Chapter 1

PREFERENCE MODELLING

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Abstract This chapter provides the reader with a presentation of preference modelling fundamental notions as well as some recent results in this field. Preference modelling is an inevitable step in a variety of fields: economy, sociology, psychology, mathematical programming, even medicine, archaeology, and obviously decision analysis. Our notation and some basic definitions, such as those of binary relation, properties and ordered sets, are presented at the beginning of the chapter. We start by discussing different reasons for constructing a model or preference. We then go through a number of issues that influence the construction of preference models. Different formalisations besides classical logic such as fuzzy sets and non-classical logics become necessary. We then present different types of preference structures reflecting the behavior of a decision-maker: classical, extended and valued ones. It is relevant to have a numerical representation of preferences: functional representations, value functions. The concepts of thresholds and minimal representation are also introduced in this section. We also deal with the problem of how to extend a preference relation over a set A of "objects" to the set of all subsets of A. In section 8, we briefly explore the concept of deontic logic (logic of preference) and other formalisms associated with "compact representation of preferences" introduced for special purposes. We end the chapter with some concluding remarks.

Keywords: preference modelling, decision aiding, uncertainty, fuzzy sets, ordered relations, binary relations, preference extensions

Introduction

The purpose of this chapter is to present fundamental notions of preference modelling as well as some recent results in this field. Basic references on this issue can be considered: Fishburn, 1970, Krantz et al., 1971, Roberts, 1979, Fishburn, 1985, Roubens and Vincke, 1985, Kacprzyk and Roubens, 1988, Tanguiane, 1991, Trotter, 1992, Pirlot and Vincke, 1997, Fishburn, 1999, Aleskerov et al., 2007.

The chapter is organised as follows: The purpose for which formal models of preference and more generally of objects comparison are studied, is introduced in section 1. In section 2, we analyse the information used when such models are established and introduce different sources and types of uncertainty. Our notation and some basic definitions, such as those of binary relation, properties and ordered sets, are presented in section 3. Besides classical logic, different formalisms can be used in order to establish a preference model, such as fuzzy sets and non-classical logics. These are discussed in section 4. In section 5, we then present different types of preference structures reflecting the behavior of a decision-maker: classical, extended and valued ones. It appears relevant to have a numerical representation of preferences: functional representations, value functions and intervals. These are discussed in section 6. The concepts of thresholds and minimal representation are also introduced in this section. In section 7, we present different approaches to the analysis of the problem of how to extend a preference relation over a set A of "objects" to the set of all subsets of A. Finally, after briefly exploring the concept of deontic logic (logic of preference) and other related issued in section 8, we end the chapter with some concluding remarks

1. Purpose

Preference modelling is an inevitable step in a variety of fields. Scientists build models in order to better understand and to better represent a given situation; such models may also be used for more or less operational purposes (see Bouyssou et al., 2000). It is often the case that it is necessary to compare objects in such models, basically in order to either establish if there is an order between the objects or to establish whether such objects are "near". Objects can be everything, from candidates to time intervals, from computer codes to medical patterns, from prospects (lotteries) to production systems. This is the reason why preference modelling is used in a great variety of fields such as economy (Armstrong, 1939, Armstrong, 1948, Armstrong, 1950, Debreu, 1959), sociology, psychology (Coombs and Smith, 1973, Chisholm and Sosa, 1966b, Kahneman and Tversky, 1979, Kahneman et al., 1981, Broome, 1991), political science (Barthélemy et al., 1982, Sen, 1986), artificial intelligence (Doyle and Wellman, 1992), computer science (Scott, 1982, Trotter, 1992, Fishburn, 1999), temporal logic (see Allen, 1983) and the interval satisfiability problem (Golumbic and Shamir, 1993; Pe'er and Shamir, 1997) mathematical programming (Perny and Spanjaard, 2005, Perny and Spanjaard, 2002), electronic business, medicine and biology (Benzer, 1962, Carrano, 1988, Karp, 1993, Nagaraja, 1992, Janowitz et al., 2003), archaeology (Hodson et al., 1971), and obviously decision analysis.

In this chapter, we are going to focus on preference modelling for decision aiding purposes, although the results have a much wider validity.

Throughout this chapter, we consider the case of somebody (possibly a decision-maker) who tries to compare objects taking into account different points of view. We denote the set of alternatives A[1], to be labelled $a, b, c, ...$ and the set of points of view J, labelled $j = 1, 2, ..., m$. In this framework, a data $g_j(a)$ corresponds to the evaluation of the alternative a from the point of view $j \in J$.

As already mentioned, comparing two objects can be seen as looking for one of the two following possible situations:
- object a is "before" object b, where "before" implies some kind of order between a and b, such an order referring either to a direct preference (a is preferred to b) or being induced from a measurement and its associated scale (a occurs before b, a is longer, bigger, more reliable, than b);
- object a is "near" object b, where "near" can be considered either as indifference (object a or object b will do equally well for some purpose), or as a similarity, or again could be induced by a measurement (a occurs simultaneously with b, they have the same length, weight, reliability).

The two above-mentioned "attitudes" (see Ngo The, 2002) are not exclusive. They just stand to show what type of problems we focus on. From a decision aiding point of view we traditionally focus on the first situation. Ordering relations is the natural basis for solving ranking or choice problems. The second situation is traditionally associated with problems where the aim is to be able to put together objects sharing a common feature in order to form "homogeneous" classes or categories (a classification problem).

The first case we focus on is the ordering relation: given the set A, establishing how each element of A compares to each other element of A from a "preference" point of view enables to obtain an order which might be used to make either a choice on the set A (identify the best) or to rank the set A. Of course, we have to consider whether it is possible to establish such an ordering relation and of what type (certain, uncertain, strong, weak etc.) for all pairs of elements of A. We also have to establish what "not preference" represents (indifference, incomparability etc.). In the following sections (namely in section 5), we are going to see that different options are available, leading to different, so called, preference structures.

In the second case we focus on the "nearness" relation since the issue here is to put together objects which ultimately are expected to be "near" (whatever the concept of "near" might represent). In such a case, there is also the problem how to consider objects which are "not near". Typical situations in this case include the problems of grouping, discriminating and assigning (Hand, 1981)). A further distinction in such problems concerns the fact that the categories within which the objects might be associated could already exist or not and the fact that such categories might be ordered or not. Putting objects into non pre-existing non ordered categories is the typical classification problem, conversely, assigning objects to pre-existing ordered categories is known as the "sorting" problem (Pawlak and Słowiński, 1994; Zopounidis and Doumpos, 2002; Perny, 1998).

It should be noted that although preference relations have been naturally associated to ranking and choice problem statements, such a separation can be argued. For instance, there are sorting procedures (which can be seen as classification problems) that use preference relations instead of "nearness" ones (Yu, 1992; Massaglia and Ostanello, 1991; Moscarola and Roy, 1977). The reason is the following: in order to establish that two objects belong to the same category we usually either try to check whether the two objects are "near" or whether they are near a "typical" object of the category (see for instance Perny, 1998). If, however, a category is described, not through its typical objects, but through its boundaries, then, in order to establish if an object belongs to such a category it might make sense to check whether such an object performs "better" than the "minimum", or "least" boundary of the category and that will introduce the use of a preference relation.

In Ngo The, 2002 it is claimed that decision aiding should not exclusively focus on preference relations, but also on "nearness relations", since quite often the problem statement to work with in a problem formulation is that of classification (on the existence of different problem statements and their meaning the reader is referred to Vincke, 1992; Roy, 1996; Roy and Bouyssou, 1993; Dias and Tsoukiàs, 2004).

2. Nature of Information

As already mentioned, the purpose of our analysis is to present the literature associated with objects comparison for either a preference or a nearness relation. Nevertheless, such an operation is not always as intuitive as it might appear. Building up a model from reality is always an abstraction (see Bouyssou, 1989). This can always be affected by the presence of uncertainty due to our imperfect knowledge of the world, our limited capability of observation and/or discrimination, the inevitable errors occurring in any human activity etc. (Roy, 1988). We call such an uncertainty exogenous. Besides, such an activity might generate uncertainty since it creates an approximation of reality, thus concealing some features of reality. We call this an endogenous uncertainty (see Tsoukiàs, 1997).

As pointed out in Vincke, 2001, preference modelling can be seen as either the result of direct comparison (asking a decision-maker to compare two objects and to establish the relation between them) from which it might be possible to infer a numerical representation, or as the result of the induction of a preference relation from the knowledge of some "measures" associated to the compared objects.

In the first case, uncertainty can arise from the fact that the decision-maker might not be able to clearly state a preference relation for any pair of actions. We do not care why this may happen, we just consider the fact that the decision-maker may reply when asked if "x is preferred to y": yes, no, I do not know, yes and no, I am not sure, it might be, it is more preference than indifference, but ... etc.. The problem in such cases is how to take such replies into account when defining a model of preferences.

In the second case, we may have different situations such as: incomplete information (missing values for some objects), uncertain information (the value of an object lies within an interval to which an uncertainty distribution might be associated, but the precise value is unknown), ambiguous information (contradictory statements about the present state of an object). The problem here is how to establish a preference model on the basis of such information and to what extent the uncertainty associated with the original information will be propagated to the model and how.

Such uncertainties can be handled through the use of various formalisms (see section 4 of this chapter). Two basic approaches can be distinguished (see also Dubois and Prade, 2001).

1. Handling uncertain information and statements. In such a case, we consider that the concepts used in order to model preferences are well-known and that we could possibly be able to establish a preference relation without any uncertainty, but we consider this difficult to do in the present situation with the available information. A typical example is the following: we know that x is preferred to y if the price of x is lower than the price of y, but we know very little about the prices of x and y. In such cases we might use an uncertainty distribution (classical probability, ill-known probabilities, possibility distributions, see Fishburn, 1970; Cohen and Jaffray, 1980; Jaffray and Wakker, 1993; Dubois and Prade, 1988) in order to associate a numerical uncertainty with each statement.

2. Handling ambiguous concepts and linguistic variables. With such a perspective we consider that sentences such as "x is preferred to y" are ill-defined, since the concept of preference itself is ill-defined, independently from the available information. A typical example is a sentence of the type: "the largest the difference of price between x and y is, the strongest the preference is". Here we might know the prices of x and y perfectly, but the concept of preference is defined through a continuous valuation. In such cases, we might use a multi-valued logic such that any preferential sentence obtains a truth value representing the "intensity of truth" of such a sentence. This should not be confused with the concept of "preference intensity", since such a concept is based on the idea of "measuring" preferences (as we do with temperature or with weight) and there is no "truth" dimension (see Krantz et al., 1971; Roberts, 1979; Rescher, 1969; Keeney and Raiffa, 1976). On the other hand such a subtle theoretical distinction can be transparent in most practical cases since often happens that similar techniques are used under different approaches.

3. Notation and Basic Definitions

The notion of binary relation appears for the first time in De Morgan's study (De Morgan, 1864) and is defined as a set of ordered pairs in Peirce's works (Peirce, 1880, Peirce, 1881, Peirce, 1883). Some of the first work dedicated to the study of preference relations can be found in Dushnik and Miller, 1941 and in Scott and Suppes, 1958 (more in general the concept of models of arbitrary relations will be introduced in Tarski, 1954, Tarski, 1955). Throughout this chapter, we adopt Roubens' and Vincke's notation (Roubens and Vincke, 1985).

Definition 3.1 (Binary Relation)**.** *Let A be a finite set of elements (a, b, c,...,n), a binary relation R on the set A is a subset of the cartesian product A×A, that is, a set of ordered pairs (a, b) such that a and b are in A: $R \subseteq A \times A$.*

For an ordered pair (a, b) which belongs to R, we indifferently use the notations:

$$(a, b) \in R \text{ or } aRb \text{ or } R(a, b)$$

Let R and T be two binary relations on the same set A. Some set operations are:

The Inclusion :	$R \subseteq T$ if $aRb \longrightarrow aTb$
The Union :	$a(R \cup T)b$ iff aRb or(inclusive) aTb
The Intersection :	$a(R \cap T)b$ iff aRb and aTb
The Relative product :	$a(R.T)b$ iff $\exists c \in A :$ aRc and cTb
	$(aR^2b$ iff $aR.Rb)$

When such concepts apply we respectively denote (R^a), (R^s), $(\hat{R})$ the asymmetric, the symmetric and the complementary part of binary relation R:

$$aR^ab \text{ iff } aRb \text{ and } not(bRa)$$
$$aR^sb \text{ iff } aRb \text{ and } bRa$$
$$a\hat{R}b \text{ iff } not(aRb) \text{ and } not(bRa)$$

The complement (R^c), the converse (the dual)$(\overline{R})$ and the co-dual (R^{cd}) of R are respectively defined as follows:

$$aR^cb \text{ iff } not(aRb)$$
$$a\overline{R}b \text{ iff } bRa$$
$$aR^{cd}b \text{ iff } not(bRa)$$

The relation R is called

reflexive,	if	$aRa, \quad \forall a \in A$
irreflexive,	if	$aR^c a, \quad \forall a \in A$
symmetric,	if	$aRb \longrightarrow bRa, \quad \forall a, b \in A$
antisymmetric,	if	$(aRb, bRa) \longrightarrow a = b, \quad \forall a, b \in A$
asymmetric,	if	$aRb \longrightarrow bR^c a, \quad \forall a, b \in A$
complete,	if	$(aRb \quad \text{or} \quad bRa), \quad \forall a \neq b \in A$
strongly complete,	if	$aRb \quad \text{or} \quad bRa, \quad \forall a, b \in A$
transitive,	if	$(aRb, bRc) \longrightarrow aRc, \quad \forall a, b, c \in A$
negatively transitive,	if	$(aR^c b, bR^c c) \longrightarrow aR^c c, \quad \forall a, b, c \in A$
negatively transitive,	if	$aRb \longrightarrow (aRc \quad \text{or} \quad cRb), \quad \forall a, b, c \in A$
semitransitive,	if	$(aRb, bRc) \longrightarrow (aRd \quad \text{or} \quad dRc), \quad \forall a, b, c, d \in A$
Ferrers relation,	if	$(aRb, cRd) \longrightarrow (aRd \quad \text{or} \quad cRb), \quad \forall a, b, c, d \in A$

The equivalence relation E associated with the relation R is a reflexive, symmetric and transitive relation, defined by:

$$aEb \text{ iff } \quad \forall a \in A \begin{cases} aRc & \Longleftrightarrow & bRc \\ cRa & \Longleftrightarrow & cRb \end{cases}$$

A binary relation R may be represented by a direct graph (A, R) where the nodes represent the elements of A, and the arcs, the relation R. Another way to represent a binary relation is to use a matrix M^R; the element M^R_{ab} of the matrix (the intersection of the line associated to a and the column associated to b) is 1 if aRb and 0 if $not(aRb)$.

