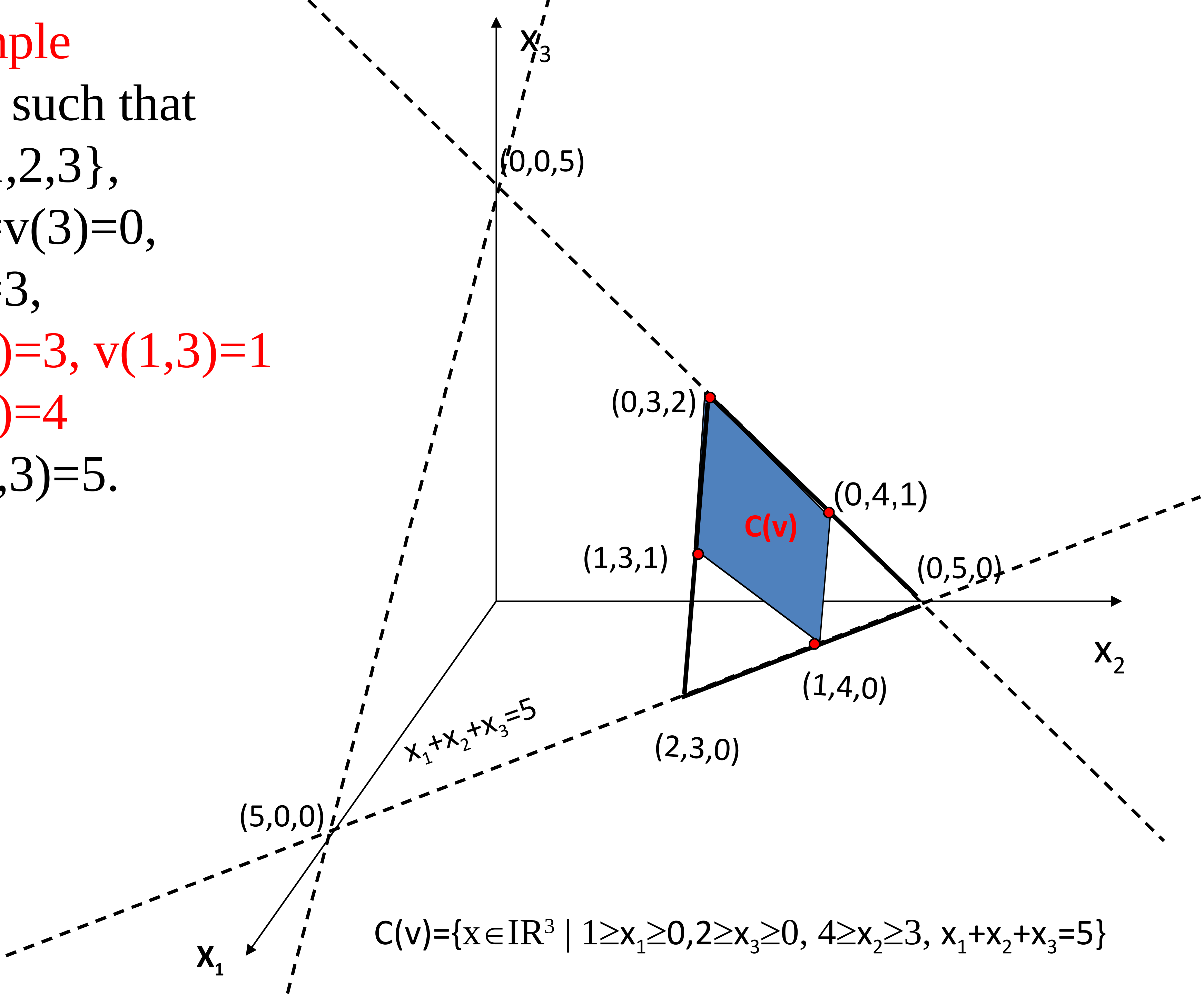
v(1)=v(3)=0,

v(2)=3,

v(1,2)=3, v(1,3)=1

v(2,3)=4

v(1,2,3)=5.



