Geometric Representations of Graphs

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Bipartite planar graphs: representation by contacts of segments

vertices $\rightarrow$ segments
edges $\rightarrow$ contact points
same topology
Bipartite planar graphs: Contacts of segments $\rightarrow$ 2 trees on 2 pages

Let $G$ be a planar graph such that for any subgraph $H$ of $G$ (with $n(H) > 1$):

- $m(H) \leq 2n(H) - 2$ then $G$ is representable by a contact family of pseudo-segments.
- $m(H) \leq 2n(H) - 3$ then $G$ is representable by a contact family of segments.

K4 is not representable by a contact family of segments
Planar graphs:
Representation by contacts of triangles → contacts of T

Exponential size

Linear size
Vertex packing algorithm →
• straight line drawing on a linear size grid
• representation by contacts of triangles
Incidence graph of a graph / a contact system

Indegree $\leq 1$

Indegree $= 2$

$(2,\leq 1)$ orientation
Planar Linear Hypergraphs: Representation by contacts of segments and/or triangles
(Vertices are represented by segments or triangles Edged by contact points)

\( H \text{ linear} \Leftrightarrow \) 2 edges share at most 1 vertex
Planar Linear Hypergraphs:
Representation by contacts of segments and/or triangles
(Edges are represented by segments or triangles
Vertices by contact points)
Our Hypergraph H
(linear, planar)
Incidence graph $\mathbf{R}$ of a planar linear hypergraph $\mathbf{H}$: planar bipartite graph without cycle of length 4

(white vertex $\rightarrow$ triangle/segment
black vertex $\rightarrow$ contact point)

$\mathbf{H}$ planar $\Leftrightarrow$ $\mathbf{R}$ planar
Incidence graph

3-Orientation

Splitting some vertices

Symlifying $\rightarrow (2, \leq 1)$ Orientation
Construction of a \((2, \leq 1)\)-orientation:

- White vertices will get exactly 2 incoming edges.
- Black vertices will get at most 1 incoming edge.

Make all faces of length 6

Add a vertex \(r\) incident to the black vertices of the external face

Double all edges
\textbf{\(\lambda\)-orientation of a multigraph}

\textit{Lemma}:
Let \(G\) be a multigraph, let \(\lambda\) be a mapping from \(V(G)\) to \(\mathbb{N}\). Then there exists an orientation of \(G\) such that each vertex \(v \in V(G)\) has indegree bounded by \(\lambda(v)\) if and only if
\[\forall A \subseteq V(G) : |E(G[A])| \leq \sum_{v \in A} \lambda(v)\]
Moreover, this orientation is such that each vertex \(v\) has indegree \(\lambda(v)\) if and only if we also have the global condition
\[|E(G)| = \sum_{v \in V(G)} \lambda(v)\]

3-orient the graph

We define \(\lambda(v) = 3\) for the original vertices and \(\lambda(r) = 0\) for the extra vertex.
Using Euler formula, the previous lemma applies.
Types of Vertices

Type I
- Both incoming to white
- Both incoming to black
- Otherwise

Type II
- Both incoming to white
- One incoming to black
- One incoming to white
Split **white** vertices of type 2

Type I

Type II

SPLIT!
Finally we get a \((2,\leq 1)\)-Orientation
(2, \leq 1)-Orientation

White: indegree = 2  
Black: indegree \leq 1
Contacts of Pseudo-Segments
Stretching the Pseudo-Segments
Eventually...
Thank you for your attention…