Example 3.1. *Let R be a binary relation defined on a set A, such that the set A and the relation R are defined as follows:*
$A = \{a, b, c, d\}$ *and* $R = \{(a, b), (b, d), (b, c), (c, a), (c, d), (d, b)\}$

The graphical and matrix representation of R are given in figures 1.1 and 1.2.

4. Languages

Preference models are formal representations of comparisons of objects. As such they have to be established through the use of a formal and abstract language capturing both the structure of the world being described and the manipulations of it. It seems natural to consider formal logic as such a language. However, as already mentioned in the previous sections, the real world might be such that classical formal logic might appear too rigid to allow the definition of useful and expressive models. For this purpose, in this section, we introduce some further formalisms which extend the expressiveness of classical logic, while keeping most of its calculus properties.

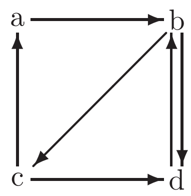

Figure 1.1. Graphical representation of R

	a	b	c	d
a	0	1	0	0
b	0	0	1	1
c	1	0	0	1
d	0	1	0	0

Figure 1.2. Matrix representation of R

4.1 Classic Logic

The interested reader can use two references: Mendelson, 1964; van Dalen, 1983 as introductory books to the use and the semantics of classical logic. All classic books mentioned in this chapter, implicitly or explicitly use classical logic, since binary relations are just sets and the calculus of sets is algebraically equivalent to truth calculus. Indeed the semantics of logical formulas as established by Tarski, 1954, Tarski, 1955, show the equivalence between membership of an element to a set and truth of the associate sentence.

Building a binary preference relation, a valuation of any proposition takes the values $\{0, 1\}$:

$$\mu(aRb) = 1 \text{ iff } aRb \text{ is true}$$
$$\mu(aRb) = 0 \text{ iff } aRb \text{ is false.}$$

The reader will note that all notations introduced in the previous section are based on the above concept. He/she should also note that when we write "a preference relation P is a subset of $A \times A$", we introduce a formal structure where the universe of discourse is $A \times A$ and P is the model of the sentence "x in relation P with y", that is, P is the set of all elements of $A \times A$ (ordered pairs of x and y) for which the sentence is true.

The above semantic can be in sharp contrast with decision analysis experience. For this purpose we will briefly introduce two more semantics: fuzzy sets and four-valued logic.

4.2 Fuzzy Sets

In this section, we provide a survey of basic notions of fuzzy set theory. We present definitions of connectives and several valued binary relation properties in order to be able to use this theory in the field of decision analysis. Basic references for this section include Zimmermann, 1985, Dubois and Prade, 1988, Słowiński, 1998, Fodor and Roubens, 1994.

Fuzzy sets were first introduced by Zadeh (Zadeh, 1965, Zadeh, 1975). The concept and the associated logics were further developed by other researchers: Gougen, 1969, Kaufmann, 1975, Kaufmann and Gupta, 1985, Mizumoto and Tanaka, 1976a, Mizumoto and Tanaka, 1976b, Nahmias, 1976, Nguyen et al., 1978, Dubois and Prade, 1978.

Fuzzy measures can be introduced for two different uses: either they can represent a concept imprecisely known (although well defined) or a concept which is vaguely perceived such as in the case of a linguistic variable. In the first case they represent possible values, while in the second they are better understood as a continuous truth valuation (in the interval $[0, 1]$). To be more precise:
- in the first case we associate a possibility distribution (an ordinal distribution of uncertainty) to classical logic formulas;
- in the second case we have a multi-valued logic where the semantics allow values in the entire interval $[0, 1]$.
A fuzzy set can be associated either with the set of alternatives considered in a decision aiding model (consider the case where objects are represented by fuzzy numbers) or with the preference relations. In decision analysis we may consider four possibilities [2]:

- Alternatives with crisp values and crisp preference relations
- Alternatives with crisp values and fuzzy preference relations
- Alternatives with fuzzy values and crisp preference relations (defuzzification, Li and Lee, 1988 with gravity center, Yager, 1981 with means interval)
- Alternatives with fuzzy values and fuzzy preference relations (possibility graphs, Dubois and Prade, 1983; four fuzzy dominance index, Roubens and Vincke, 1988); in this chapter we are going to focus on fuzzy preference relations

In the following we introduce the definitions required for the rest of the chapter.

Definition 4.1 (Fuzzy Set)**.** *A fuzzy set (or a fuzzy subset) F on a set* Ω *is defined by the result of an application:*

$$\mu_F : \Omega \longrightarrow [0, 1]$$

where $\forall x \in \Omega$, $\mu(x)$ *is the membership degree of* x *to* F.

Definition 4.2 (Negation)**.** *A function n: [0, 1]* $\longrightarrow$ *[0, 1] is a negation if and only if it is non-increasing and :*
n(0) = 1 and n(1) = 0

If the negation n is strictly decreasing and continuous then it is called *strict.*

In the following we investigate the two basic classes of operators, the operators for the intersection (triangular norms called t-norms) and the union (triangular conorms called t-conorms or s-norms) of fuzzy sets:

Definition 4.3 (t-norm)**.** *A function T: [0, 1]*^{2} $\longrightarrow$ *[0, 1] is a triangular norm (t-norm), if and only if it satisfies the four conditions:*

Equivalence Condition: $T(1, x) = x \ \forall x \in [0, 1]$
T is commutative: $T(x, y) = T(y, x) \ \forall x, y \in [0, 1]$
T is nondecreasing in both elements: $T(x, y) \leq T(u, v)$ *for all* $0 \leq x \leq u \leq 1$ *and* $0 \leq y \leq v \leq 1$
T is associative: $T(x, T(y, z)) = T(T(x, y), z) \ \forall x, y, z \in [0, 1]$

The function T defines a general class of intersection operators for fuzzy sets.

Definition 4.4 (t-conorm)**.** *A function S: [0, 1]*^{2} $\longrightarrow$ *[0, 1] is a (t-conorm), if and only if it satisfies the four conditions:*

Equivalence Condition: $S(0, x) = x \ \forall x \in [0, 1]$
S is commutative: $S(x, y) = S(y, x) \ \forall x, y \in [0, 1]$
S is nondecreasing in both elements: $S(x, y) \leq S(u, v)$ *for all* $0 \leq x \leq u \leq 1$ *and* $0 \leq y \leq v \leq 1$
S is associative: $S(x, S(y, z)) = S(S(x, y), z) \ \forall x, y, z \in [0, 1]$

T-norms and t-conorms are related by duality. For suitable negation operators[3] pairs of t-norms and t-conorms satisfy the generalisation of the De Morgan law:

Definition 4.5 (De Morgan Triplets)**.** *Suppose that T is a t-norm, S is a t-conorm and n is a strict negation.* $\langle T, S, n\rangle$ *is a De Morgan triple* *if and only if:*

$$n(S(x,y)) = T(n(x), n(y))$$

Such a definition extends De Morgan's law to the case of fuzzy sets. There exist different proposed De Morgan triplets: Dombi, 1982, Frank, 1979, Schweizer and Sklar, 1983, Yager, 1980, Dubois and Prade, 1980, Weber, 1983, Yu, 1985.

The more frequent t-norms and t-conorms and negations are presented in table 1.1.

Table 1.1. Principal t-norms, t-conorms and negations

Names	*t-norms*	*t-conorms*
Zadeh	$min(x,y)$	$max(x,y)$
probabilistic	$x*y$	$x+y-xy$
Lukasiewicz	$max(x+y-1,0)$	$min(x+y,1)$
Hamacher($\gamma > 0$)	$(xy)/$	$(x+y+xy-(1-\gamma)xy)/$
	$(\gamma+(1-\gamma)(x+y-xy))$	$(1-(1-\gamma)xy)$
Yager($p > 0$)	$max(1-((1-x)^p+(1-y)^p)^{1/p},0)$	$min((x^p+y^p)^{1/p},1)$
Weber($\lambda > -1$)	$max((x+y-1+\lambda xy)/(1+\lambda),0))$	$min(x+y+\lambda xy,1)$
drastic	x if $y=1$	x if $y=0$
	y if $x=1$	y if $x=0$
	0 ifnot	1 ifnot

We make use of De Morgan's triplet $\langle T, S, n\rangle$ in order to extend the definitions of the operators and properties introduced above in crisp cases. First, we give the definitions of the operators of implication I_T and equivalence E_T:

$$I_T(x,y) = sup\{z \in [0,1] : T(x,z) \leq y\}$$
$$E_T(x,y) = T(I_T(x,y), I_T(y,x))$$

Since preference modelling makes use of binary relations, we extend the definitions of binary relation properties to the valued case. For the sake of simplicity $\mu(R(x,y))$ will be denoted $R(x,y)$: a valued binary relation $R(x,y)$ is $(\forall a,b,c,d \in A)$

reflexive,	if	$R(a,a) = 1$
irreflexive,	if	$R(a,a) = 0$
symmetric,	if	$R(a,b) = R(b,a)$
T-antisymmetric,	if	$a \neq b \longrightarrow T(R(a,b), R(b,a)) = 0$
T-asymmetric,	if	$T(R(a,b), R(b,a)) = 0$
S-complete,	if	$a \neq b \quad \longrightarrow \quad S(R(a,b), R(b,a)) = 1$
S-strongly complete,	if	$S(R(a,b), R(b,a)) = 1$
T-transitive,	if	$T(R(a,c), R(c,b)) \leq R(a,b)$
negatively S-transitive,	if	$R(a,b) \leq S(R(a,c), R(c,b))$
T-S-semitransitive,	if	$T(R(a,d), R(d,b)) \leq S(R(a,c), R(c,b))$
T-S-Ferrers relation,	if	$T(R(a,b), R(c,d)) \leq S(R(a,d), R(c,b))$

Different instances of De Morgan triplets will provide different definitions for each property.

The equivalence relation is one of the most-used relations in decision analysis and is defined in fuzzy set theory as follows:

Definition 4.6 (Equivalence Relation)**.** *A function* $E : [0, 1]^2 \longrightarrow [0, 1]$ *is an equivalence if and only if it satisfies:*
$E(x, y) = E(y, x) \; \forall \; x, y \in [0, 1]$
$E(0, 1) = E(1, 0) = 0$
$E(x, x) = 1 \; \forall \; x \in [0, 1]$
$x \leq x' \leq y' \leq y \Longrightarrow E(x, y) \leq E(x', y')$

In section 5.3 and chapter 11, some results obtained by the use of fuzzy set theory are represented.

4.3 Four-valued Logics

When we compare objects, it might be the case that it is not possible to establish precisely whether a certain relation holds or not. The problem is that such a hesitation can be due either to incomplete information (missing values, unknown replies, unwillingness to reply etc.) or to contradictory information (conflicting evaluation dimensions, conflicting reasons for and against the relation, inconsistent replies etc.). For instance, consider the query "is Anaxagoras intelligent?" If you know who Anaxagoras is you may reply "yes" (you came to know that he is a Greek philosopher) or "no" (you discover he is a dog). But if you know nothing you will reply "I do not know" due to your ignorance (on this particular issue). If on the other hand you came to know both that Anaxagoras is a philosopher and a dog you might again reply "I do not know", not due to ignorance, but to inconsistent information. Such different reasons for hesitation can be captured through four-valued logics allowing for different truth values for four above-mentioned cases. Such logics were first studied by Dubarle in 1963 (appeared in Dubarle, 1989) and introduced in the literature in Belnap, 1976 and Belnap, 1977. Further literature on such logics can be found in Bergstra et al., 1995, Fages and Ruet, 1997, Font and Moussavi, 1993, Fitting, 1991, Kaluzhny and Muravitsky, 1993, Thomason and Horty, 1987, Arieli and Avron, 1998, Tsoukiàs, 2002, Arieli et al., 2011.

In the case of preference modelling, the use of such logics was first suggested in Tsoukiàs, 1991 and Doherty et al., 1992. Such logics extend the semantics of classical logic through two hypotheses:
- the complement of a first order formula does not necessarily coincide with its negation;
- truth values are only partially ordered (in a bilattice), thus allowing the definition of a boolean algebra on the set of truth values.