$C(v)=\{x \in IR^3 \mid 1 \geq x_1 \geq 0, 2 \geq x_3 \geq 0, 4 \geq x_2 \geq 3, x_1+x_2+x_3=5\}$

Example (Game of pirates) Three pirates 1,2, and 3. On the other side of the river there is a treasure (10€). At least two pirates are needed to wade the river…

(N,v), N={1,2,3}, v(1)=v(2)=v(3)=0,

v(1,2)=v(1,3)=v(2,3)=v(1,2,3)=10

Suppose $(x_1, x_2, x_3) \in C(v)$. Then

efficiency $x_1 + x_2 + x_3 = 10$

stability
$$\begin{cases} x_1 + x_2 \geq 10 \\ x_1 + x_3 \geq 10 \\ x_2 + x_3 \geq 10 \end{cases}$$

$$20 = 2(x_1 + x_2 + x_3) \geq 30$$ Impossible. So $C(v) = \emptyset$.

Note that (N,v) is superadditive.

Example

(Glove game with $L=\{1,2\}$, $R=\{3\}$)

$v(1,3)=v(2,3)=v(1,2,3)=1$, $v(S)=0$ otherwise

Suppose $(x_1, x_2, x_3) \in C(v)$. Then

$x_1+ x_2+ x_3=1$

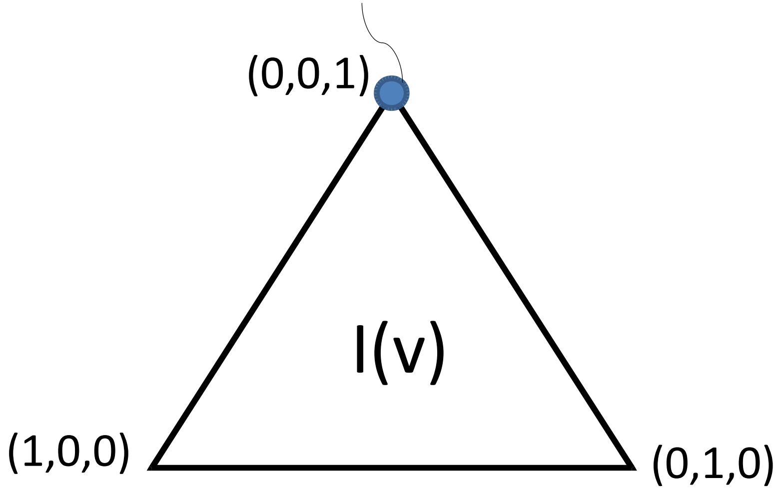
$x_1+x_3 \geq 1$

$x_2 \geq 0$

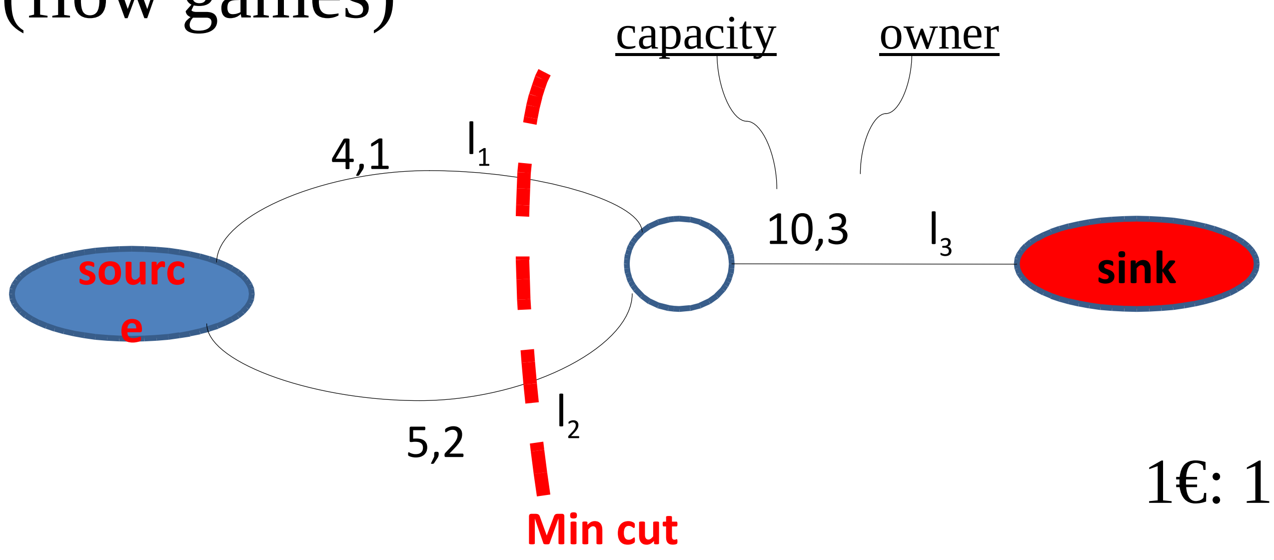
$\Rightarrow$ $x_2=0$, $x_1+x_3 =1$

$x_2+ x_3 \geq 1$ $\Rightarrow$ $x_1=0$ and $x_3=1$

So $C(v)=\{(0,0,1)\}$.



Example
(flow games)



N={1,2,3}

1€: 1 unit source → sink

S=	A	{1}	{2}	{3}	{1,2}	{1.3}	{2,3}	{1,2,3}
v(S)	0	0	0	0	0	4	5	9

Min cut $\{l_1, l_2\}$. Corresponding core element (4,5,0)

Convex games (1)

DEF. An n-persons TU-game (N,v) is convex iff

$$v(S)+v(T)\leq v(S \cup T)+v(S\cap T) \quad \text{for each } S,T\in 2^N.$$

