Parameterized Approximability of Target Set Selection

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Outline

Introduction

Parameterized Approximation

Hardness
Outline

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Parameterized Approximation

Hardness
An Example from Viral Marketing

Barack: "If three of my friends have an iPhone, I buy an iPhone too"

François: "If two of my friends have an iPhone, I buy an iPhone too"

Apple sold two iPhones without any advertisement!
An Example from Viral Marketing

- Angela
- János
- Barack
- François

Barack: "If three of my friends have an iPhone, I buy an iPhone too"
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An Example from Viral Marketing

▶ Angela: "If three of my friends have an iPhone, I buy an iPhone too"
▶ János: "If two of my friends have an iPhone, I buy an iPhone too"
▶ Barack: “If three of my friends have an iPhone, I buy an iPhone too”
▶ François: "If two of my friends have an iPhone, I buy an iPhone too"
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- Barack: “If **three** of my **friends** have an iPhone, I buy an iPhone too”
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An Example from Viral Marketing

- Goal: get the **fewest customers** with advertisement in order to **attract all customers** at the end.
An Example from Viral Marketing

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- Other applications:
  - Spreading of information/influence in social networks via word-of-mouth recommendations.
  - Diseases in populations.
  - Faults in distributed computing.
  - ...


Problem Definition

- **Diffusion** (threshold model):
  - A vertex of the graph is activated if it is in the target set or if at least \( thr(v) \) of its neighbors are activated.
Problem Definition

- **Diffusion** (threshold model):
  - A vertex of the graph is **activated** if it is in the *target set* or if at least $thr(v)$ of its *neighbors* are activated.

- **Optimization** problem [*Chen 2008*]:

**Min Target Set Selection:**

- **Input:** A graph, a threshold for each vertex.
- **Output:** A subset of vertices of *minimum cardinality* s.t. all vertices of the graph are activated at the end of the diffusion process.
Problem Definition

- **Diffusion** (threshold model):
  - A vertex of the graph is *activated* if it is in the *target set* or if at least \(thr(v)\) of its *neighbors* are activated.

- **Decision problem** [Chen 2008]:

**Target Set Selection:**

- **Input:** A graph, a threshold for each vertex, **an integer** \(k\).
- **Output:** A subset of vertices of **size at most** \(k\) s.t. all vertices of the graph are activated at the end of the diffusion process.
**Activation Process - Toy Example** (*K_7*)

Numbers in vertex = threshold of the vertex

\[ k = 3. \]
Activation Process - Toy Example \((K_7)\)

Numbers in vertex = threshold of the vertex
\(k = 3\).
Squares = target set
Activation Process - Toy Example ($K_7$)

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Types of thresholds

- General. \((1 \leq thr(v) \leq deg(v))\)
Types of thresholds

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**Types of thresholds**

- General. \(1 \leq \text{thr}(v) \leq \text{deg}(v)\)
- Constant. \(1 \leq \text{thr}(v) \leq c\)
- Majority. \(\text{thr}(v) = \lceil \text{deg}(v)/2 \rceil\).
- Unanimity. \(\text{thr}(v) = \text{deg}(v)\).
Outline

Introduction

Parameterized Approximation

Hardness
Fixed-Parameter Tractability

“Measuring complexity only in terms of the input size means ignoring any structural information about the instances”

J. Flum and M. Grohe

“Question: When will the input of a problem coming from “real life” have no more structure than its size? Answer: Never!”

R. Downey and M. Fellows
"Measuring complexity only in terms of the input size means ignoring any structural information about the instances"

J. Flum and M. Grohe

"Question : When will the input of a problem coming from "real life" have no more structure than its size?
Answer : Never!"

R. Downey and M. Fellows

"The fundamental idea is to restrict the combinatorial explosion, seemingly unavoidable, that causes the exponential growth in the running time of certain problem-specific parameters..."

R. Niedermeier
Fixed-Parameter Tractability

- Problem in FPT: any instance \((I, k)\) solved in \(f(k) \cdot |I|^c\).

Examples:
- Solution size \(k\) in a \(n\)-vertices graph.
- \(n\) voters for \(k\) candidates.
- Requests of size \(k\) in a \(n\)-sized database.
- ...
Fixed-Parameter Tractability

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  ▶ Solution size \(k\) in a \(n\)-vertices graph.
  ▶ \(n\) voters for \(k\) candidates.
  ▶ Requests of size \(k\) in a \(n\)-sized database.
  ▶ ...