The result is that using such logics, it is possible to formally characterise different states of hesitation when preferences are modelled (see Tsoukiàs and Vincke, 1995, Tsoukiàs and Vincke, 1997). Further more, using such a formalism, it becomes possible to generalise the concordance/discordance principle (used in several decision aiding methods) as shown in Tsoukiàs et al., 2002 and several characterisation problems can be solved (see for instance Tsoukiàs and Vincke, 1998). More recently (see Perny and Tsoukiàs, 1998, Fortemps and Słowiński, 2002, Arieli et al., 2006, Deschrijver et al., 2007, Öztürk and Tsoukiàs, 2007, Öztürk and Tsoukiàs, 2008, Turunen et al., 2010) it has been suggested to use the extension of such logics for continuous valuations.

5. Preference Structures

Definition 5.1 (Preference Structure)**.** *A preference structure is a collection of binary relations defined on the set A and such that:*
- for each couple a, b in A; at least one relation is satisfied
- for each couple a, b in A; if one relation is satisfied, another one cannot be satisfied.

In other terms a preference structure defines a partition[4] of the set $A \times A$. In general it is recommended to have two other hypotheses with this definition (also denoted as fundamental relational system of preferences):

- Each preference relation in a preference structure is uniquely characterised by its properties (symmetry, transitivity, etc..)

- For each preference structure, there exists a unique relation from which the different relations composing the preference structure can be deduced. Any preference structure on the set A can thus be characterised by a unique binary relation R in the sense that the collection of the binary relations are defined through the combinations of the epistemic states of this characteristic relation[5]. For instance aPb if and only if aRb and not bRa.

5.1 $\langle P, I \rangle$ Structures

The most traditional preference model considers that the decision-maker confronted with a pair of distinct elements of a set A, either:
- clearly prefers one element to the other,
or
- does not express a preference among them.

The subset of ordered pairs (a, b) belonging to $A \times A$ such that the statement "a is preferred to b" is true, is called *preference relation* and is denoted by P.

The subset of pairs (a, b) belonging to $A \times A$ such that the statement "a and b are not preferred" is true, is called (in this case) *indifference relation* and is denoted by I (I being considered the complement of $P \cup P^{-1}$ with respect to $A \times A$). We will see later on in Section 5.2.2 that this relation can be further decomposed in indifference and incomparability.

In the literature, there are two different ways of defining a specific preference structure:
- the first defines it by the properties of the binary relations of the relation set;
- the second uses the properties of the characteristic relation. In the rest of the section, we give definitions in both ways.

Definition 5.2 (⟨ P, I ⟩ Structure)**.** *A ⟨ P, I ⟩ structure on the set A is a pair $\langle P, I \rangle$ of relations on A such that:*

- *P is asymmetric,*
- *I is reflexive, symmetric.*

The characteristic relation R of a $\langle P, I \rangle$ structure can be defined as a combination of the relations P and I as :

$$aRb \text{ iff } a(P \cup I)b \tag{1.1}$$

In this case P and I can be defined from R as follows:

$$aPb \text{ iff } aRb \text{ and } bR^c a \tag{1.2}$$

$$aIb \text{ iff } aRb \text{ and } bRa \tag{1.3}$$

The construction of *orders* is of a particular interest, especially in decision analysis since they allow an easy operational use of such preference structures. We begin by representing the most elementary orders (weak order, complete order). In order to define such structures we add properties to the relations P and I (namely different forms of transitivity).

Definition 5.3 (Total Order)**.** *Let R be a binary relation on the set A, R being a characteristic relation of $\langle\ P,\ I\ \rangle$, the following definitions are equivalent:*
i. R is a total order.
ii. R is reflexive, antisymmetric, complete and transitive

$$iii.\ \begin{cases} I = \{(a, a), \forall a \in A\} \\ P \text{ is transitive} \\ P \cup I \text{ is reflexive and complete} \end{cases}$$

$$iv.\ \begin{cases} P \text{ is transitive} \\ P\ .\ I \subset P \text{ (or equivalently } I\ P \subset P) \\ P \cup I \text{ is reflexive and complete} \end{cases}$$

With this relation, we have an indifference between any two objects only if they are identical. The total order structure consists of an arrangement of objects from the best one to the worst one without any ex aequo.

In the literature, one can find different terms associated with this structure: total order, complete order, simple order or linear order.

Definition 5.4 (Weak Order)**.** *Let R be a binary relation on the set A, R being a characteristic relation of $\langle P, I\rangle$, the following definitions are equivalent:*
i. R is a weak order
ii. R is reflexive, complete and transitive

$$iii.\ \begin{cases} I \text{ is transitive} \\ P \text{ is transitive} \\ P \cup I \text{ is reflexive and complete} \end{cases}$$

This structure is also called complete preorder or total preorder. In this structure, indifference is an equivalence relation. The associated order is indeed a total order of the equivalence (indifference) classes of A.

These first two structures consider indifference (or absence of preference) as a transitive relation. This is empirically falsifiable. Literature studies on the intransitivity of indifference show this; undoubtedly the most famous is that of Luce, 1956, which gives the example of a cup of sweetened tea.[6] Before him, Armstrong, 1939, Georgescu-Roegen, 1936, Fechner, 1860, Halphen, 1955 and Poincaré, 1905 already suggested this phenomenon. For historical commentary on the subject, see Fishburn and Monjardet, 1992. Relaxing the property of transitivity of indifference results in two well-known structures: semi-orders and interval orders.

Definition 5.5 (Semiorder). *Let R be a binary relation on the set A, R being a characteristic relation of $\langle P, I \rangle$, the following definitions are equivalent:*
i. R is a semiorder
ii. R is reflexive, complete, Ferrers relation and semitransitive
iii. $\begin{cases} P.I.P \subset P \\ P^2 \cap I^2 = \emptyset \\ P \cup I \text{ is reflexive and complete} \end{cases}$
iv. $\begin{cases} P.I.P \subset P \\ P^2 I \subset P \text{ (or equivalently } IP^2 \subset P) \\ P \cup I \text{ is reflexive and complete} \end{cases}$

Definition 5.6 (Interval Order (IO)). *Let R be a binary relation on the set A, R being a characteristic relation of $\langle P, I \rangle$, the following definitions are equivalent:*
i. R is an interval order
ii. R is reflexive, complete and Ferrers relation
iii. $\begin{cases} P.I.P \subset P \\ P \cup I \text{ is reflexive and complete} \end{cases}$

A detailed study of this structure can be found in Pirlot and Vincke, 1997, Monjardet, 1978, Fishburn, 1985. It is easy to see that this structure generalises all the structures previously introduced.

Can we relax transitivity of preference? Although it might appear counterintuitive there is empirical evidence that such a situation can occur: May, 1954, Tversky, 1969. Similar work can be found in: Fishburn, 1982, Fishburn, 1991b, Fishburn, 1991a, Vind, 1991, Bouyssou, 1996, Bouyssou and Pirlot, 1999, Bouyssou and Pirlot, 2002b, Bouyssou and Pirlot, 2002a, Abbas et al., 2007.

5.2 Extended Structures

The $\langle P, I \rangle$ structures presented in the previous section neither take into account all the decision-maker's attitudes, nor all possible situations. In the literature, there are several non exclusive ways to extend such structures:

- Using sophisticated numerical representations (such as n ordered points, triangles, trapezoids, etc.);

- Introduction of several distinct preference relations representing (one or more) hesitation(s) between preference and indifference or preference intensities;

- Introduction of one or more situations of incomparability.

5.2.1 Preference Relations on n ordered points. As we showed by the end of the previous section intervals may be used in order to represent sophisticated preferences (for instance where the indifference is not necessarily transitive). The use of intervals in order to take into account imprecision and vagueness in handling preferences is well known in the literature, but a general theory on how such models behave was lacking until recently. Öztürk et al., 2011 have generalized the concept of interval by introducing the notion of n-point intervals where each object is represented by n ordered points (for more details see also Öztürk, 2005). They provided an exhaustive study of 2-point and 3-point intervals comparison and show the way to generalize such results to n-point intervals. Their results may be interpreted in two ways:

- What are the all preference structures that can be defined using n-point interval representations and satisfying some axioms?
- How to define all different ways to compare two objects represented by n-point intervals in order to obtain a $\langle P, I \rangle$- preference structure?

Their approach is based on two notions that they called a *relative position* (intuitively showing how far is the actual position of the two intervals w.r.t. to complete disjunction: one interval completely to the right of the other) and a *component set* associated with each relative position (where all redundant information is discarded and where the coding is done in a compact way).

Concerning the first point it turns out that the comparison of 2-point intervals allows to establish 3 different preference structures: 2 types of weak orders, bi-weak order and interval order. The use of 3-point intervals allows to establish 7 types of preference structures: 3 types of weak orders, 3 types of bi-weak orders, 3 types of interval orders, one three-weak order, one split-interval order, one triangle order and 2 types of intransitive preference structures. In their paper they showed also the equivalence between the usual definitions of such preferences structures, their numerical representation and the properties that characterize them. Such results confirm the descriptive power of the framework which allows to provide a complete characterization for preference structures that have never been studied before, as well as other structures well known in the literature (for instance it is possible to interpret within the same framework triangle orders and weak orders).

Concerning the second point they were also interested to the relation between n-point intervals and fuzzy numbers. In order to interpret a fuzzy number as a n-point interval one may alternatively consider ordinal fuzzy intervals as a family of α-cuts of ordinary (i.e. with continuous membership function) fuzzy numbers or intervals; the family of cuts correspond to a finite number of different values of threshold α. Using such an approach they showed how to make use of their comparison rules in order to compare fuzzy intervals and analyzed the link between their framework and the four comparison indices introduced by Dubois and Prade (Dubois and Prade, 1983) for fuzzy intervals. Three of these correspond to strict preference relations obtained for 2-point intervals while the fourth is associated with a non-strict preference relation that is an interval order. In a similar way, they investigated special fuzzy numbers having only two non-zero levels of membership. Their comparison by means of Dubois and Prade comparison indices corresponds to preference structures met in the comparison of 3-point intervals, namely three types of interval orders and one type of weak order.

5.2.2 Several Preference Relations. One can wish to give more freedom to the decision-maker and allow more detailed preference models, introducing one or more intermediate relations between indifference and preference. Such relations might represent one or more zones of ambiguity and/or uncertainty where it is difficult to make a distinction between preference and indifference. Another way to interpret such "intermediate" relations is to consider them as different "degrees of preference intensity". From a technical point of view these structures are similar and we are not going to further discuss such semantics. We distinguish two cases: one where only one such intermediate relation is introduced (usually called weak preference and denoted by Q), and another where several such intermediate relations are introduced.

1 $\langle P, Q, I \rangle$ preference structures. In such structures we introduce one more preference relation, denoted by Q which is an asymmetric and irreflexive binary relation. The usual properties of prefer-

ence structures hold. Usually such structures arise from the use of thresholds when objects with numerical values are compared or, equivalently, when objects whose values are intervals are compared. The reader who wants to have more information on thresholds can go to section 6.1. where all definitions and representation theorems are given.

$\langle P, Q, I \rangle$ preference structures have been generally discussed in Vincke, 1988. Two cases are studied in the literature:
- PQI interval orders and semi-orders (for their characterisation see Tsoukiàs and Vincke, 2003). The detection of such structures has been shown to be a polynomial problem (see Ngo The et al., 2000).
- double threshold orders (for their characterisation see Vincke, 1988, Tsoukiàs and Vincke, 1998) and more precisely pseudo-orders (see Roy and Vincke, 1984, Roy and Vincke, 1987).

One of the difficulties of such structures is that it is impossible to define P, Q and I from a single characteristic relation R as is the case for other conventional preference structures.

2 $\langle P_1, \cdots P_n \rangle$ preference structures. Practically, such structures generalise the previous situation where just one intermediate relation was considered. Again, such structures arise when multiple thresholds are used in order to compare numerical values of objects. The problem was first introduced in Cozzens and Roberts, 1982 and then extensively studied in Roubens and Vincke, 1984, Doignon et al., 1984, Doignon et al., 1986, see also Valadares Tavares, 1988, Moreno, 1992, Abbas and Vincke, 1993, Doignon and Falmagne, 1997. Typically such structures concern the coherent representation of multiple interval orders. The particular case of multiple semi-orders was studied in Doignon, 1987.

5.2.3 Incomparability. In the classical preference structures presented in the previous section, the decision-maker is supposed to be able to compare all alternatives, the absence of preference being considered indifference (we can have aPb, bPa or aIb). But certain situations, such as lack of information, uncertainty, ambiguity, multi-dimensional and conflicting preferences, can create incomparability between alternatives. Within this framework, the partial structures use a third symmetric and irreflexive relation J ($aJb \iff not(aPb)$, $not(bPa)$, $not(aIb)$, $not(aQb)$, $not(bQa)$), called incomparability, to deal with this kind of situation. To have a partial structure $\langle P, I, J \rangle$ or $\langle P, Q, I, J \rangle$, we add to the definitions of the preceding structures (total order, weak order, semi-order, interval order and pseudo-order), the relation of incomparability ($J \neq \emptyset$); and we obtain respectively partial order, partial preorder (quasi-order), partial semi-order, partial interval order and partial pseudo-order (Roubens and Vincke, 1985).

