# Parameterized Approximability of Influence in Social Networks

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# Outline

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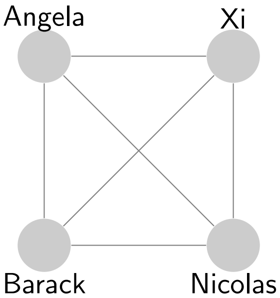
Introduction

Hardness

FPT Approximation

# An Example from Viral Marketing



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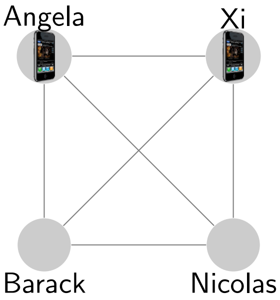


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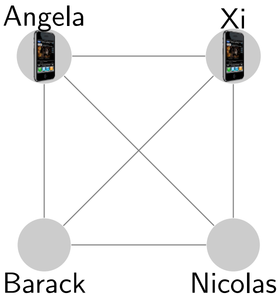


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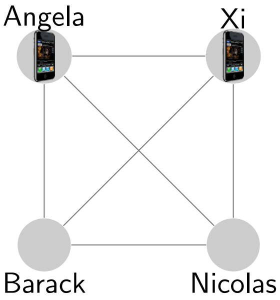


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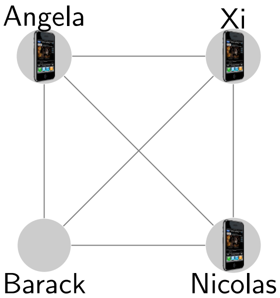


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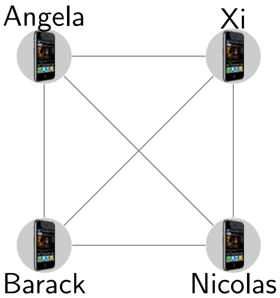


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- Apple sold two iPhones without any advertisement!

# An Example from Viral Marketing

- The first customers (target set) had an iPhone (Apple gave bonuses, free phones...).
- Goal: get the fewest customers with advertisement in order to **attract the maximum number** of customers at the end.
- What is the target set of customers to attract?

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- Goal: get the fewest customers with advertisement in order to **attract the maximum number** of customers at the end.
- What is the target set of customers to attract?
- Other applications:
  - Spreading of information/influence in social networks via word-of-mouth recommendations.
  - Diseases in populations.
  - Faults in distributed computing.
  - ...

# Problem Definition

- **Diffusion** (threshold model):
  - A vertex of the graph is **activated** if it is in the **target set** or if at least $thr(v)$ of its **neighbors** are activated.
- **Optimization** problem [Kempe et al. 2003]:

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**Max Influence:**

- **Input:** A graph, a threshold for each vertex, an integer $k$.
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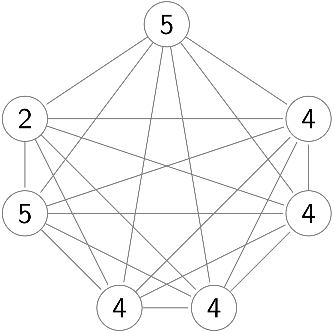
**Decision Influence:**

- **Input:** A graph, a threshold for each vertex, an integer $k$, **an integer $l$**.
- **Output:** A subset of vertices of size at most $k$ s.t. the number of activated vertices is **at least $l$**.

# Activation Process - Example




Numbers in vertex = threshold of the vertex

$k = 3$, maximize the number of activated vertices.

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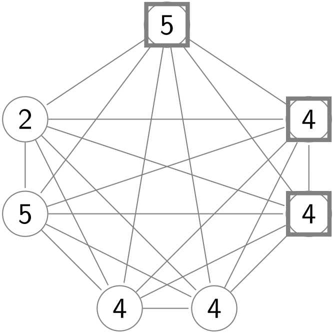


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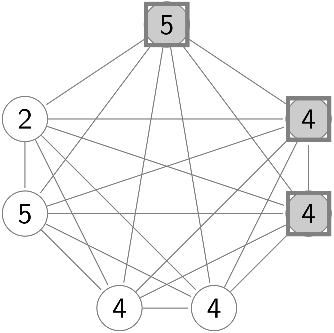