This condition is also known as *supermodularity*. It can be rewritten as

$$v(T)-v(S\cap T)\leq v(S \cup T)-v(S) \quad \text{for each } S,T\in 2^N$$

For each $S,T\in 2^N$, let $C=(S\cup T)\setminus S$. Then we have:

$$v\big(C\cup(S\cap T)\big)-v(S\cap T)\leq v(C \cup S)-v(S)$$

Interpretation: the marginal contribution of a coalition C to a disjoint coalition S does not increase if S becomes smaller

Convex games (2)

- It is easy to show that supermodularity is equivalent to

$$v(S \cup \{i\})-v(S) \leq v(T \cup \{i\})-v(T)$$

for all $i \in N$ and all $S,T \in 2^N$ such that $S \subseteq T \subseteq N \setminus \{i\}$

- interpretation: player's marginal contribution to a large coalition is not smaller than her/his marginal contribution to a smaller coalition (which is stronger than superadditivity)
    - Clearly all convex games are superadditive ($S \cap T = \emptyset$…)
    - A superadditive game can be not convex (try to find one)
- An important property of convex games is that they are (*totally*) balanced, and it is “easy” to determine the core (coincides with the Weber set, i.e. the convex hull of all marginal vectors…)

Example

(N,v) such that

N={1,2,3},

v(1)=v(3)=0,

v(2)=3,

v(1,2)=3, v(1,3)=1

v(2,3)=4

v(1,2,3)=5.

Check it is convex

Marginal vectors

123→(0,3,2)