Definition 5.7 (Partial Order)**.** *Let R be a binary relation ($R = P \cup I$) on the set A, R being a characteristic relation of $\langle P, I, J\rangle$, the following definitions are equivalent:*
i. R is a partial order.
ii. R is reflexive, antisymmetric, transitive

$$iii. \begin{cases} P \text{ is asymmetric, transitive} \\ I \text{ is reflexive, symmetric} \\ J \text{ is irreflexive and symmetric} \\ I = \{(a, a), \forall a \in A\} \end{cases}$$

Definition 5.8 (quasi-order)**.** *Let R be a binary relation ($R = P \cup I$) on the set A, R being a characteristic relation of $\langle P, I, J\rangle$, the following definitions are equivalent:*
i. R is a quasi-order.
ii. R is reflexive, transitive

$$iii. \begin{cases} P \text{ is asymmetric, transitive} \\ I \text{ is reflexive, symmetric and transitive} \\ J \text{ is irreflexive and symmetric} \\ (P.I \cup I.P) \subset P \end{cases}$$

A fundamental result (Dushnik and Miller, 1941, Fishburn, 1985) shows that every partial order (resp. partial preorder) on a finite set can be obtained as an intersection of a finite number of total orders (resp. total preorders, see Bossert et al., 2002).

A further analysis of the concept of incomparability can be found in Tsoukiàs and Vincke, 1995 and Tsoukiàs and Vincke, 1997. In these papers it is shown that the number of preference relations that can be introduced in a preference structure, so that it can be represented through a characteristic binary relation, depends on the semantics of the language used for modelling. In other terms, when classical logic is used in order to model preferences, no more than three different relations can be established (if one characteristic relation is used). The introduction of a four-valued logic allows to extend the number of independently defined relations to 10, thus introducing different types of incomparability (and hesitation) due to the different combination of positive and negative reasons (see Tsoukiàs et al., 2002). It is therefore possible, with such a language, to consider an incomparability due to ignorance separately from one due to conflicting information.

5.3 Valued Structures

In this section, we present situations where preferences between objets are defined by a valued preference relation such that $\mu(R(a,b))$ represents either the intensity or the credibility of the preference of a over b[7] or the proportion of people who prefer a to b or the number of times that a is preferred to b. In this section, we make use of results cited in Fodor and Roubens, 1994 and Perny and Roubens, 1998. To simplify the notation, the valued relation $\mu(R(a,b))$ is denoted $R(a,b)$ in the rest of this section. We begin by giving a definition of a valued relation:

Definition 5.9 (Valued Relation)**.** *:*
A valued relation R on the set A is a mapping from the cartesian product $A \times A$ *onto a bounded subset of* $\mathbb{R}$*, often the interval [0,1].*

Remark 5.1. *A valued relation can be interpreted as a family of crisp nested relations. With such an interpretation, each* α*-cut level of a fuzzy relation corresponds to a different crisp nested relation.*

In this section, we show some results obtained by the use of fuzzy set theory as a language which is capable to deal with uncertainty. The seminal paper by Orlovsky (Orlovsky, 1978) can be considered as the first attempt to use fuzzy set theory in preference modelling. Roy in Roy, 1977 will also make use of the concept of fuzzy relations in trying to establish the nature of a pseudo-order. In his paper Orlovsky defines the strict preference relation and the indifference relation with the use of Lukasiewicz and min t-norms. After him, a number of researchers were interested in the use of fuzzy sets in decision aiding, most of these works are published in the journal Fuzzy Sets and Systems.

In the following we give some definitions of fuzzy ordered sets. We derive the following definitions from the properties listed in section 4.2:

Definition 5.10 (Fuzzy Total Order)**.** *A binary relation R on the set A, is a fuzzy total order iff:*
- R is antisymmetric, strongly complete and T-transitive

Definition 5.11 (Fuzzy Weak Order)**.** *A binary relation R on the set A is a fuzzy weak order iff:*
- R is strongly complete and transitive

Definition 5.12 (Fuzzy Semi-order)**.** *A binary relation R on the set A is a fuzzy semi-order iff:*
- R is strongly complete, a Ferrers relation and semitransitive

Definition 5.13 (Fuzzy Interval Order (IO)). *A binary relation R on the set A is a fuzzy interval order iff:*
- R is a strongly complete Ferrers relation

Definition 5.14 (Fuzzy Partial Order). *A binary relation R on the set A is a fuzzy partial order iff:*
- R is antisymmetric reflexive and T-transitive

Definition 5.15 (Fuzzy Partial Preorder). *A binary relation R on the set A is a fuzzy partial preorder iff:*
- R is reflexive and T-transitive

All the above definitions are given in terms of the characteristic relation R. The second step is to define valued preference relations (valued strict preference, valued indifference and valued incomparability) in terms of the characteristic relation (Fodor, 1992, Fodor, 1994, Fodor and Roubens, 1994, Ovchinnikov and Roubens, 1991, Perny and Roy, 1992). For this, equations 1.1- 1.3 are interpreted in terms of fuzzy logical operations:

$$P(a,b) = T[R(a,b), nR(b,a)] \tag{1.4}$$

$$I(a,b) = T[R(a,b), R(b,a)] \tag{1.5}$$

$$R(a,b) = S[P(a,b), I(a,b)] \tag{1.6}$$

However, it is impossible to obtain a result satisfying these three equations using a De Morgan triplet. Alsina, 1985, Fodor and Roubens, 1994 present this result as an impossibility theorem that proves the non-existence of a single, consistent many-valued logic as a logic of preference. A way to deal with this contradiction is to consider some axioms to define $\langle P, I, J\rangle$. Fodor, Ovchinnikov, Roubens propose to define three general axioms that they call Independence of Irrelevant Alternatives (IA), Positive Association (PA), Symmetry (SY). With their axioms, the following propositions hold:

Proposition 5.1 (Fuzzy Weak Order). *if $\langle P, I\rangle$ is a fuzzy weak order then:*
- P is a fuzzy strict partial order
- I is a fuzzy similarity relation (reflexive, symmetric, transitive)

Proposition 5.2 (Fuzzy Semi-order). *if $\langle P, I\rangle$ is a fuzzy semi-order then:*
- P is a fuzzy strict partial order
- I is not transitive

Proposition 5.3 (Fuzzy Interval Order (IO)). *if $\langle P, I\rangle$ is a fuzzy interval order then:*
- P is a fuzzy strict partial order
- I is not transitive

De Baets, Van de Walle and Kerre (De Baets et al., 1995, Van De Walle et al., 1996, Van De Walle et al., 1998) define the valued preference relations without considering a characteristic relation:

P is T-asymmetric ($P \cap_T P^{-1}$)= $\emptyset$
I is reflexive and J is irreflexive ($I(a,a)$=1, $J(a,a)$=0 $\forall a \in A$)
I and J are symmetric ($I = I^{-1}$, $J = J^{-1}$)
$P \cap_T I = \emptyset$, $P \cap_T J = \emptyset$, $I \cap_T J = \emptyset$
$P \cup_T P^{-1} \cup_T I \cup_T = A \times A$

With a continuous t-norm and without zero divisors, these properties are satisfied only in crisp case. To deal with this problem, we have to consider a continuous t-norm with zero divisor.

In multiple criteria decision aiding, we can make use of fuzzy sets in different ways. One of these helps to construct a valued preference relation from the crisp values of alternatives on each criteria. As an example we cite the work of Perny and Roy (Perny and Roy, 1992). They define a fuzzy outranking relation R from a real valued function θ defined on $\mathbb{R} \times \mathbb{R}$, such that $R(a,b) = \theta(g(a), g(b))$ verifies the following conditions for all a, b in A:

$$\forall y \in X, \quad \theta(x,y) \quad \text{is a nondecreasing function of } x \tag{1.7}$$
$$\forall x \in X, \quad \theta(x,y) \quad \text{is a nonincreasing function of } y \tag{1.8}$$
$$\forall z \in X, \quad \theta(z,z) = 1 \tag{1.9}$$

The resulting relation R is a fuzzy semi-order (i.e. reflexive, complete, semi-transitive and Ferrers fuzzy relation). Roy (1978) proposed in Electre III to define the outranking relation R characterized by a function θ for each criterion as follows:

$$\theta(x,y) = \frac{p(x) - min\{y - x, p(x)\}}{p(x) - min\{y - x, q(x)\}}$$

Where $p(x)$ and $q(x)$ are thresholds of the selected criteria.
We may work with alternatives representing some imprecision or ambiguity for a criterion. In this case, we make use of fuzzy sets to define the evaluation of the alternative related to the criterion. In the

ordered pair $\{x, \mu_j^a\}$, μ_j^a represents the grade of membership of x for alternative a related to the criterion j. The fuzzy set μ is supposed to be normal ($sup_x(\mu_j^a) = 1$) and convex ($\forall x, y, z \in \mathbb{R}$, $y \in [x, z]$, $\mu_j^a(y) \leq min\{\mu_j^a(x), \mu_j^a(z)\}$). The credibility of the preference of a over b is obtained from the comparison of the fuzzy intervals (normal, convex fuzzy sets) of a and b with some conditions:

- The method used should be sensitive to the specific range and shape of the grades of membership.

- The method should be independent of the irrelevant alternatives.

-The method should satisfy transitivity.

Fodor and Roubens (Fodor and Roubens, 1994) propose the use of two procedures.

In the first one, the credibility of the preference of a over b for j is defined as the possibility that $a \geq b$:

$$\Pi_j(a \geq b) = \bigvee_{x \geq y} [\mu_j^a(x) \wedge \mu_j^b(y)] = sup_{x \geq y}[min(\mu_j^a(x), \mu_j^b(y))] \quad (1.10)$$

The credibility as defined by 1.10 is a fuzzy interval order (Π_j is reflexive, complete and a Ferrers relation) and

$$min(\Pi_j(a, b), \Pi_j(b, a)) = sup_x min(\mu_j^a(x), \mu_j^b(x))$$

In the case of a symmetrical fuzzy interval (μ^a), the parameters of the fuzzy interval can be defined in terms of the valuation $g_j(a)$ and thresholds $p(g_j(a))$ and $q(g_j(a))$. Some examples using trapezoidal fuzzy numbers can be found in the work of Fodor and Roubens.

The second procedure proposed by Fodor and Roubens makes use of the shapes of membership functions, satisfies the three axioms cited at the beginning of the section (PA, SY and SY) and gives the credibility of preference and indifference as follows:

$$P_j(a, b) = R_j^d(a, b) = 1 - \Pi_j(b \geq a) = N_j(a > b) \quad (1.11)$$

$$I_j(a, b) = min[\Pi_j(a \geq b), \Pi_j(b \geq a)] \quad (1.12)$$

Where Π (the possibility degree) and N (the necessity degree) are two dual distributions of the possibility theory that are related to each other with the equality: $\Pi(A) = 1 - N(A)$ (see Dubois and Prade, 2001 for an axiomatic definition of the theory of possibility).

6. Domains and Numerical Representations

In this section we present several results concerning the numerical representation of the preference structures introduced in the previous section (see also Aleskerov et al., 2007). This is an important operational problem. Given a set A and a set of preference relations holding between the elements of A, it is important to know whether such preferences fit a precise preference structure admitting a numerical representation. If this is the case, it is possible to replace the elements of A with their numerical values and then work with these. Otherwise, when to the set A is already associated a numerical representation (for instance a measure), it is important to test which preference structure should be applied in order to faithfully interpret the decision-maker's preferences (Vincke, 2001).

6.1 Representation Theorems

Theorem 6.1 (Total Order). *Let $R = \langle P, I \rangle$ be a reflexive relation on a finite set A, the following definitions are equivalent:*

i. R is a total order structure (see 5.3)

ii. $\exists$ g: $A \mapsto \mathbb{R}^+$ satisfying for $\forall a, b \in A$:

$$\begin{cases} aPb \quad \textit{iff} \quad g(a) > g(b) \\ a \neq b \quad \Longrightarrow \quad g(a) \neq g(b) \end{cases}$$

iii. $\exists$ g: $A \mapsto \mathbb{R}^+$ satisfying for $\forall a, b \in A$:

$$\begin{cases} aRb \quad \textit{iff} \quad g(a) > g(b) \\ a \neq b \quad \Longrightarrow \quad g(a) \neq g(b) \end{cases}$$

In the infinite not enumerable case, it can be impossible to find a numerical representation of a total order. For a detailed discussion on the subject, see Beardon et al., 2002. The necessary and sufficient conditions to have a numerical representation for a total order are present in many works: Debreu, 1954, Fishburn, 1970, Krantz et al., 1971, Briges and Mehta, 1995.

Theorem 6.2 (Weak Order). *Let $R = \langle P, I \rangle$ be a reflexive relation on a finite set A, the following definitions are equivalent:*

i. R is a weak order structure (see 5.4)

ii. $\exists\ g\colon A \mapsto \mathbb{R}^+$ *satisfying for* $\forall a, b \in A$*:*

$$\begin{cases} aPb & \text{iff} \quad g(a) > g(b) \\ aIb & \text{iff} \quad g(a) = g(b) \end{cases}$$

iii. $\exists\ g\colon A \mapsto \mathbb{R}^+$ *satisfying for* $\forall a, b \in A$*:*

aRb *iff* $g(a) \geq g(b)$

Remark 6.1. *Numerical representations of preference structures are not unique. All monotonic strictly increasing transformations of the function g can be interpreted as equivalent numerical representations*[8].