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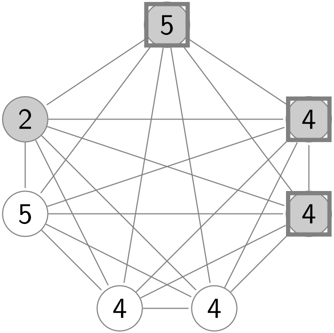


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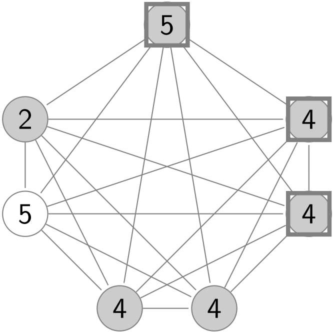


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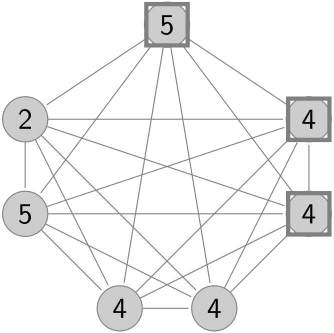


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# Thresholds

- Different types of thresholds:
  - General.
  - Constant.
  - Majority ($thr(v) = \lceil degree(v)/2 \rceil$).
  - Unanimity ($thr(v) = degree(v)$).

# Cardinality Constraint Problems

- Our problem is part of a larger class of problems formalized in [Cai 2008].
- Find a solution of **cardinality** $k$ (given in the input) s.t. an objective is **maximized** (or minimized).
- Examples:
  - Max Vertex Cover: Find $k$ vertices s.t. the number covered edges is maximum.
    - Classical Vertex Cover is FPT.
    - Decision version of Max Vertex Cover is W[1]-hard.
  - Max Dominating Set.
  - Same problems with minimization.
  - ...

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- Can achieve better ratios if we remove the polynomial-time constraint [Cai et al. 2006, Chen et al. 2006, Downey et al. 2006].

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  - Algorithm achieve a $f(n)$-approximation.
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- Pertinent for cardinality constraint problems!
  - Time parameterized by $k$.
  - Minimize/Maximize the objective.
- Example for Max Vertex Cover:
  - No polynomial-time approximation scheme, W[1]-hard.
  - Admits a fpt-time approximation scheme [Marx 2008].

# Known results

- (Of course), NP-hard, even in bipartite graphs and thresholds=2 [Chen 2009].
- **Hard to approximate** within $O(2^{\log^{1-\epsilon} n})$, even if thresholds are bounded by 2 and the graph is bipartite [Chen 2009].
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- Our problem is hard to approximate and W[2]-hard:
  - Can we have better approximation ratio if we allow fpt-time?

# Outline

# Reduction

**Dominating Set**

- **Input:** A graph, an integer $k$.
- **Output:** A subset of the vertices of size at most $k$ s.t. each vertex of the graph has at least a neighbor in the solution.

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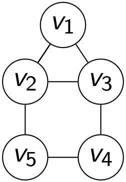

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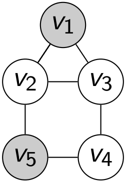

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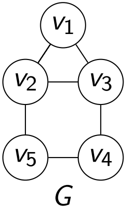

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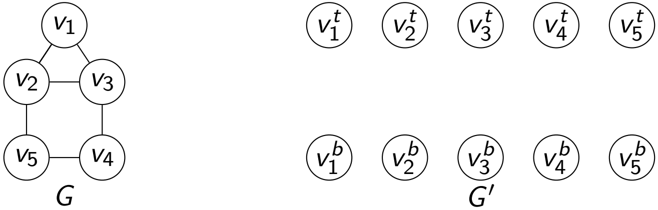
- From an instance of DOMINATING SET with parameter $k$.
- Build an instance of our problem.
  - Two copies of the vertex set .