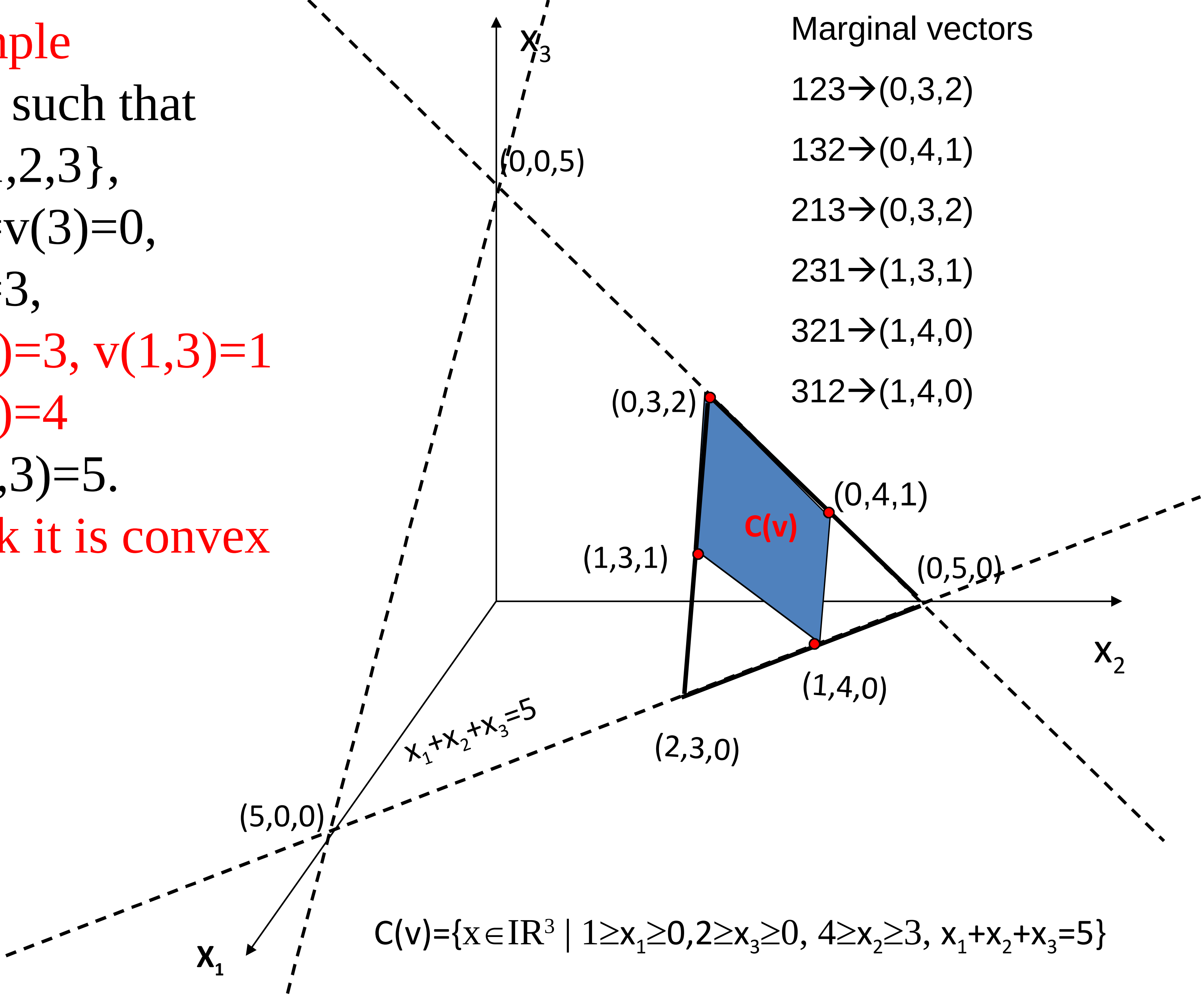
132→(0,4,1)

213→(0,3,2)

231→(1,3,1)

321→(1,4,0)

312→(1,4,0)



$C(v)=\{x\in IR^3 \mid 1\geq x_1\geq 0, 2\geq x_3\geq 0, 4\geq x_2\geq 3, x_1+x_2+x_3=5\}$

How to share v(N)…

- The Core of a game can be used to exclude those allocations which are *not stable*.
- But the core of a game can be a bit "*extreme*" (see for instance the glove game)
- Sometimes the core is *empty* (pirates)
- And if it is not empty, there can be many allocations in the core (*which is the best*?)

An axiomatic approach (Shapley (1953)

- Similar to the approach of Nash in bargaining: *which properties an allocation method should satisfy in order to divide v(N) in a reasonable way*?
- Given a subset **C** of $\mathbf{G}^N$ (class of all TU-games with N as the set of players) a *(point map) solution* on **C** is a map Φ:**C** $\rightarrow \mathrm{IR}^N$.
- For a solution Φ we shall be interested in various properties…

Symmetry

PROPERTY 1(SYM) Let $v \in G^N$ be a TU-game.

Let $i, j \in$ □ If $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \in 2^{N \setminus \{i,j\}}$,

then $\Phi_i(v) = \Phi_j(v)$.

EXAMPLE

We have a TU-game $(\{1,2,3\},v)$ s.t. $v(1) = v(2) = v(3) = 0$,

$v(1, 2) = v(1, 3) = 4$, $v(2, 3) = 6$, $v(1, 2, 3) = 20$.

Players 2 and 3 are symmetric. In fact:

$v(\emptyset \cup \{2\}) = v(\emptyset \cup \{3\}) = 0$ and $v(\{1\} \cup \{2\}) = v(\{1\} \cup \{3\}) = 4$

If Φ satisfies SYM, then $\Phi_2(v) = \Phi_3(v)$

Efficiency

PROPERTY 2 (EFF) Let $v \in G^N$ be a TU-game.