Intransitivity of indifference or the appearance of intermediate hesitation relations is due to the use of thresholds that can be constant or dependent on the value of the objects under comparison (in this case values of the threshold might obey further coherence conditions).

Theorem 6.3 (Semi-Order)**.** *Let* $R = \langle P, I \rangle$ *be a binary relation on a finite set A, the following definitions are equivalent:*

i. R is a semi-order structure (see 5.5)

ii. $\exists\ g\colon A \mapsto \mathbb{R}^+$ *and a constant* $q \geq 0$ *satisfying* $\forall a, b \in A$*:*

$$\begin{cases} aPb & \text{iff} \quad g(a) > g(b) + q \\ aIb & \text{iff} \quad |g(a) - g(b| \leq q \end{cases}$$

iii. $\exists\ g\colon A \mapsto \mathbb{R}^+$ *and a constant* $q \geq 0$ *satisfying* $\forall a, b \in A$*:*

aRb *iff* $g(a) \geq g(b) - q$

iv. $\exists\ g\colon A \mapsto \mathbb{R}^+$ *and* $\exists q : \mathbb{R} \mapsto \mathbb{R}^+$ *satisfying* $\forall a, b \in A$*:*

$$\begin{cases} aRb & \text{iff} \quad g(a) \geq g(b) - q(g(b)) \\ (g(a) > g(b)) & \longrightarrow \quad (g(a) + q(g(a)) \geq g(b) + q(g(b))) \end{cases}$$

For the proofs of these theorems see Scott and Suppes, 1958, Fishburn, 1985, Krantz et al., 1989, Pirlot and Vincke, 1997.

The threshold represents a quantity for which any difference smaller than this one is not significant for the preference relation. As we can see, the threshold is not necessarily constant, but if it is not, it must satisfy the inequality which defines a coherence condition.

Here too, the representation of a semi-order is not unique and all monotonic increasing transformations of g appear as admissible representations provided the condition that the function q also obeys the same transformation [9].

Theorem 6.4 (PI Interval Order)**.** *Let $R = \langle P, I \rangle$ be a binary relation on a finite set A, the following definitions are equivalent:*

i. R is an interval order structure (see 5.6)

ii. $\exists\ g$: $A \mapsto \mathbb{R}^+$ satisfying $\forall a, b \in A$:

$$\begin{cases} aPb \quad \textit{iff} \quad g(a) > g(b) + q(b) \\ aIb \quad \textit{iff} \quad \begin{array}{l} g(a) \leq g(b) + q(b) \\ g(b) \leq g(a) + q(a) \end{array} \end{cases}$$

It should be noted that the main difference between an interval order and a semi-order is the existence of a coherence condition on the value of the threshold. One can further generalise the structure of interval order, by defining a threshold depending on both of the two alternatives. As a result, the asymmetric part appears without circuit: Abbas and Vincke, 1993, Agaev and Aleskerov, 1993,Subiza, 1994, Abbas, 1995, Diaye, 1999, Aleskerov et al., 2007. For extensions on the use of thresholds see Fishburn, 1997, Moore, 1966, Hansen, 1992. The special case where a "frontier" has to be explicitly considered instead of threshold is discussed in Bouyssou and Th. Marchant, 2011. For the extension of the numerical representation of interval orders in the case A is infinite not denumerable see Fishburn, 1973, Chateauneuf, 1987, Briges and Mehta, 1995, Oloriz et al., 1998, Nakamura, 2002, Bosi et al., 2007.

We can now see the representation theorems concerning preference structures allowing an intermediate preference relation (Q). Before that, let us mention that numerical representations with thresholds are equivalent to numerical representations of intervals. It is sufficient to note that associating a value $g(x)$ and a strictly positive value $q(g(x))$ to each element x of A is equivalent to associating two values: $l(x) = g(x)$ (representing the left extreme of an interval) and $r(x) = g(x) + q(g(x))$ (representing the right extreme of the interval to each x; obviously: $r(x) > l(x)$ always holds).

Theorem 6.5 (PQI Interval Orders)**.** *Let $R = \langle P, Q, I \rangle$ be a relation on a finite set A, the following definitions are equivalent:*

i. R is a PQI interval Order

ii. There exists a partial order L such that:
1) $I = L \cup R \cup I_d$ where $I_d = \{(x, x), x \in A\}$ and $R = L^{-1}$;
2) $(P \cup Q \cup L).P \subset P$; *3) $P.(P \cup Q \cup R) \subset P$;*
4) $(P \cup Q \cup L).Q \subset P \cup Q \cup L$; *5) $Q.(P \cup Q \cup R) \subset P \cup Q \cup R$.*

iii. $\exists\ l,r\colon A \mapsto \mathbb{R}^+$ *satisfying:*

$$\begin{cases} r(a) \geq l(a) & \\ aPb & \text{iff} \;\; l(a) > r(b) \\ aQb & \text{iff} \;\; r(a) > r(b) \geq l(a) \geq l(b) \\ aIb & \text{iff} \;\; r(a) \geq r(b) \geq l(a) \quad \text{or} \quad r(b) \geq r(a) \geq l(a) \geq l(b) \end{cases}$$

For proofs, further theory on the numerical representation and algorithmic issues associated with such a structure see Tsoukiàs and Vincke, 2003, Ngo The et al., 2000, Ngo The and Tsoukiàs, 2005.

Theorem 6.6 (Double Threshold Order)**.** *Let $R = \langle P, Q, I\rangle$ be a relation on a finite set A, the following definitions are equivalent:*

i. R is a double Threshold Order (see Vincke, 1988)

ii. $$\begin{cases} Q.I.Q \subset Q \cup P \\ P.I.P \subset P \\ Q.I.P \subset P \\ P.Q^{-1}.P \subset P \end{cases}$$

iii. $\exists\ g,q,p\colon A \mapsto \mathbb{R}^+$ *satisfying:*

$$\begin{cases} aPb & \text{iff} \;\; g(a) > g(b) + p(b)) \\ aQb & \text{iff} \;\; g(b) + p(b) \geq g(a) > g(b) + q(b) \\ aIb & \text{iff} \;\; g(b) + q(b) > g(a) > g(b) - q(a) \end{cases}$$

Theorem 6.7 (pseudo-order)**.** *Let $R = \langle P, Q, I\rangle$ be a relation on a finite set A, the following definitions are equivalent:*

i. R is a pseudo-order

ii. $$\begin{cases} \text{is a double threshold order} \\ \langle (P \cup Q), I\rangle \text{ is a semi-order} \\ \langle P, (Q \cup I \cup Q^{-1})\rangle \text{is a semi-order} \\ P.I.Q \subset P \end{cases}$$

iii. $$\begin{cases} \text{is a double threshold order} & \\ g(a) > g(b) \Longleftrightarrow & g(a) + q(a) > g(b) + q(b) \\ & g(a) + p(a) > g(b) + p(b) \end{cases}$$

A pseudo-order is a particular case of double threshold order, such that the thresholds fulfil a coherence condition. It should be noted however, that such a coherence is not sufficient in order to obtain two constant thresholds. This is due to different ways in which the two functions can be defined (see Doignon et al., 1986). For the existence of multiple constant thresholds see Doignon, 1987.

For partial structures of preference, the functional representations admit the same formulas, but equivalences are replaced by implications. In the following, we present a numerical representation of a partial order and a quasi-order examples:

Theorem 6.8 (Partial Order)**.** *If $\langle P, I, J\rangle$ presents a partial order structure,then $\exists$ g: $A \mapsto \mathbb{R}^+$ such that:*
$$\left\{ \quad aPb \quad \Longrightarrow \quad g(a) > g(b) \right.$$

Theorem 6.9 (Partial Weak order)**.** *If $\langle P, I, J\rangle$ presents a partial weak order structure, then $\exists$ g: $A \mapsto \mathbb{R}^+$ such that:*
$$\left\{ \begin{array}{lll} aPb & \Longrightarrow & g(a) > g(b) \\ aIb & \Longrightarrow & g(a) = g(b) \end{array} \right.$$

The detection of the dimension of a partial order [10] is a NP hard problem (Doignon et al., 1984, Fishburn, 1985).

Remark 6.2. *In the preference modelling used in decision aiding, there exist two different approaches: In the first one, the evaluations of alternatives are known (they can be crisp or fuzzy) and we try to reach conclusions about the preferences between the alternatives. For the second one, the preferences between alternatives (pairwise comparison) are given by an expert (or by a group of experts), and we try to define an evaluation of the alternatives that can be useful. The first approach uses the inverse implication of the equivalences presented above (for example for a total order we have $g(a) > g(b) \longrightarrow aPb$); and the second one the other implication of it (for the same example, we have $aPb \longrightarrow g(a) > g(b)$)*

Remark 6.3. *There is a body of research on the approximation of a preference structure by another one; here we cite some studies on the research of a total order with a minimum distance to a tournament (complete and antisymmetric relation): Slater, 1961, Bermond, 1972, Monjardet, 1979, Barthélémy and Monjardet, 1981, Barthélémy et al., 1989, Charon-Fournier et al., 1992, Hudry and Woirgard, 1996,*

6.2 Minimal Representation

In some decision aiding situations, the only available preferential information can be the kind of preference relation holding between each pair of alternatives. In such a case we can try to build a numerical representation of each alternative by choosing a particular functional representation of the ordered set in question and associating this with the known qualitative relations.

This section aims at studying some minimal or parsimonious representations of ordered sets, which can be helpful for this kind of situation. Particularly, given a countable set A and a preference relation $R \subseteq A \times A$, we are interested to find a numerical representation $\hat{f} \in \mathcal{F} = \{f : A \mapsto \mathbb{R},\ f \text{ homomorph to } R\}$, such that for all $x \in A$, $\hat{f}$ is minimal.

6.2.1 Total Order, Weak order. The way to build a minimal representation for a total order or a weak order is obvious since the preference and the indifference relations are transitive: The idea is to minimise the value of the difference $g(a) - g(b)$ for all a, b in A. To do this we can define a unit $k = \min_{a,b \in A}(g(a) - g(b))$ and the minimal evaluation $m = \min_{a \in A}(g(a))$. The algorithm will be:

- Choose any value for k and m , e.g. $k = 1$, $m = 0$;
- Find the alternative i which is dominated by all the other alternatives j in A and evaluate it by $g(i) = m$
- For all the alternatives l for which we have lIi, note $g(l) = g(i)$
- Find the alternative i' which is dominated by all the alternatives j' in $A - \{i\}$ and evaluate it by $g(i') = m + k$
- For all the alternatives l' for which we have $l'Ii'$, note $g(l') = g(i')$
- Stop when all the alternatives are evaluated

6.2.2 Semi-order. The first study on the minimal representation of semi-orders was done in Pirlot, 1990 who proved its existence and proposed an algorithm to build it. One can find more information about this in Doignon, 1988, Mitas, 1994, Pirlot and Vincke, 1997 and Ngo The, 2002. Pirlot uses an equivalent definition of the semi-order which uses a second positive constant: *Total Semi-order*: A reflexive relation $R = (P, I)$ on a finite set A is a semi-order iff there exists a real function g, defined on A, a non negative constant q and a positive constant ε such that $\forall a, b \in A$

$$\begin{cases} aPb & \text{iff} \quad g(a) > g(b) + q + \varepsilon \\ aIb & \text{iff} \quad |g(a) - g(b)| \leq q \end{cases}$$

Such a triple $(g,\ q,\ \varepsilon)$ is called an $\varepsilon - representation$ of (P, I). Any representation $(g,\ q)$, as in the definition of semi-order given in 5.1, yields an ε-representation where

$$\varepsilon = min_{(a,b)\in P}(g(a) - g(b) - q)$$

Let (A, R) be an associated to the semi-order $R = (P, I)$, we denote $G(q, \varepsilon)$ the valued graph obtained by giving the value $(q + \varepsilon)$ to the arcs P and $(-q)$ to the arcs I.

Theorem 6.10. *: If R=(P, I) is a semi-order on the finite set A, there exists an ε-representation with threshold q iff:*

$$\frac{q}{\varepsilon} \geq \alpha = max_C\{\frac{|C \cap P|}{|C \cap I| - |C \cap P|},\ C\ circuit\ of\ (A,\ R)\}$$

where $|C \cap P|$ (resp. $|C \cap I|$), represents the number of arcs P (resp. I) in the circuit C of the graph (A, R).

An algorithm to find a numerical representation of a semi-order is as follows:

- Choose any value for εk, e.g. $\varepsilon = 1$;
- Choose a large enough value of $\frac{q}{\varepsilon}$(e.g. $\frac{q}{\varepsilon} = |P|$;
- Solve the maximal value path problem in the graph $G(q, \varepsilon)$ (e.g. by using the Bellman algorithm, see Lawler, 1976).

Denote by $g_{q,\varepsilon}$, the solution of the maximal path problem in $G(q, \varepsilon)$; we have:

$$g_{q,\varepsilon} \leq g(a) \forall a \in A$$

Example 6.1. *We consider the example given by Pirlot and Vincke (see Pirlot and Vincke, 1997):*
Let $S = (P, I)$ be a semiorder on $A = \{a, b, c\}$ defined by $P = \{(a, c)\}$.