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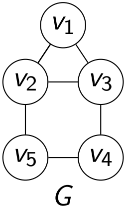


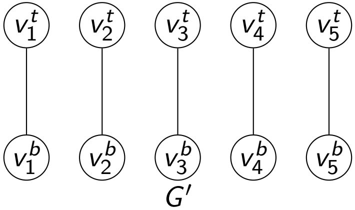

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- From an instance of DOMINATING SET with parameter $k$.
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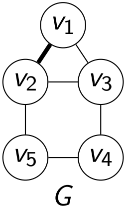


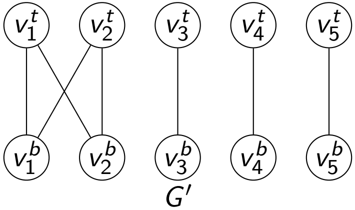

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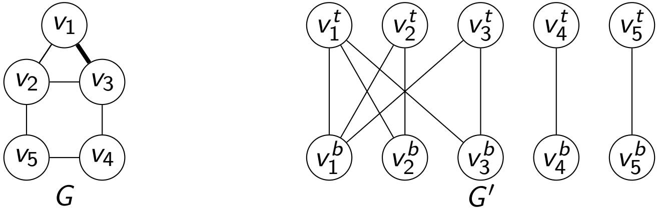

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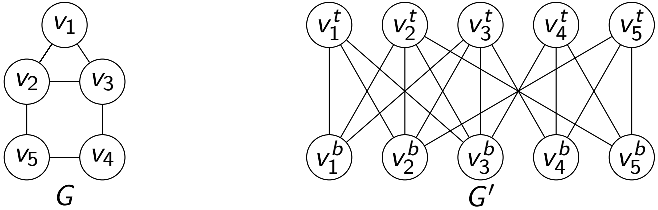

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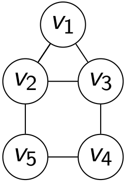


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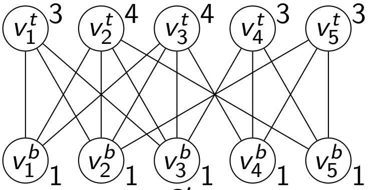


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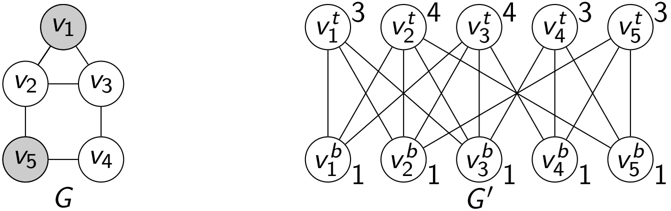


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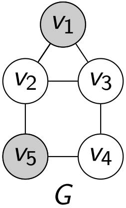


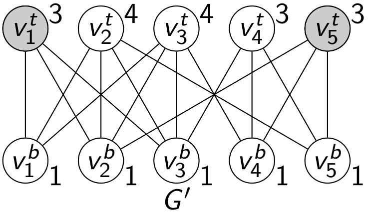


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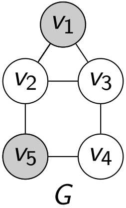


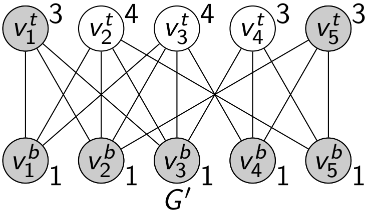


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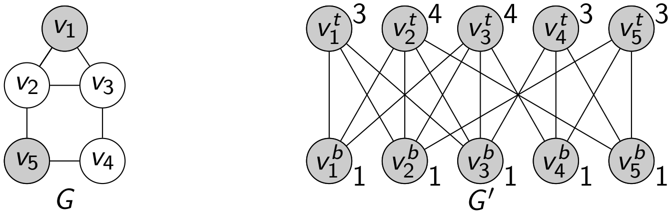


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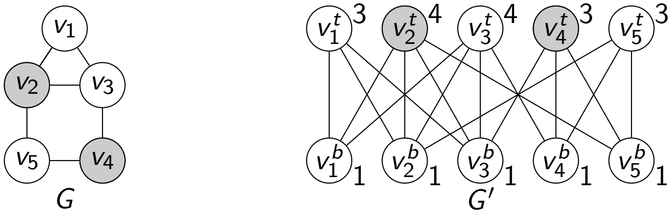