▶ Complexity classes: \(\text{FPT} \subseteq W[1] \subseteq W[2] \subseteq \ldots\)

presumably \(\not\subseteq \text{FPT}\)
Polynomial Approximation (minimization)

\[ \text{Solutions} \]

\[ \text{Opt} \quad f(n) \times \text{Opt} \]
Coping with the hardness

<table>
<thead>
<tr>
<th></th>
<th>Time</th>
<th>Solution Quality</th>
</tr>
</thead>
<tbody>
<tr>
<td>FPT</td>
<td><img src="image" alt="FPT Time" /></td>
<td><img src="image" alt="FPT Solution" /></td>
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FPT-Approximation

- A (minimization) problem is \textit{fpt-}\(\rho\)-\textit{approximable} if for any input \((I, k)\):
  - If \(\text{opt}(I) \leq k\), computes a solution of value bounded by \(\rho(k) \cdot k\) in time \(f(k)|I|^{O(1)}\),
  - Otherwise, output can be arbitrary.
Example for treewidth

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<tr>
<td>FPT</td>
<td>$2^{O(k^2)} \cdot n$</td>
<td>1</td>
<td>[Bodlaender 96]</td>
</tr>
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<td>poly($n$)</td>
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<td>[Feige et al. 05]</td>
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## Example for treewidth

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<tr>
<td><strong>FPT Approx.</strong></td>
<td>$2^{O(k)} \cdot n$</td>
<td>$5$</td>
<td>[Bodlaender et al. 13]</td>
</tr>
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FPT-inapproximability

- Not fpt-$\rho$-approximation for any function $\rho$:  
  - INDEPENDENT DOMINATING SET [Downey et al. 2008]
  - WEIGHTED CIRCUIT SAT [Chen et al. 2006]
FPT-inapproximability

- Not fpt-$\rho$-approximation for any function $\rho$:
  - Independent Dominating Set [Downey et al. 2008]
  - Weighted Circuit Sat [Chen et al. 2006]
- Not-monotone problems...
  - If optimum is $k$ and every feasible solution has cost $k \rightarrow$ as hard as decision.
FPT-inapproximability - Monotone problems

- **Clique** or **Dominating Set** remain open and challenging.
FPT-inapproximability - Monotone problems

- **Clique** or **Dominating Set** remain open and challenging.
- **Monotone Weighted Circuit Sat** (no negation) is not fpt-approximable. [Marx 2013]
Outline

Introduction

Parameterized Approximation

Hardness
Known results for Target Set Selection

- **Hard to approximate** in poly-time within $O(2^{\log^{1-\epsilon} n})$ [Chen 2009].
- **W[2]-hard** for parameter solution size [Nichterlein et al. 2012].
Known results for Target Set Selection

- **Hard to approximate** in poly-time within $O(2^{\log^{1-\epsilon} n})$ [Chen 2009].

- **W[2]-hard** for parameter solution size [Nichterlein et al. 2012].

- TSS is hard to approximate and W[2]-hard:
  - Can we have better approximation ratio if we allow fpt-time?
Directed edge gadget

$u$
General idea - Bipartite graphs

- From Monotone Weighted Circuit Sat.

```
\[ \begin{array}{cccc}
  v_1 & v_2 & v_3 & v_4 \\
\end{array} \]
```

\[
\begin{array}{cccc}
  v_1 & v_2 & v_3 & v_4 \\
  1 & 1 & 1 & 1 \\
  2 & 2 & 2 & 2 \\
\end{array}
\]
Results

- Reduction with one-to-one correspondence between solutions.
- As Monotone Circuit Sat, Target Set Selection is not fpt-$\rho$-approximable, for any function $\rho$.
- Can be extended to majority and constant threshold via additional gadgets.
**Unanimity Thresholds**

- All the neighbors of a vertex must be activated.
  - Only one round!
Unanimity Thresholds

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Unanimity Thresholds

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Target Set
**Unanimity Thresholds**

- All the neighbors of a vertex must be activated.
  - Only one round!

\[ \begin{align*}
1 & \quad 1 \\
2 & \quad 2 \\
4 & \quad 4 \\
1 & \quad 1 \\
2 & \quad 2 \\
2 & \quad 2
\end{align*} \]

- Equivalent to **Vertex Cover**: in FPT.
Conclusion

▶ Hard hard hard hard.
Conclusion

- Hard hard hard hard hard.
- Dual of the problem?
  - Unanimity thresholds:
    - Equivalent to \textsc{Independent Set}: \textsc{W[1]}-hard.
  - Majority or constant thresholds?
  - fpt-approximation?
Köszönöm!