The first inequality of 6.2.2 gives the following equations:

$$\begin{aligned} g(a) &\geq g(c) + q + \varepsilon \\ g(a) &\geq g(b) - q \\ g(b) &\geq g(a) - q \\ g(b) &\geq g(c) - q \\ g(c) &\geq g(b) - q \end{aligned}$$

Figure 1.3 shows the graphical representation of this semiorder.

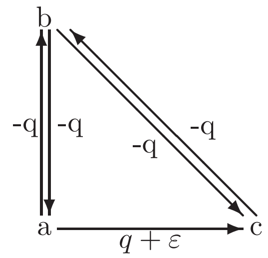

Figure 1.3. Graphical representation of the semiorder

		a	b	c
q=1	g_1=1	2	1	0
q=1	g_2=1	9.5	8.5	7.5
q=2.5	g_3=1	3.5	1	0
q=2.5	g_4=1	10.5	8.5	7
q=2.5	g_5=1	3.5	2.5	0

Table 1.2. Various ε-representations with ε=1

As the non-trivial circuit $C = \{(a,c),(c,b),(b,a)\}$ is $-q+\varepsilon$ ($-q+\varepsilon = (q+\varepsilon)+(-q)+(-q)$), necessary and sufficient conditions for the existence of an ε-representation is $q \geq \varepsilon$.
The table1.2 provides an example of possible numerical representation of this semiorder:

Definition 6.1. *A representation (g^*, q^*, ε) is minimal in the set of all non-negative ε-representations $(g,\ q,\ \varepsilon)$ of a semiorder iff $\forall a \in A$ $g^*(a) \leq g(a)$.*

Theorem 6.11. *The representation $(g_{q^*,\varepsilon}, q^*, \varepsilon)$ is minimal in the set of all ε-representations of a semiorder R.*

6.2.3 Interval Order. An interval can be represented by two real functions l and r on the finite set A which satisfy:

$$(\forall a \in A,\ l(a) \leq r(a))^{11}$$

Definition 6.2. *A reflexive relation $R = (P \cup I)$ on a finite set A is an interval order iff there exists a pair of functions l, r: $A \longrightarrow R^+$ and a positive constant ε such that $\forall a, b \in A$*

$$\begin{cases} aPb & \textit{iff} \quad l(a) > r(b) + q + \varepsilon \\ aIb & \textit{iff} \quad l(a) \geq r(b) \quad \textit{and} \quad l(b) \geq r(a) \end{cases}$$

Such a triplet (l, r, ε) is called an ε-representation of the interval order $P \cup I$.

Definition 6.3. *The ε-representation (l^*, r^*, ε) of the interval order $P \cup I$ is minimal iff for any other ε-representation (l, r, ε) we have, $\forall a \in A$,*

$$l^*(a) \leq l(a)$$
$$r^*(a) \leq r(a)$$

Theorem 6.12. *For any interval order $P \cup I$, there exists a minimal ε-representation (l^*, r^*, ε); the values of l^* and r^* are integral multiples of ε.*

6.2.4 *PQI* Interval Order. Ngo The and Tsoukias (Ngo The and Tsoukiàs, 2005) have extended the results concerning the minimal representation of interval orders to the case of *PQI* interval orders. After presenting some real life examples which showed that it does not make sense to have a minimal representation of a *PQI* interval orders, they studied the problem through an instance of a *PQI* interval orders which is a separated *PQI* interval orders (it corresponds to the presentation of the condition *ii* of Theorem 6.5 where the indifference is separated into three relations, the identity, a partial order and its inverse). They obtained a result enabling to order the endpoints of intervals using an ε-representation like in the case of interval orders and they proposed two algorithms : the first one determining a general numerical representation (in $O(n^2)$) and the second one minimising the first one (in $O(n)$).

7. Extending preferences to sets

The problem of how to extend a preference relation over a set A of “objects” (e.g., alternatives, opportunities, candidates, etc.) to the set of all subsets of A is a very general problem inspired to many individual and collective decision making situations. Consider, for instance, the comparison of the stability of groups in coalition formation theory, or the ranking of likely sets of events in the axiomatic analysis of subjective probability, or the evaluation of equity of sets of rights inside a society, or the comparison of assets in portfolio analysis. In those situations, and in many others, a ranking of the single elements of a (finite) universal set A is not sufficient to compare the subsets of A. On the other hand, for many practical problems, only the information about preferences among single objects is available. Consequently, a central question is:

how to derive a ranking over the set of all subsets of A in a way that is "compatible" with the primitive ranking over the single elements of A?

This question has been carried out in the tradition of the literature on extending an order on a set A to its *power set* (denoted by 2^A) with the objective to axiomatically characterise families of ordinal preferences over subsets (see, for instance, Barberà et al., 1984, Barberà et al., 2004, Bossert, 1995, Bossert et al., 1994, Geist and Endriss, 2011, Fishburn, 1992, Kannai and Peleg, 1984, Kreps, 1979). In this context, an order $\succcurlyeq$ on the power set 2^A is required to be an *extension* of a primitive order R on A. This means that the relative ranking of any two singleton sets according to $\succcurlyeq$ must be the same as the relative ranking of the corresponding alternatives according to R (i.e., for each $a, b \in A$, $\{a\} \succcurlyeq \{b\} \Leftrightarrow aRb$).

The interpretation of the properties used to characterise extensions is deeply interconnected to the meaning that is attributed to sets. According to the survey of Barberà et al., 2004, the main contributions from the literature on ranking sets of objects may be grouped in three main classes of problems: 1) *complete uncertainty*, where a decision-maker is asked to rank sets which are considered as formed by mutually exclusive objects (i.e., only one object from a set will materialise), and taking into account that he cannot influence the selection of an object from a set (see, for instance, Barberà et al., 1984, Kannai and Peleg, 1984, Nitzan and Pattanaik, 1984); 2) *opportunity sets*, where sets contain again mutually exclusive objects but, in this case, a decision maker compares sets taking into account that he will be able to select a single element from a set (see, for example, Bossert et al., 1994, Kreps, 1979, Puppe, 1996, Puppe, 1997); 3) *sets as final outcomes*, where each set contains objects that are assumed to materialise simultaneously (if that set is selected; for instance, see Bossert, 1995, Fishburn, 1992, Roth, 1985).

In order to better clarify the differences between these three classes of problems, and to stress the importance of the nature of problems in the selection of intuitive axioms, consider the following example. Let $A = \{a, b\}$ be a universal set of two alternatives. Suppose that a decision-maker prefers a over b. Then under the *complete uncertainty* interpretation, it is reasonable to expect that the decision-maker will prefer set $\{a\}$ to $\{a, b\}$, since the possibility that alternative b materialises does exist if set $\{a, b\}$ is selected. But under the interpretation of *opportunity sets*, the two sets $\{a\}$ and $\{a, b\}$ could be simply considered indifferent. Finally, under the interpretation of *sets as final outcomes*, if objects are goods, one could guess that to have $\{a, b\}$ is better, because the decision-maker will receive both b and a. But the judgement depends on the nature of a and b and on possible effects of incompatibility between the two objects.

Let R be a binary relation on the set A, being R the characteristic relation of a preference structure $\langle P, I\rangle$. In the following, in order to rank the elements of 2^A, we use a binary relation $\succcurlyeq$ on the set 2^A, being $\succcurlyeq$ the characteristic relation of a preference structure $\langle \succ, \sim\rangle$. For example, assume that a linear order R on the set A is given. For each $S \in 2^A \setminus \{\emptyset\}$, we denote by $\max(S, R)$ the *best element* of S with respect to R such that $\max(S)Rb$ for each $b \in S$, and by $\min(S, R)$ the *worst element* of S with respect to R such that $bR\min(S)$ for each $b \in S$. Perhaps the two simplest extensions of R are the *MAX extension* and the *MIN extension*, which are defined, respectively, as a binary relation $\succcurlyeq^{\max}$ on 2^A such that $(S \succcurlyeq^{\max} T) \Leftrightarrow (\max(S)R\max(T))$, and as a binary relation $\succcurlyeq^{\min}$ on 2^A such that $(S \succcurlyeq^{\min} T) \Leftrightarrow (\min(S)R\min(T))$, for each $S, T \in 2^A \setminus \{\emptyset\}$.

7.1 Complete uncertainty

In this section we introduce some axioms used in the literature in order to characterise extensions under complete uncertainty. In this context, a decision-maker is assumed to face a decision problem of establishing a ranking over all possible sets of outcomes, provided that the objects of a set are interpreted as mutually exclusive outcomes, and a final outcome is selected at a later stage according to a random procedure. As an example in this class, consider the problem faced by a policy maker that must compare different public policies, where a public policy may bring, after a certain period of time, to alternative (mutually exclusive) outcomes, whose realisation may be influenced by unforeseen contingencies.

Historically, one of the most studied axioms for extensions in this class of problems is the *dominance* property, that is referred to as the *Gärdenfors principle* in Kannai and Peleg, 1984, in recognition of the use of this axiom in Gärdenfors, 1976. This property requires that adding an element which is better (worse) than all elements in a given set $S \in 2^A$ according to a preference relation R on the universal set A, leads to a set that is better (worse) than the original set according to preference relation $\succcurlyeq$ over 2^A.

Definition 7.1 (Dominance, DOM)**.** *Let R be a binary relation on A. A binary relation $\succcurlyeq$ on 2^A satisfies the dominance property (with respect to R) iff for all $S \in 2^A$ and for all $a \in A$,*

$$(i)\ \ [aPb \text{ for all } b \in S] \Rightarrow S \cup \{a\} \succ S;$$
$$(ii)\ \ [bPa \text{ for all } b \in S] \Rightarrow S \succ S \cup \{a\}.$$

It is important to note that, if R on A is reflexive and antisymmetric and $\succcurlyeq$ on 2^A is reflexive and transitive, than the property of dominance for $\succcurlyeq$ (w.r.t. R) implies that $\succcurlyeq$ is an extension of R (i.e., if aPb, then the DOM property implies that $\{a\} \succ \{a,b\}$ and also $\{a,b\} \succ \{b\}$; so, by transitivity, $\{a\} \succ \{b\}$).

Another important axiom which has extensively been used in the literature is the *independence* property (introduced by Kannai and Peleg, 1984 with the name of *monotonicity axiom*). It requires that if there exists a strict preference between two sets $S, T \in 2^A$ and the same alternative $a \in A$ is added to both sets, then the ranking between the two formed sets must exist (according to $\succcurlyeq$) and cannot be reversed.

Definition 7.2 (Independence, IND)**.** *Let R be a binary relation on A. A binary relation $\succcurlyeq$ on 2^A satisfies the independence property (with respect to R) iff for all $S, T \in 2^A$, for all $a \in A \setminus (S \cup T)$,*

$$S \succ T \Rightarrow (S \cup \{a\}) \succcurlyeq (T \cup \{a\}).$$

The following proposition (Kannai and Peleg, 1984), says that if a reflexive and transitive relation $\succcurlyeq$ on 2^A satisfies DOM (i.e. $\succcurlyeq$ is an extension of R on A) and IND, then any set $A \in 2^A \setminus \{\emptyset\}$ is indifferent (with respect to $\succcurlyeq$) to the set consisting of the best element and the worst element in A (according to the primitive linear order R).

Theorem 7.1. *Let R a linear order on A and let $\succcurlyeq$ be a reflexive and transitive relation on 2^A. If $\succcurlyeq$ satisfies DOM and IND (w.r.t. R), then*

$$S \simeq \{\max(S,R), \min(S,R)\}$$

for all $S \in 2^A \setminus \{\emptyset\}$.

For a proof of this theorem see Kannai and Peleg, 1984, Barberà et al., 2004. Both DOM and IND are quite intuitive properties for extensions when objects are mutually exclusive. Surprisingly, the following proposition shows that DOM and IND properties are incompatible when completeness of the ranking on the 2^A is assumed (and $|A| \geq 6$).

Theorem 7.2. *Let R be a linear order on A, with $|A| \geq 6$. There exists no total preorder $\succcurlyeq$ on 2^A which satisfies DOM and IND.*

For a proof of this theorem see Kannai and Peleg, 1984, Barberà et al., 2004. Other (possibility or impossibility) results can be obtained by modifying axioms IND and DOM (Barberà and Pattanaik, 1984,Bossert et al., 2000, Geist and Endriss, 2011), or weakening the assumption that $\succcurlyeq$ is a total preorder on 2^A (Nitzan and Pattanaik, 1984, Pattanaik and

Peleg, 1984). Many other extensions have been proposed and axiomatically studied in the literature for problems under complete uncertainty Barberà et al., 2004. In particular, we refer to the *lexi-max* and *lexi-min* extensions (Pattanaik and Peleg, 1984, Bossert, 1995), which are obtained, respectively, as the lexicographical generalizations of the MAX and the MIN extensions, and the *median-based* extensions (Nitzan and Pattanaik, 1984), where the relative ranking of the median alternatives is used as the criterion for comparing two sets.

7.2 Opportunity sets

For this family of problems, the elements in 2^A are interpreted as sets of opportunities from which a decision-maker is allowed to select precisely one element. Note that the substantial difference from the context of complete uncertainty is that for opportunity sets the choice of an outcome from a set is left to the decision-maker, whereas in the context of complete uncertainty the selection procedure is based on a random device that cannot be influenced by the decision-maker. An example of opportunity set is the set of consumption bundles that a consumer may afford given his budget and the market price of goods in the bundle. Another example could be the sets of candidates (e.g., corresponding to different parties) that are available to a voter in a particular election (Gravel, 2008).