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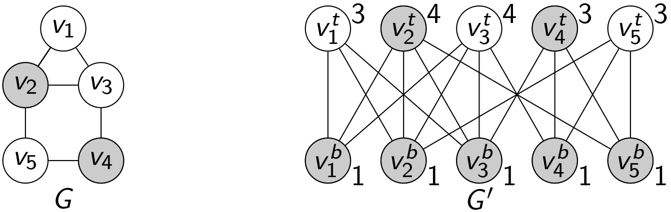


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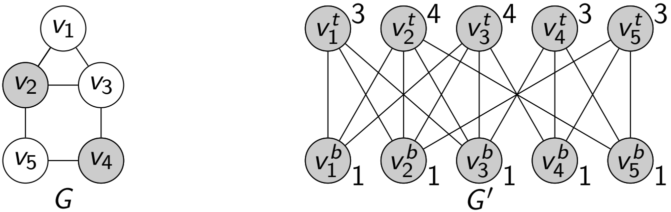


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# Parameterized Intractibility

- With **additional gadgets**, we can prove that our problem **cannot be approximated** within $n^{1-\epsilon}$ in **fpt-time** unless FPT = W[2].
- Even if:
  - The graph is bipartite.
  - Majority thresholds.
  - Thresholds are at most 2.
  - All the activated vertices (including the target set) are counted.

# Outline

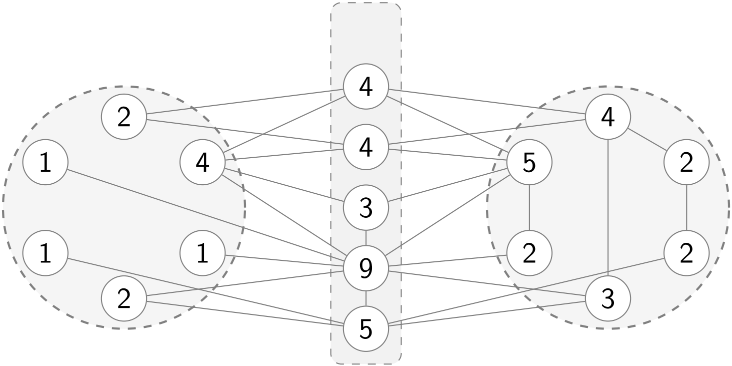
Introduction

Hardness

**FPT Approximation**

# Unanimity Thresholds

- All the neighbors of a vertex must be activated.
  - Only one round!



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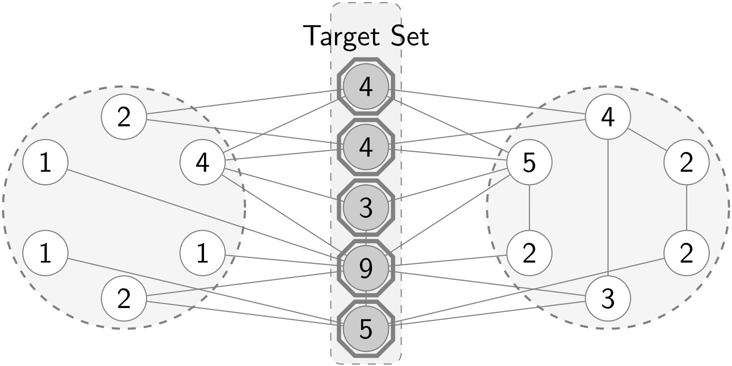

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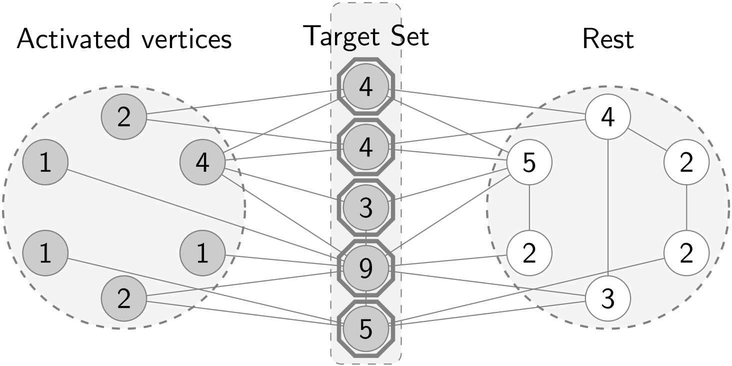