$\sum_{i \in N} \Phi_i(v) = v(N)$, i.e., $\Phi(v)$ is a *pre-imputation*.

Null Player Property

DEF. Given a game $v \in \mathbf{G}^N$, a player $i \in N$ s.t.

$v(S \cup i) = v(S)$ for all $S \in 2^N$ will be said to be a null player.

PROPERTY 3 (NPP) Let $v \in G^N$ be a TU-game. If $i \in N$ is a null player, then $\Phi_i(v) = 0$.

EXAMPLE We have a TU-game $(\{1,2,3\},v)$ such that $v(1) = 0$, $v(2) = v(3) = 2$, $v(1, 2) = v(1, 3) = 2$, $v(2, 3) = 6$, $v(1, 2, 3) = 6$. Player 1 is null. Then $\Phi_1(v) = 0$

EXAMPLE We have a TU-game ({1,2,3},v) such that v(1) =0, v(2) = v(3) = 2, v(1, 2) = v(1, 3) = 2, v(2, 3) = 6, v(1, 2, 3) = 6. On this particular example, if Φ satisfies NPP, SYM and EFF we have that

$\Phi_1(v) = 0$ by NPP

$\Phi_2(v)= \Phi_3(v)$ by SYM

$\Phi_1(v)+\Phi_2(v)+\Phi_3(v)=6$ by EFF

So $\Phi=(0,3,3)$

But our goal is to characterize Φ on $\mathbf{G}^N$**.** One more property is needed.

Additivity

PROPERTY 4 (ADD) Given $v,w \in \mathbf{G}^N$,

$$\Phi(v)+\Phi(w)=\Phi(v+w).$$

.EXAMPLE Two TU-games v and w on N={1,2,3}

v		w		v+w
v(1) =3	+	w(1) =1	=	v+w(1) =4
v(2) =4		w(2) =0		v+w(2) =4
v(3) = 1		w(3) = 1		v+w(3) = 2
v(1, 2) =8		w(1, 2) =2		v+w(1, 2) =10
v(1, 3) = 4		w(1, 3) = 2		v+w(1, 3) = 6
v(2, 3) = 6		w(2, 3) = 3		v+w(2, 3) = 9
v(1, 2, 3) = 10		w(1, 2, 3) = 4		v+w(1, 2, 3) = 14

Φ Φ Φ

Theorem 1 (Shapley 1953)

There is a unique map ϕ defined on $\mathbf{G}^N$ that satisfies EFF, SYM, NPP, ADD. Moreover, for any $i \in N$ we have that

$$\text{¤}_i(v) = \frac{1}{n!} \sum_{\pm \in \Pi} m_i^{\pm}(v)$$

Here Π is the set of all permutations $\sigma: N \to N$ of N, while $m^{\sigma}_i(v)$ is the marginal contribution of player i according to the permutation σ, which is defined as:

$v(\{\sigma(1), \sigma(2), \ldots, \sigma(j)\}) - v(\{\sigma(1), \sigma(2), \ldots, \sigma(j-1)\})$,

where j is the unique element of N s.t. $i = \sigma(j)$.

Probabilistic interpretation: (the “room parable”)

- Players gather one by one in a room to create the “grand coalition”, and each one who enters gets his marginal contribution.
- Assuming that all the different orders in which they enter are equiprobable, the Shapley value gives to each player her/his expected payoff.

Example

(N,v) such that
N={1,2,3},
v(1)=v(3)=0,
v(2)=3,
v(1,2)=3,
v(1,3)=1,
v(2,3)=4,
v(1,2,3)=5.

Permutation	1	2	3
1,2,3	0	3	2
1,3,2	0	4	1
2,1,3	0	3	2
2,3,1	1	3	1
3,2,1	1	4	0
3,1,2	1	4	0
Sum	3	21	6
$\phi(v)$	3/6	21/6	6/6

Example

(N,v) such that
N={1,2,3},
v(1)=v(3)=0,
v(2)=3,
v(1,2)=3, v(1,3)=1
v(2,3)=4
v(1,2,3)=5.

Marginal vectors
123→(0,3,2)
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