In Kreps, 1979, a characterisation of the MAX extension $\succcurlyeq^{\max}$ for opportunity sets[12] is provided. The axiom of *extension robustness* used in Kreps, 1979 requires that adding to a set $A \in 2^A$ a set $B \in 2^A$ that is at most as good as A determines a set that is indifferent to A.

Definition 7.3 (Extension Robustness, EXT ROB)**.** *A binary relation $\succcurlyeq$ on 2^A satisfies the extension robustness property if and only if for all $S, T \in 2^A$,*

$$S \succcurlyeq T \Rightarrow S \sim (S \cup T).$$

One of the main results in Kreps, 1979 is that a binary relation $\succcurlyeq$ on 2^A satisfies the EXT ROB property if and only if there exists a linear order R on A such that $\succcurlyeq$ coincides with $\succcurlyeq^{\max}$, the MAX extension on R.

The MAX extension has been subject to some criticism when used to compare sets of opportunities. A certain branch of the economic literature, illustrated by the contributions of Baharad and Nitzan, 2000, Bossert, 1996, Bossert et al., 1994, Dutta and Sen, 1996, Gravel, 1994, Gravel, 1998, Gravel et al., 1997, Klemisch-Ahlert, 1993, Pattanaik and Xu, 1990 have attempted to define rankings of opportunity sets without explicitly refer to the future choice behavior of a decision-maker. The problem of ranking opportunity sets in this context amounts to define what it means for a set of opportunities to offer more *freedom of choice* than another. We do not enter here in the philosophical debate on the concept of "freedom" (see, for instance, Gravel, 2008) and how its definition may be related to the nature of different constraints (physical, economical, legal, etc. Hayek, 1960, Gravel, 2008). Moreover, there is no unity in the opportunity sets literature about the notion of freedom to be used for ranking opportunity sets. According to Baujard, 2006, different notions of freedom have been proposed: freedom of choice *per se*, introduced by the seminal article of Pattanaik and Xu, 1990, where the absence of preference information means that a measure of freedom can only reflect quantitative aspects of opportunity sets; freedom as *autonomy*, which keeps into account the autonomy of the decision-makers in making choices and where the autonomy is defined according to the independence of the choices of a decision-maker of his conditioning or of the will of other decision-makers (Jones and Sugden, 1982); freedom as the valuation of *exercise of choice*, where the significance of the choices is evaluated according to some notion of diversity or similarity among alternatives (e.g., see Pattanaik and Xu, 2000, Rosenbaum, 2000); *negative* freedom, where ranking is aimed to represent the measure of absence of coercion or oppression imposed by other decision-makers on individual choices rather than any other constraints (Van Hees, 1998).

7.3 Sets as final outcomes

In this section, the problem of how to rank sets of elements that materialise simultaneously is considered. For instance, consider the formation of coalitions that should work jointly for a common goal, or the election of new members to join an organisation, or many situations where matching problems arise. A standard application of this kind of problems is the *college admissions problem* (Roth, 1985, Gale and Shapley, 1962), where colleges need to rank sets of students based on their ranking of individual applicants.

We start with the introduction of the *fixed cardinality ranking* approach (Roth, 1985), where the number of elements in ranked sets is fixed *a priori*. For instance, in the college admission problem, where the objective is to evaluate individual students for the admissibility to the first class, colleges are assumed to have a fixed quota $q \in \mathbb{N}$ specifying the maximum number of students they can admit. Therefore, matching analysis concentrates on the preferences of colleges over sets of students of size q. In order to analyse this kind of problems, Roth, 1985 introduced the property of *responsiveness*, which requires that if one element

a in a set A is replaced by another element b, then the ranking between the new set $A \setminus \{a\} \cup \{b\}$ and the original set A is determined by the ranking between a and b according to R. In the following, we denote by $\mathcal{A}_q$ the set of all subsets of A of cardinality $q \in \{1, \dots, |A|\}$, that is $\mathcal{A}_q = \{S \in 2^A \text{ s.t. } |S| = q\}$.

Definition 7.4 (Responsiveness, RESP)**.** *Let R be a binary relation on the set A. A binary relation $\succcurlyeq_q$ on $\mathcal{A}_q$ satisfies the responsiveness property on $\mathcal{A}_q$ (and with respect to R) iff for all $S \in \mathcal{A}_q$, for all $a \in A$ and for all $b \in A \setminus S$ we have that*

$$\big[[S \succcurlyeq_q (S \setminus \{a\}) \cup \{b\}] \Leftrightarrow aRb\big] \text{ and } \big[[(S \setminus \{a\}) \cup \{b\} \succcurlyeq_q S] \Leftrightarrow bRa\big].$$

Clearly, the RESP property is aimed at preventing complementarity effects. As shown in Bossert, 1995, the RESP property was used by Bossert, 1995 to characterize[13] the family of *lexicographic rank-ordered extensions*, which generalise the idea of lexi-min and lexi-max orderings.

Another simple way to generate rankings of sets as final results is to look for a utility representation of the ranking over sets (Fishburn, 1970, Fishburn, 1967, Roberts, 1979). In particular, it is interesting to study under which conditions an extension $\succcurlyeq$ of R is additively representable (De Finetti, 1931, Fishburn, 1970).

A still different approach was introduced by Fishburn, 1992, where the information available to establish an extension is not only a primitive ranking on the universal set A, but also a *signed ordering* on the "complements" of the alternatives in A is available. Looking at A as a set of possible candidates for a committee, for instance, the model based on signed ordering allows for the consideration of comparisons like "it is more important to prevent a candidate a from being in the committee than having candidate b in the committee", or "leaving candidate a off the committee is preferred to leaving b off the committee", etc. Properties of signed orderings and conditions for their extensions in this richer informational content are presented in Fishburn, 1992.

Recently, Moretti and Tsoukiàs, 2012, introduced a new class of orderings of sets as final results, and they called the elements of this class *Shapley extensions*, for their attitude to preserve the ranking provided by the *Shapley value* (Shapley, 1953; Moretti and Patrone, 2008) of associated coalitional games. In general, Shapley extensions do not need to satisfy the RESP property (even if an axiomatic characterization using this property on the class of monotonic total preorders is provided in Moretti and Tsoukiàs, 2012) and therefore they can be used to keep into account possible complementarity effects among objects.

7.4 An overview to related theories

The problem of electing a committee is also well-studied in *voting theory* (Brams and Fishburn, 2002; Brams et al., 2007; Kilgour et al., 2006). In such situations, voters face the problem to choose from a finite set A of candidates a nonempty subset K of committee members. For a general discussion on voting methods see, for example, Chevaleyre et al., 2009 and Lang and Xia, 2009. Here we focus on the aspects of the problem which are directly related to the extension of preference of voters over single candidates to subsets of candidates. Following an example illustrated by Uckelman, 2009, suppose that a group of voters are invited to elect a committee of three persons from the set of five candidates $A = \{1, 2, 3, 4, 5\}$. Now, suppose that a voter believes that 1 and 2 are the best candidates: we may represent this fact with a preference structure $\langle P, I\rangle$ on A such that $1\ I\ 2\ P\ 3\ I\ 4\ I\ 5$. Consequently, it could be reasonable to assume that any committee containing one of them is better than any committee with neither. In addition, suppose also that such a voter also believes that 1 and 2 will fight if they are on the committee together (so, any committee with both of them is worse than any committee with neither). Thus, the voter would rank the committees in the following way:

$$\begin{aligned}&\{1,3,4\} \sim \{1,3,5\} \sim \{1,4,5\} \sim \{2,3,4\} \sim \{2,3,5\} \sim \{2,4,5\}\\ &\succ \{3,4,5\} \succ \{1,2,3\} \sim \{1,2,4\} \sim \{1,2,5\}.\end{aligned} \tag{1.13}$$

Which criterion can be adopted to extend a characteristic relation R of the preference structure $\langle P, I\rangle$ on single candidates, in order to end up in a characteristic relation $\succcurlyeq$ of the preference structure $\langle \succ, \sim\rangle$ on committees? Put in a more general way, how to consider the fact that committee membership for one candidate is not necessarily independent of the question of committee membership for some other candidate? Several approaches have been proposed in literature to extend preference of voters. In Kilgour et al., 2006 and Brams et al., 2007, voters are assumed to rank committees according to their *Hamming distance* from their top preferences, where the top preference of a voter is the committee it most prefers. Let n be the number of voters and $k \leq n$ be the number of seats in the committee. A *ballot* is a binary k-vector, $(p_1, p_2, \ldots, p_k)$, where p_i equals 0 or 1, for each $i \in \{1, \ldots, k\}$. These binary vectors indicate the approval or disapproval of each candidate by a voter. For instance, ordering candidates increasingly, the committee $\{1, 3, 4\}$ corresponds to the ballot $(1, 0, 1, 1, 0)$ (shortly, 10110). The Hamming distance $d(p, q)$ between two binary k-vectors p and q is the number of components on which they differ. Note that the Hamming distance between $\{1, 3, 4\}$ and

$\{2,4,5\}$ is $d(10110, 01011) = 4$, whereas the distance between $\{1,3,4\}$ and $\{1,2,3\}$ is $d(10110, 11100) = 2$. Consequently, the ordering induced by the Hamming distance from $\{1,3,4\}$ is $\{1,3,4\} \succ \{1,2,3\} \succ \{2,3,4\}$, thus putting an optimal committee last and one least favoured committee in second place, which does not represent the true ranking $\langle \succ, \sim \rangle$ introduced in (1.13).

In *financial theory*, the goal of *portfolio management* is to allocate resources and budgets to a group of *assets* (e.g., stocks, projects, initiatives etc.) that maximise the return and minimise the risk. Typically, the answer to the investment problem is not the selection of the most preferred assets: a diversified portfolio will likely have less risk than the weighted average risk of its constituent assets (see, for instance, Samuelson, 1967). Therefore, the problem to extend preferences over single assets to a preference over portfolios of assets is very important in practical investment problems. Since the pioneering paper of Markowitz, 1968, where the classical model of *Mean-Variance optimisation* has been developed, many different techniques for portfolio management have been proposed in the area of multi-criteria analysis (Zopounidis, 1999, Salo et al., 2011).

We conclude this section with a short introduction to some applications in *artificial intelligence* which require the specification of preferences over sets of information items that a computer should be able to process (desJardins et al., 2006). For instance, web search engines are designed to retrieve information relative to a particular query on the World Wide Web, presenting the retrieved information as a list of *hits* (e.g., web pages, images, media files, etc.). Since search engines operate according to predefined algorithms or procedures, efficient methods to specify and compute the relevance of sets of hits to a specified query are demanded.

In order to specify preferences of decision-makers on sets of items, one possibility is to assume that preferences are numerical and to use compact representation of such valuations as, for instance, the *bidding languages* for *combinatorial auctions* (Nisan, 2000; Uckelman, 2009). An alternative approach is provided by *ordinal preferences* and methods that have been introduced in literature for elicitation and compact representation of ordinal preferences over combinatorial domains. A well-known language for eliciting and representing ordinal preferences over combinatorial domains is known under the name of (*Ceteris Paribus*) *CP-nets* (Boutilier et al., 2004b), which is tailored for representing preference relations on the domain of each variable conditioned by the values of the variables it depends on.

More recently, richer (and more sophisticated) approaches have been introduced: TCP-nets (Brafman et al., 2006), which extend CP-nets by allowing statements of *conditional importance* between single variables; *conditional preference theories* (Wilson, 2004), which further extend TCP-nets; *conditional importance* networks (CI-nets) (Bouveret et al., 2009), that also generalise TCP nets, with the further simplification that CI-nets do not include any conditional preference statements on the values of the variables.

Specific solution for information retrieval problems have been also introduced. A new language has been developed in desJardins et al., 2006, namely (*Depth and Diversity*) *DD-pref*, that allows for set-based preference learning starting from the specification of few examples in a numerical form and keeping into account possible effects of interaction among single items. Effects of complementarity have been modelled in Zhai et al., 2003 as a trading off "relevance" against "novelty" of information; or by measuring the "marginal relevance" (Carbonell and Goldstein, 1998), with the objective to minimise redundancy in a set of information items corresponding to a certain query.

8. Logic of Preferences

The increasing importance of preference modelling immediately interested people from other disciplines, particularly logicians and philosophers. The strict relation with deontic logic (see Åqvist, 1986) raised some questions such as:

- does a general logic exist where any preferences can be represented and used?
- if yes, what is the language and what are the axioms?
- is it possible, via this formalisation, to give a definition of bad or good as absolute values?

It is clear that this attempt had a clear positivist and normative objective: to define the one well-formed logic that people should follow when expressing preferences. The first work on the subject is the one by Halldén, 1957, but it is Von Wright's book (von Wright, 1963) that tries to give the first axiomatisation of a logic of preferences. Inspired by this work some important contributions have been made (Houthakker, 1965, Chisholm and Sosa, 1966b, Chisholm and Sosa, 1966a, Rescher, 1967, Hansson, 1966b, Hansson, 1966a). Influence of this idea can also be found in Jeffrey, 1965 and Rescher, 1969, but in related fields (statistics and value theory respectively). The discussion apparently was concluded by von Wright, 1972, but Huber, 1974, Huber, 1979 continued on later on Halldin, 1986 and Widmeyer, 1988, Widmeyer, 1990 also worked on this.