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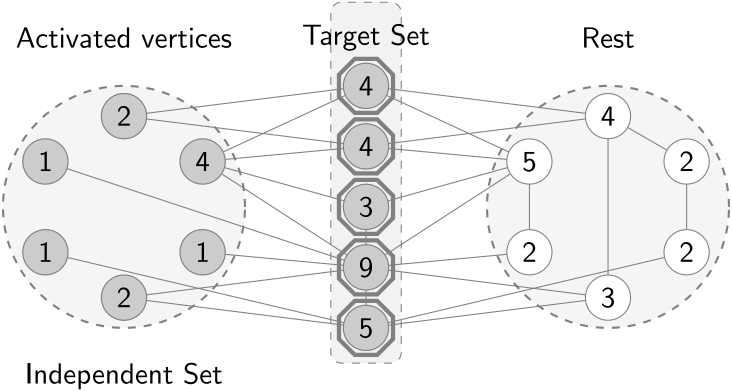

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- **Cannot** be approximated within $n^{1-\epsilon}$ in **polynomial time**...
- But it is $2^k$-**approximable in polynomial time**!
- Therefore, it is $r(n)$-**approximable in fpt-time**, for any strictly increasing function $r$.

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- All together: **approximation algorithm in fpt-time** ($\log_2(n)$-approximation in time $O^*(2^{k2^k})$).
- Generalization: replace $\log_2(n)$ by any strictly increasing function of n.
  - A worse running time implies a better ratio.

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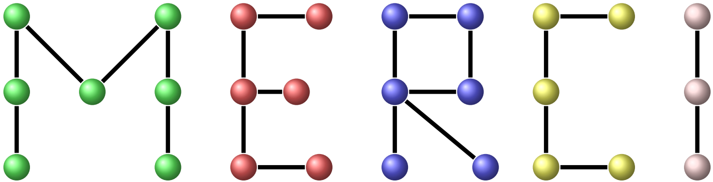
- Apply to all cardinality constraint problems.
- We can make a E-reduction from DENSEST $k$-SUBGRAPH to our problem with unanimity thresholds.
- $\Rightarrow$ DENSEST $k$-SUBGRAPH is also $r(n)$-approximable in fpt-time, for any increasing function $r$.

# Conclusion

- Problem hard to approximate, even in fpt-time.
- If the thresholds are unanimity, the problem is a bit easier.
- In the paper, more positive results (approximation, fpt) if we focus on bounded degree graphs.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  - Solution size $k$ in a $n$-vertices graph.
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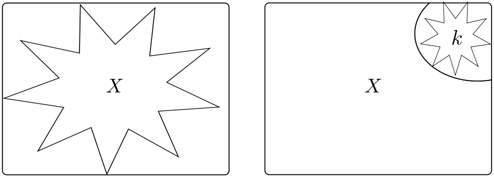

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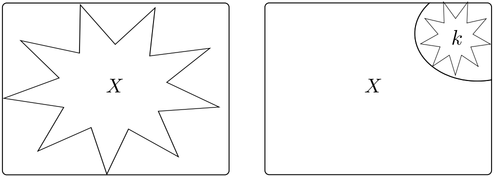


- Complexity classes: $\text{FPT} \subseteq \overbrace{\text{W[1]} \subseteq \text{W[2]} \subseteq \dots}^{\text{presumably} \not\subset \text{FPT}}$

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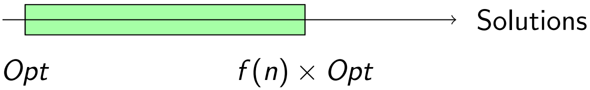
- With a bound on the error.
- An algorithm is a $f(n)$-**approximation** if it runs in polynomial-time and if the cost of the returned solutions is bounded in the worst case by $r \times Opt$ (for minimization problems).



# Unanimity Thresholds - A $2^k$ approximation in poly-time

- Find the largest set of "false-twins" with all vertices degree bounded by $k$.
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- There is at most $2^k$ different false-twins set.