The general idea can be presented as follows. At least two questions should be clarified: preferences among what? How should preferences be understood? Von Wright (von Wright, 1963) argues that preferences can be distinguished as extrinsic and intrinsic. The first ones are derived as *a reason from a specific purpose*, while the second ones are *self-referential* to an actor expressing the preferences. In this sense intrinsic preferences are the expression of the actor's system of values of the actor. Moreover, preferences can be expressed for different things, the most general being (following Von Wright) *"states of affairs"*. That is, the expression "a is preferred to b" should be understood as the preference of a state (a world) where a occurs (whatever a represents: sentences, objects, relations etc.) over a state where b occurs. On this basis Von Wright expressed a theory based on five axioms:

$\mathrm{A}^W 1.\ \forall x, y\ \ p(x, y)\ \rightarrow\ \neg p(y, x)$
$\mathrm{A}^W 2.\ \forall x, y, z\ \ p(x, y)\ \wedge\ p(y, z)\ \rightarrow\ p(x, z)$
$\mathrm{A}^W 3.\ p(a, b)\ \equiv\ p(a \wedge \neg b, \neg a \wedge b)$
$\mathrm{A}^W 4.\ p(a \vee b, c)\ \equiv\ p(a \wedge b \wedge \neg c, \neg a \wedge \neg b \wedge c)\ \wedge\ p(a \wedge \neg b \wedge \neg c, \neg a \wedge \neg b \wedge c)\ \wedge\ p(\neg a \wedge b \wedge \neg c, \neg a \wedge \neg b \wedge c)$
$\mathrm{A}^W 5.\ p(a, b)\ \equiv\ p(a \wedge c, b \wedge c)\ \wedge\ p(a \wedge \neg c, b \wedge \neg c)$

The first two axioms are asymmetry and transitivity of the preference relation, while the following three axioms face the problem of combinations of states of affairs. The use of specific elements instead of the variables and quantifiers reflects the fact that von Wright considered the axioms not as logical ones, but as "reasoning principles". This distinction has important consequences on the calculus level. In the first two axioms, preference is considered as a binary relation (therefore the use of a predicate), in the three "principles", preference is a proposition. Von Wright does not make this distinction directly, considering the expression aPb ($p(a, b)$ in our notation) as a well-formed formulation of his logic. However, this does not change the problem since the first two axioms are referred to the binary relation and the others are not. The difference appears if one tries to introduce quantifications; in this case the three principles appear to be weak. The problem with this axiomatisation is that empirical observation of human behavior provides counterexamples of these axioms. Moreover, from a philosophical point of view (following the normative objective that this approach assumed), a logic of intrinsic preferences about general states of affairs should allow to define what is good (the always preferred?) and what is bad (the always not preferred?). But this axiomatisation fails to enable such a definition.

Chisholm and Sosa (Chisholm and Sosa, 1966b) rejected axioms A^W3 to A^W5 and built an alternative axiomatisation based on the concepts of *"good"* and *"intrinsically better"*. Their idea is to postulate the concept of good and to axiomatise preferences consequently. So a *good* state of affairs is one that is always preferred to its negation ($p(a, \neg a)$); Chisholm and Sosa, use this definition only for its operational potential as they argue that it does not capture the whole concept of "good"). In this case we have:

A^S1. $\forall x, y \ \ p(x, y) \ \rightarrow \ \neg p(y, x)$
A^S2. $\forall x, y, z \ \ \neg p(x, y) \ \wedge \ \neg p(y, z) \ \rightarrow \ \neg p(x, z)$
A^S3. $\forall x, y \ \ \neg p(x, \neg x) \ \wedge \ \neg p(\neg x, x) \ \wedge \ \neg p(y, \neg y) \ \wedge \ \neg p(\neg y, y) \ \rightarrow \ \neg p(y, x) \ \wedge \ \neg p(x, y)$
A^S4. $\forall x, y \ \ p(x, y) \ \wedge \ \neg p(y, \neg y) \ \wedge \ \neg p(\neg y, y) \ \rightarrow \ p(x, \neg x)$
A^S5. $\forall x, y \ \ p(y, \neg x) \ \wedge \ \neg p(y, \neg y) \ \wedge \ \neg p(\neg y, y) \ \rightarrow \ p(x, \neg x)$

Again in this axiomatisation there are counterexamples of the axioms. The assumption of the concept of good can be argued as it allows circularities in the definitions of preferences between combinations of states of affairs. This criticism leaded (Hansson, 1966b) to consider only two fundamental, universally recognised axioms:

A^H1. $\forall x, y, z \ \ s(x, y) \ \wedge \ s(y, z) \ \rightarrow \ s(x, z)$
A^H2. $\forall x, y \ \ s(x, y) \ \vee \ s(y, x)$

where s is a "large preference relation" and two specific preference relations are defined, p (strict preference) and i (indifference):

D^H1. $\forall x, y \ \ p(x, y) \ \equiv \ s(x, y) \ \wedge \ \neg s(y, x)$
D^H2. $\forall x, y \ \ i(x, y) \ \equiv \ s(x, y) \ \wedge \ s(y, x)$

He also introduces two more axioms, although he recognises their controversial nature:

A^H3. $\forall x, y, z \ \ s(x, y) \ \wedge \ s(x, z) \ \rightarrow \ s(x, y \vee z)$
A^H4. $\forall x, y, z \ \ s(x, z) \ \wedge \ s(y, z) \ \rightarrow \ s(x \vee y, z)$

Von Wright in his reply (von Wright, 1972), trying to argue for his theory, introduced a more general frame to define intrinsic *"holistic"* preferences or as he called them *"ceteris paribus"* preferences. In this approach he considers a set S of states where the elements are the ones of A (n elements) and all the 2^n combinations of these elements. Given two states s and t (elementary or combinations of m states of S) you have i ($i = 2^{n-m}$) combinations C_i of the other states. You call an s-world any state that holds when s holds. A combination C_i of states is also a state so you can define it in the same way a C_i-world. Von Wright gives two definitions (strong and weak) of preference:

1. (strong): s is preferred to t under the circumstances C_i **iff** every C_i-world that is also an s-world and not a t-world is preferred to every C_i-world that is also t-world and not s-world.
2. (weak): s is preferred to t under the circumstances C_i **iff** some C_i-world that is an s-world is preferred to a C_i-world that is a t-world, but a C_i-world that is a t-world that is preferred to a C_i-world that is an s-world does not exist.

Now s is *"ceteris paribus"* preferred to t **iff** it is preferred under all C_i. We leave the discussion to the interested reader, but we point out that, with these definitions, it is difficult to axiomatise both transitivity and complete comparability unless they are assumed as necessary truths for "coherence" and "rationality" (see von Wright, 1972).

It can be concluded that the philosophical discussion about preferences failed the objective to give a unifying frame of generalised preference relations that could hold for any kind of states, based on a well-defined axiomatisation (for an interesting discussion see Mullen, 1979). It is still difficult (if not impossible) to give a definition of good or bad in absolute terms based on reasoning about preferences and the properties of these relations are not unanimously accepted as axioms of preference modelling. For more recent advances in deontic logic see Nute, 1997.

More recently, Von Wright's ideas and the discussion about "logical representation of preferences" attracted attention again. This is due to problems found in the field of Artificial Intelligence field due to essentially two reasons:

- the necessity to introduce some "preferential reasoning" (see Boutilier, 1994, Boutilier et al., 1999, Brafman and Friedman, 2001, Doyle, 1989, Doyle, 1990, Doyle, 1994, Kraus et al., 1990, Lehmann, 2001, Shoham, 1987);
- the large dimension of the sets to which such a reasoning might apply,

thus demanding a compact representation of preferences (see Benferhat and Kaci, 2001, Benferhat et al., 2002a, Benferhat et al., 2002b, Domshlak and Brafman, 2002, Lafage and Lang, 2000)

Even if these motivations may appear different, the link between them is surprisingly strong as they use related languages. In fact, in both of these cases, the idea is to propose a language allowing a succinct representation of the problem without enumerating a prohibitive number of alternatives and being as close as possible to the way that a decision maker expresses his preferences in a natural language. The two common approaches consist on the use of the propositional logic or a graphical language for the representation of preferences which may be given as an ordinal data (generally a preorder) or as an utilitarian preferences.

Concerning the propositional logic, a survey may be found in Lang, 2006. In this field some authors have been interested on the use of penalties or weigthed bases with propositional formulae (see de Saint-Cyr et al., 1994, Pinkas, 1995, Had92, Langetal94, Benferhat et al., 2001, Chevaleyre et al., 2006, Coste-Marquis et al., 2004, Öztürk and Marquis, 2009, among others) others have proposed the use of distance between logical worlds (see Lafage and Lang, 2000, Lafage and Lang, 2001).

Graphical languages have been proposed for qualitative and quantitative preferences specially when the set of alternatives is defined as the cartesian product of finite domains and when there are some interactions between criteria. *Generalized additive decomposable* (*GAI*) utility functions have been introduced by Fishburn, 1970 in order to represent interaction between criteria by preserving some decompasability of the model. One of the earliest studies to exploit separable preferences in a graphical model is the extension of influence diagrams (see J. A. Tatman and R. D. Shachter, 1990), then Bacchus and A. J. Grove, 1995 have introduced the GAI-nets, the first graphical model based on conditional independence structure. The elicitation in GAI-nets have been adressed in Braziunas and Boutilier, 2005, Braziunas and Boutilier, 2007 and Ch. Gonzales and Perny, 2004. Another important research line is about *CP-nets* which propose a qualitative graphical representation of preferences interpreting conditional independence of preference statements under a *ceteris paribus* (all else being equal) principle. The idea of using *ceteris paribus* principale is due to Von Wright (vonWright63) and have became to be used by AI researches for twenty years, firstly by Doyle (Doyle, 1989, Doyle, 1990), and then others have been interested in different aspects such as elicitation, consistency, computation of a result, ... (for more details see Boutilier et al., 2004a, Domshlak, 2002, Goldsmith et al., 2005).

9. Conclusion

We hope that this chapter on preference modelling, gave the non-specialist reader a general idea of the field by providing a list of the most important references of a very vast and technical literature. In this chapter, we have tried to present the necessary technical support for the reader to understand the following chapters. One can note that our survey does not interpret all the questions related to preference modelling. Let us mention some of them:
- How to get and validate preference information (von Winterfeldt and Edwards, 1986), Bana e Costa and Vansnick, 1999
- The relation between preference modelling and the problem of meaningfulness in measurement theory (Roberts, 1979)
- Statistical analysis of preferential data (Coombs, 1964, Green et al., 1988)
- Interrogations on the relations between preferences and the value system, and the nature of these values(Broome, 1991, Cowan and Fishburn, 1988, Tsoukiàs and Vincke, 1992, von Wright, 1963).

Notes

1. we can use the word *action* instead of alternative

2. Lets take an example: Imagine that we have to choose one car between two. We have to know the performance of each car in order to establish the relation of preference:

- in the first case, the performance of each car is known and noted between 1 and 10 ($p(car1) = 8$ and $p(car2) = 5$); the relation of preference is known too (car1 is preferred to car2: $car1Pcar2$ ($\mu(car1Pcar2) = 1$))
- in the second case, the performance of each car is known and noted between 1 and 10 ($p(car1) = 8$ and $p(car2) = 7$); we are not sur about the preference relation that is why the relation of preference is fuzzy ($\mu(car1Pcar2) = 0.75$)
- in the third case, the performance of each car is fuzzy (in this case the performances of each car will be defined by fuzzy numbers; in this case we can use triangular or trapezoidal fuzzy number to represent the performance); the relation of preference is crisp (car1 is preferred to car2: $car1Pcar2$ ($\mu(car1Pcar2) = 1$))
- in the fourth case, the performance of each car is fuzzy (in this case the performances of each car will be defined by fuzzy numbers); the preference relation is also fuzzy (($\mu(car1Pcar2) = 0.75$))

3. a suitable one can be the complement operator defined: $n(\mu(x)) = 1 - \mu(x)$

4. to have a partition of the set $A \times A$, the inverse of the asymmetric relation must be considered too.

5. While several authors prefer using both of them, there are others for which one is sufficient. For example Fishburn does not require the use of preference structures with a characteristic relation

6. one can be indifferent between a cup of tea with n milligrams of sugar and one with $n+1$ milligrams of sugar, if one admits the transitivity of the indifference, after a certain step of transitivity, one will have the indifference between a cup of tea with n milligram of sugar and that with $n+N$ milligram of sugar with N large enough, even if there is a very great difference of taste between the two; which is contradictory with the concept of indifference

7. this value can be given directly by the decision-maker or calculated by using different concepts, such values (indices) are widely used in many MCDA methods such as ELECTRE, PROMETHEE (**?**,**?**)

8. the function g defines an ordinal scale for both structures

9. but in this case the scale defined by g is more complex than an ordinal scale

10. when it is a partial order of dimension 2, the detection can be made in a polynomial time

11. One can imagine that $l(a)$ represents the evaluation of the alternative a $(g(a))$ which is the left limit of the interval and $r(a)$ represents the value of $(g(a)+q(a))$ which is the right limit of the interval. One can remark that a semi-order is an interval order with a constant length.

12. The MAX extension is also known, in the context of opportunity sets, as the *indirect-utility criterion*, i.e. the criterion to rank sets is the best possible choice that can be made.

13. Together with another property called *fixed-Cardinality neutrality*, saying that the labelling of the alternatives is irrelevant in establishing the ranking among sets of fixed cardinality q.

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